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Adiabatic invariants of generalized Lutzky type for disturbed holonomic nonconservative systems*

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Based on the definition of higher-order adiabatic invariants of a mechanical system, a new type of adiabatic invariants, i.e. generalized Lutzky adiabatic invariants, of a disturbed holonomic nonconservative mechanical system are obtained by investigating the perturbation of Lie symmetries for a holonomic nonconservative mechanical system with the action of small disturbance. The adiabatic invariants and the exact invariants of the Lutzky type of some special cases, for example, the Lie point symmetrical transformations, the special Lie symmetrical transformations, and the Lagrange system, are given. And an example is given to illustrate the application of the method and results.

Keywords: analytical mechanics, disturbed holonomic nonconservative system, Lie symmetrical perturbation, adiabatic invariant of Lutzky type

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1. Introduction

Symmetrical perturbation and adiabatic invariants for a dynamical system play an important role in the fields of mathematics, mechanics and physics.[1−8] In 1917, Burgers first proposed adiabatic invariants referring to a special kind of Hamilton systems.[1] A classical adiabatic invariant is a certain physical quantity that changes more slowly than some parameter of the system which varies very slowly.[8] In fact, the parameter varying very slowly is equivalent to the action of small disturbance. For a mechanical system, there exists intimate relations among the integrability of the system, the variations of its symmetries, and the invariants under the action of small disturbance. Therefore, research on symmetrical perturbation and adiabatic invariants is of great significance.[3,8] More and more attention has been paid to research on symmetrical perturbation and adiabatic invariants of mechanical systems, and some important results have been reported in recent years. But the adiabatic invariants obtained by all these studies belong to the type of Noether and the type of Hojman.[3,7−19]

In 1979, Lutzky applied the Lie's theory to the differential equations of motion for a mechanical system, and obtained a new type of conserved quantities of the dynamical system under Lie point symmetrical transformations,[20−23] i.e. the Lutzky conserved quantities. In recent 20 years, researchers developed the Lutzky's method and obtained some important results (see Refs.[24–30]). But the invariants of Lutzky type obtained by all these studies belong to the conserved quantities of Lutzky type for the dynamical system without perturbations. Recently, Refs.[31, 32] gave the adiabatic invariants of Lutzky type of a Lagrange system.

In this paper, we further study the Lie symmetrical perturbation under general infinitesimal transformations of groups for holonomic nonconservative system with the action of small disturbance, and will present a new type of adiabatic invariants; we refer to the new adiabatic invariants as adiabatic invariants of generalized Lutzky type of disturbed holonomic nonconservative system. Also, we present an example to illustrate the application of the results.

2. Disturbed equations of holonomic nonconservative system

Here we consider a holonomic nonconservative mechanical system whose configuration is determined

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by the n generalized coordinates q_s $(s = 1, \ldots, n)$, and the differential equations of motion of the system can be expressed as the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \qquad (s = 1, \ldots, n), \tag{1}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is Lagrangian, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonpotential generalized forces. In general, suppose that the system is not singular, then all the generalized accelerations can be solved from Eqs.(1)

$$\ddot{q}_s = \frac{M_{sk}}{D}\left(\frac{\partial L}{\partial q_k} - \frac{\partial^2 L}{\partial \dot{q}_k \partial t} - \frac{\partial^2 L}{\partial \dot{q}_k \partial q_j}\dot{q}_j + Q_k\right), \tag{2}$$

and a relation between $\ddot{q}_s$ and L can be easily given by

$$\frac{\partial \ddot{q}_s}{\partial \dot{q}_s} - \frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{sk}}{D}Q_k\right) + \frac{\mathrm{d}}{\mathrm{d}t}\ln D = 0 \qquad (s, k = 1, \ldots n), \tag{3}$$

where

$$D = \det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{4}$$

M_{sk} are the cofactors of the matrix element $\partial^2 L/\partial \dot{q}_s \partial \dot{q}_k$. If the generalized accelerations of the system are

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n), \tag{5}$$

then Eqs.(3) give

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} - \frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{sk}}{D}Q_k\right) + \frac{\mathrm{d}}{\mathrm{d}t}\ln D = 0 \qquad (s, k = 1, \ldots, n). \tag{6}$$

Suppose the holonomic nonconservative system given by Eqs.(1) is disturbed by small forces of perturbation εW_s, then the equations of motion of the system can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \varepsilon W_s \quad (s = 1, \ldots, n). \tag{7}$$

Expanding Eqs.(7), we can obtain all generalized accelerations of the disturbed system as follows:

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\frac{M_{sk}}{D}W_k \quad (s, k = 1, \ldots, n). \tag{8}$$

Under the action of small forces of perturbation εW_s, the original motion of the system may vary. Assuming that the disturbed Eqs.(8) are small perturbation on the basis of the undisturbed Eqs.(5) of the system, we have from Eqs.(3)

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} - \frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{sk}}{D}Q_k\right) + \varepsilon\frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right) + \frac{\mathrm{d}}{\mathrm{d}t}\ln D = 0 \quad (s, k = 1, \ldots, n), \tag{9}$$

where ε is an infinitesimal parameter.

3. Lie symmetrical perturbation of disturbed holonomic nonconservative system

We now study the Lie symmetrical perturbation for the holonomic nonconservative mechanical system under the action of small disturbance, and obtain the Lie symmetrical determining equations of the disturbed system.

Introduce a single-parameter infinitesimal transformation with respect to time and coordinates

$$t^* = t + \varepsilon\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{10}$$

where τ, ξ_s are infinitesimal generators. Taking the infinitesimal generator vector

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \tag{11}$$

and its first prolonged vector

$$X^{(1)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s}. \tag{12}$$

After the action of small forces of perturbation, the invariance of Eqs.(8) under the infinitesimal transformations (10) leads to the satisfaction of the determining equations

$$\ddot{\xi}_s - \dot{q}_s\ddot{\tau} - 2\dot{\tau}\left(\alpha_s + \varepsilon\frac{M_{sk}}{D}W_k\right) = X^{(1)}(\alpha_s) + \varepsilon X^{(1)}\left(\frac{M_{sk}}{D}W_k\right) \qquad (s, k = 1, \ldots, n). \tag{13}$$

Under the action of small forces of perturbation εW_s, the original symmetries and invariants of the system may vary. Assuming that the disturbed generators $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of infinitesimal transformations are small perturbation on the basis of the generators of symmetrical transformation of undisturbed system, we have

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \ldots,$$
$$\xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \ldots. \tag{14}$$

Substituting Eqs.(14) into vector (12), we have

$$X^{(1)} = \varepsilon^m X_m^{(1)} \quad (m = 0, 1, 2, ...), \tag{15}$$

where

$$X_m^{(1)} = \tau^m \frac{\partial}{\partial t} + \xi_s^m \frac{\partial}{\partial q_s} + (\dot{\xi}_s^m - \dot{q}_s \dot{\tau}^m) \frac{\partial}{\partial \dot{q}_s} \quad (m = 0, 1, 2, \ldots). \tag{16}$$

Substituting Eqs.(14) and (15) into Eqs.(13), and comparing the coefficients of ε^m on the left side of equality with those on the right side, we obtain Lie symmetrical determining equations of the disturbed system (8) as follows:

$$\begin{aligned} &\ddot{\xi}_s^m - \dot{q}_s \ddot{\tau}^m - 2\dot{\tau}^m \alpha_s - 2\dot{\tau}^{m-1} \frac{M_{sk}}{D} W_k \\ =&X_m^{(1)}(\alpha_s) + X_{m-1}^{(1)} \left(\frac{M_{sk}}{D} W_k \right) \\ &(s, k = 1, \ldots, n), \end{aligned} \tag{17}$$

where, we set $\tau^{-1} = 0$ and $\xi_s^{-1} = 0$ when $m = 0$.

4. A new type of non-Noether adiabatic invariants

The following theorem will present a new type of non-Noether adiabatic invariants directly led by Lie symmetrical perturbation of a disturbed holonomic nonconservative system.

Definition 1 If $I_z = (t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to t is in direct proportion to ε^{z+1}, then I_z is called a z-th-order adiabatic invariant of a mechanical system.

Theorem 1 For a holonomic nonconservative system which is disturbed by small forces of perturbation εW_s, if the generators τ^m and ξ_s^m of infinitesimal transformations (10) satisfy the Lie symmetrical determining equations (17), and there exists a gauge function $G_L = G_L(t, \boldsymbol{q}, \dot{\boldsymbol{q}},)$ satisfying the following relation

$$\frac{\partial}{\partial \dot{q}_s} \left(\frac{M_{sk}}{D} Q_k \right) = \dot{G}_L, \tag{18}$$

then the disturbed holonomic nonconservative system possesses the following new type of z-th-order adiabatic invariant

$$\begin{aligned} I_z = \sum_{m=0}^{z} \varepsilon^m \Bigg[&\frac{\partial \xi_s^m}{\partial q_s} + \frac{\partial \dot{\xi}_s^m}{\partial \dot{q}_s} - \dot{q}_s \left(\frac{\partial \tau^m}{\partial q_s} + \frac{\partial \dot{\tau}^m}{\partial \dot{q}_s} \right) \\ &- \alpha_s \frac{\partial \tau^m}{\partial \dot{q}_s} - n\dot{\tau}^m + X_m^{(1)}(\ln D) \\ &- X_m^{(1)}(G_L) \Bigg]. \end{aligned} \tag{19}$$

Proof In fact, for an arbitrary function $\phi(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, if the infinitesimal generators τ and ξ_s satisfy the Lie symmetrical determining equations (17) of the system, we can easily verify that there is a relation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} X^{(1)}(\phi) = X^{(1)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi \right) + \dot{\tau} \frac{\mathrm{d}}{\mathrm{d}t} \phi. \tag{20}$$

Substituting Eqs.(14) and (15) into Eq.(20), and comparing the coefficients of ε^m on the left side of equality with those on the right side, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} X_m^{(1)}(\ln D) = X_m^{(1)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \ln D \right) + \dot{\tau}^m \frac{\mathrm{d}}{\mathrm{d}t} \ln D, \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_m^{(1)}(G_L) = X_m^{(1)}(\dot{G}_L) + \dot{\tau}^m \dot{G}_L. \tag{22}$$

Taking the partial derivations of determining equations (13) for $\dot{q}_s$, we can obtain by way of simple deduction

$$\begin{aligned} &\frac{\partial}{\partial \dot{q}_s} \left[\ddot{\xi}_s - \dot{q}_s \ddot{\tau} - 2\alpha_s \dot{\tau} - 2\varepsilon \dot{\tau} \frac{M_{sk}}{D} W_k - X^{(1)}(\alpha_s) - \varepsilon X^{(1)} \left(\frac{M_{sk}}{D} W_k \right) \right] \\ =&\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \dot{\xi}_s}{\partial \dot{q}_s} - \dot{q}_s \left(\frac{\partial \tau}{\partial q_s} + \frac{\partial \dot{\tau}}{\partial \dot{q}_s} \right) - \alpha_s \frac{\partial \tau}{\partial \dot{q}_s} - n\dot{\tau} \right] \\ &- \dot{\tau} \frac{\partial}{\partial \dot{q}_s} \left(\alpha_s + \varepsilon \frac{M_{sk}}{D} W_k \right) - X^{(1)} \left(\frac{\partial \alpha_s}{\partial \dot{q}_s} \right) - \varepsilon X^{(1)} \left[\frac{\partial}{\partial \dot{q}_s} \left(\frac{M_{sk}}{D} W_k \right) \right] \\ =&0. \end{aligned} \tag{23}$$

Substituting Eqs.(14) and (15) into Eq.(23), and comparing the coefficients of ε^m on the left side of equality with those on the right side, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\xi_s^m}{\partial q_s}+\frac{\partial\dot{\xi}_s^m}{\partial\dot{q}_s}-\dot{q}_s\left(\frac{\partial\tau^m}{\partial q_s}+\frac{\partial\dot{\tau}^m}{\partial\dot{q}_s}\right)-\alpha_s\frac{\partial\tau^m}{\partial\dot{q}_s}-n\dot{\tau}^m\right]\\=&\dot{\tau}^m\frac{\partial\alpha_s}{\partial\dot{q}_s}+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)+X_m^{(1)}\left(\frac{\partial\alpha_s}{\partial\dot{q}_s}\right)+X_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right].\end{aligned}\tag{24}$$

Differentiating I_z with respect to time t, and using Eqs.(9), (18), (21), (22) and (24), we obtain

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\sum_{m=0}^{z}\varepsilon^m\left\{\dot{\tau}^m\frac{\partial\alpha_s}{\partial\dot{q}_s}+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)+X_m^{(1)}\left(\frac{\partial\alpha_s}{\partial\dot{q}_s}\right)+X_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]\right.\\&\left.+X_m^{(1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln D\right)+\dot{\tau}^m\frac{\mathrm{d}}{\mathrm{d}t}\ln D-X_m^{(1)}\left(\dot{G}_L\right)-\dot{\tau}^m\dot{G}_L\right\}\\=&\sum_{m=0}^{z}\varepsilon^m\left\{X_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]-\varepsilon X_m^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]\right.\\&\left.+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)-\varepsilon\dot{\tau}^m\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right\}\\=&-\varepsilon^{z+1}\left\{X_z^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]-\dot{\tau}^z\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right\}.\end{aligned}\tag{25}$$

This shows that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, so I_z is a z-th-order adiabatic invariant of the system from Definition 1.

Expression (19) is a new type of adiabatic invariants directly led by Lie symmetries for a disturbed holonomic nonconservative system. It reveals the intrinsic relation between the invariants and the Lie symmetrical perturbation of a disturbed holonomic nonconservative system. We refer to the invariants as adiabatic invariants of generalized Lutzky type of the disturbed holonomic nonconservative system.

Taking $m=0$, we can obtain the Lie symmetrical determining equations of the undisturbed system (5) from Eqs.(17) as follows:

$$\begin{aligned}&\ddot{\xi}_s^0-\dot{q}_s\ddot{\tau}^0-2\dot{\tau}^0\alpha_s=X_0^{(1)}(\alpha_s)\\&(s,k=1,\ldots,n).\end{aligned}\tag{26}$$

Theorem 2 For a holonomic nonconservative system without perturbation, if the generators τ^0 and ξ_s^0 of infinitesimal transformations (10) satisfy the Lie symmetrical determining equations (26), and there exists a gauge function $G_L=G_L(t,\boldsymbol{q},\dot{\boldsymbol{q}},)$ satisfying the following relation

$$\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}Q_k\right)=\dot{G}_L,\tag{27}$$

then the undisturbed holonomic nonconservative system possesses the following exact invariant

$$\begin{aligned}I_0=&\frac{\partial\xi_s^0}{\partial q_s}+\frac{\partial\dot{\xi}_s^0}{\partial\dot{q}_s}-\dot{q}_s\left(\frac{\partial\tau^0}{\partial q_s}+\frac{\partial\dot{\tau}^0}{\partial\dot{q}_s}\right)\\&-\alpha_s\frac{\partial\tau^0}{\partial\dot{q}_s}-n\dot{\tau}^0+X_0^{(1)}(\ln D)-X_0^{(1)}(G_L)\\=&\text{const.}\end{aligned}\tag{28}$$

Expression (28) is an exact invariant directly derived from Lie symmetries for a undisturbed holonomic nonconservative system. We refer to the invariants as conserved quantities of generalized Lutzky type of the undisturbed holonomic nonconservative system.[28]

Particularly, if the transformations are Lie point symmetrical ,such as

$$t^*=t+\varepsilon\tau(t,\boldsymbol{q}),\quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q}),\tag{29}$$

then Theorem1 becomes:

Theorem 3 For a holonomic nonconservative system disturbed by small forces of perturbation εW_s, if the generators τ^m and ξ_s^m of infinitesimal transformations (29) correspond to a Lie point symmetry, and there exists a gauge function $G_L=G_L(t,\boldsymbol{q},\dot{\boldsymbol{q}},)$ satisfying the following relation:

$$\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}Q_k\right)=\dot{G}_L,\tag{30}$$

then the disturbed holonomic nonconservative system possesses the following new type of zth-order adiabatic invariants

$$I_z = \sum_{m=0}^{z} \varepsilon^m \left[2\left(\frac{\partial \xi_s^m}{\partial q_s} - \dot{q}_s \frac{\partial \tau^m}{\partial q_s} \right) - n\dot{\tau}^m + X_m^{(1)} (\ln D) - X_m^{(1)} (G_L) \right]. \tag{31}$$

Expression (31) is a new type of adiabatic invariants directly given by Lie point symmetries for a disturbed holonomic nonconservative system. We refer to the invariants as adiabatic invariants of generalized Lutzky type of the disturbed holonomic nonconservative system under the Lie point symmetrical transformations.

If the transformations are special Lie symmetrical transformations with time not varied, such as

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{32}$$

then Theorem1 becomes:

Theorem 4 For a holonomic nonconservative system disturbed by small forces of perturbation εW_s, if the generators ξ_s^m of infinitesimal transformations (32) correspond to a special Lie symmetry, and there exists a gauge function $G_L = G_L(t, \boldsymbol{q}, \dot{\boldsymbol{q}},)$ satisfying the following relation

$$\frac{\partial}{\partial \dot{q}_s} \left(\frac{M_{sk}}{D} Q_k \right) = \dot{G}_L, \tag{33}$$

then the disturbed holonomic nonconservative system possesses the following new type of z-th-order adiabatic invariants

$$I_z = \sum_{m=0}^{z} \varepsilon^m \left[\frac{\partial \xi_s^m}{\partial q_s} + \frac{\partial \dot{\xi}_s^m}{\partial \dot{q}_s} + X_m^{(1)} (\ln D) - X_m^{(1)} (G_L) \right]. \tag{34}$$

Expression (34) is a new type of adiabatic invariants directly given by special Lie symmetries for a disturbed holonomic nonconservative system. We refer to the invariants as adiabatic invariants of Lutzky type of the disturbed holonomic nonconservative system under the special Lie symmetrical transformations.

For the Lagrange system of $Q_s = 0$, Theorems 1, 3 and 4 become, respectively,

Theorem 5[32] For a Lagrange system disturbed by small forces of perturbation εW_s, if the generators τ^m and ξ_s^m of infinitesimal transformations (10) satisfy the Lie symmetrical determining equations (17), then the disturbed Lagrangian system possesses the following z-th-order adiabatic invariants

$$I_z = \sum_{m=0}^{z} \varepsilon^m \left[\frac{\partial \xi_s^m}{\partial q_s} + \frac{\partial \dot{\xi}_s^m}{\partial \dot{q}_s} - \dot{q}_s \left(\frac{\partial \tau^m}{\partial q_s} + \frac{\partial \dot{\tau}^m}{\partial \dot{q}_s} \right) - \alpha_s \frac{\partial \tau^m}{\partial \dot{q}_s} - n\dot{\tau}^m + X_m^{(1)} (\ln D) \right]. \tag{35}$$

Theorem 6 For a Lagrange system disturbed by small forces of perturbation εW_s, if the generators τ^m and ξ_s^m of infinitesimal transformations (29) correspond to a Lie point symmetry, then the disturbed Lagrangian system possesses the following new type of z-th-order adiabatic invariants

$$I_z = \sum_{m=0}^{z} \varepsilon^m \left[2\left(\frac{\partial \xi_s^m}{\partial q_s} - \dot{q}_s \frac{\partial \tau^m}{\partial q_s} \right) - n\dot{\tau}^m + X_m^{(1)} (\ln D) \right]. \tag{36}$$

Theorem 7[31] For a Lagrange system disturbed by small forces of perturbation εW_s, if the generators ξ_s^m of infinitesimal transformations (32) correspond to a special Lie symmetry, then the disturbed Lagrangian system possesses the following z-th-order adiabatic invariants

$$I_z = \sum_{m=0}^{z} \varepsilon^m \left[\frac{\partial \xi_s^m}{\partial q_s} + \frac{\partial \dot{\xi}_s^m}{\partial \dot{q}_s} + X_m^{(1)} (\ln D) \right]. \tag{37}$$

Expression (36) is a new type of adiabatic invariants directly given by Lie point symmetries for a disturbed Lagrange system. We refer to the invariants as adiabatic invariants of Lutzky type of the disturbed Lagrange system under the Lie point symmetrical transformations.

For Theorem 3, taking $m = 0$, we can obtain the conserved quantities of Lutzky type of the undisturbed holonomic nonconservative system under the Lie point symmetrical transformations.[27]

For Theorem 4, taking $m = 0$, we can obtain a new type of conserved quantities of Lutzky type of the undisturbed holonomic nonconservative system under the special Lie symmetrical transformations.

For Theorem 5, taking $m = 0$, we can obtain the conserved quantities of generalized Lutzky type of the undisturbed Lagrange system.[28]

For Theorem 6, taking $m = 0$, we can obtain the conserved quantities of Lutzky type of the undisturbed Lagrange system under the Lie point symmetrical transformations.[21,22]

For Theorem 7, taking $m = 0$, we can obtain the conserved quantities of Lutzky type of the undisturbed Lagrange system under the special Lie symmetrical transformations.[23−26]

5. An example

As an example, we study the adiabatic invariants of a holonomic nonconservative system, of which the Lagrangian and the nonpotential generalized forces are, respectively,

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - q_2, Q_1 = \dot{q}_1, Q_2 = \dot{q}_2. \tag{38}$$

Firstly, we seek the exact invariant of the system without perturbations.

Equations (1) give

$$\ddot{q}_1 = \dot{q}_1 = \alpha_1, \quad \ddot{q}_2 = \dot{q}_2 - 1 = \alpha_2. \tag{39}$$

The determining equations (26) give

$$\begin{aligned} &\ddot{\xi}_1^0 - \dot{q}_1\ddot{\tau}^0 - 2\dot{\tau}^0\dot{q}_1 = \dot{\xi}_1^0 - \dot{q}_1\dot{\tau}^0, \\ &\ddot{\xi}_2^0 - \dot{q}_2\ddot{\tau}^0 - 2\dot{\tau}^0(\dot{q}_2 - 1) = \dot{\xi}_2^0 - \dot{q}_2\dot{\tau}^0, \end{aligned} \tag{40}$$

and these have solution

$$\begin{aligned} \tau^0 =& q_1, \\ \xi_1^0 =& q_1 + q_2 t + \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + q_1^2 - q_2^2 - t^2), \\ \xi_2^0 =& q_1 + (q_2 - q_1)t + q_1 q_2 \\ &+ \frac{1}{2}(\dot{q}_2^2 - \dot{q}_1^2 + q_1^2 - q_2^2 - t^2). \end{aligned} \tag{41}$$

From Eq.(27) we obtain

$$G_L = 2t. \tag{42}$$

According to Theorem 2, we obtain an exact invariant of generalized Lutzky type for the system from generators (41) as follows:

$$I_0 = \dot{q}_2 - \dot{q}_1 - q_2 + q_1 + t = \text{const.} \tag{43}$$

Using the inversed Noether theorem,[8] we know that the exact invariant (43) is non-Noether one.

Lastly, we study the adiabatic invariants of the system with the action of small disturbance.

Suppose the system is perturbed by the small quantities as follows:

$$\varepsilon W_1 = -\varepsilon t, \quad \varepsilon W_2 = \varepsilon t. \tag{44}$$

The disturbed equations (8) give

$$\ddot{q}_1 = \dot{q}_1 - \varepsilon t, \quad \ddot{q}_2 = \dot{q}_2 - 1 + \varepsilon t. \tag{45}$$

Using generators (41) and disturbed equations (45), and taking $m = 1$, the determining equations (17) give

$$\begin{aligned} &\ddot{\xi}_1^1 - \dot{q}_1\ddot{\tau}^1 - 2\dot{\tau}^1\dot{q}_1 + 2\dot{q}_1 t \\ =&\dot{\xi}_1^1 - \dot{q}_1\dot{\tau}^1 - q_1, \\ &\ddot{\xi}_2^1 - \dot{q}_2\ddot{\tau}^1 - 2\dot{\tau}^1(\dot{q}_2 - 1) - 2\dot{q}_1 t \\ =&\dot{\xi}_2^1 - \dot{q}_2\dot{\tau}^1 + q_1, \end{aligned} \tag{46}$$

and these have solution

$$\begin{aligned} &\tau^1 = t^2, \quad \xi_1^1 = q_1 t, \\ &\xi_2^1 = (\dot{q}_1 + \dot{q}_2)t^2 - q_1 t. \end{aligned} \tag{47}$$

According to Theorem 1, we obtain a 1-st order adiabatic invariant of generalized Lutzky type for the system from generators (41) and (47) as follows:

$$I_1 = \dot{q}_2 - \dot{q}_1 - q_2 + q_1 + t - \frac{1}{2}\varepsilon t^2. \tag{48}$$

Furthermore, we can find more higher order adiabatic invariants of the system.

References

[1] Burgers J M 1917 *Ann. Phys, Lpz.* **52** 195

[2] Kruskal M 1962 *J. Math. Phys.* **3** 806

[3] Djukic D J S 1981 *Int. J. Non-Linear Mech.* **16** 489

[4] Notte J, Falans J, Chu R and Wurtele J S, 1993 *Phys. Rev. Lett.* **70** 3900

[5] Ostrovsky V N and Prudov N V 1995 *J. Phys.* B **20** 4435

[6] Wang L and Kevorkian J 1996 *Phys. Plasma* **3** 1162

[7] Zhao Y Y and Mei F X 1996 *Acta Mech. Sin.* **28** 207 (in Chinese)

[8] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) p728 (in Chinese)

[9] Chen X W, Zhang R C and Mei F X 2000 *Acta Mech. Sin.* **16** 282

[10] Chen X W and Mei F X 2000 *Chin. Phys.* **9** 721

[11] Chen X W and Li Y M 2005 *Chin. Phys.* **14** 663

[12] Chen X W, Li Y M and Zhao Y H 2005 *Phys. Lett.* A **337** 274
[13] Zhang Y and Mei F X 2003 *Acta Phys. Sin.* **52** 2368 (in Chinese)
[14] Zhang Y, Fan C X and Mei F X 2006 *Acta Phys. Sin.* **55** 3237 (in Chinese)
[15] Zhang Y 2006 *Chin. Phys.* **15** 1935
[16] Luo S K and Guo Y X 2007 *Commun. Theor. Phys.* (Beijing, China) **47** 133
[17] Zhang Y 2007 *Acta Phys. Sin.* **56** 1855 (in Chinese)
[18] Luo S K 2007 *Chin. Phys. Lett.* **24** 3017
[19] Jing H X, Li Y C and Xia L L 2007 *Acta Phys. Sin.* **56** 3043 (in Chinese)
[20] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973
[21] Lutzky M 1979 *Phys. Lett.* A **72** 86
[22] Lutzky M 1979 *Phys. Lett.* A **75** 8
[23] Lutzky M 1995 *J. Phys.* A: *Math. Gen.* **28** L637
[24] Lutzky M 1998 *J. Phys.* A: *Math. Gen.* **31** L721
[25] Lutzky M 1999 *Int. J. Non-Linear Mech.* **33** 393
[26] Lutzky M 1999 *Int. J. Non-Linear Mech.* **34** 387
[27] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[28] Zhang Y 2006 *Acta Phys. Sin.* **55** 2109 (in Chinese)
[29] Mei F X and Xu X J 2005 *Chin. Phys.* **14** 449
[30] Ge W K and Zhang Y 2006 *Acta Phys. Sin.* **55** 3833 (in Chinese)
[31] Luo S K 2007 *Acta Phys. Sin.* **56** 5580 (in Chinese)
[32] Luo S K 2007 *Chin. Phys. Lett.* **24** 2463