Vol 17 No 1, January 2008
1674-1056/2008/17(01)/0350-06

Chinese Physics B

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Optimized transfer trajectories in the earth–moon system*

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(Received 14 April 2007; revised manuscript received 30 May 2007)

Investigations of low energy transfer trajectories are important for both celestial mechanics and astronautics. Methodologies using the theories from dynamical systems are developed in recent years. This paper investigates the dynamics of the earth–moon system. Low energy transfer trajectories are solved numerically by employing a hybrid strategy: first, a genetic hide and seek method performs a search in large domain to confine the global minimum $f(\eta)$ (objective function) region; then, a deterministic Nelder–Mead method is utilized to refine the minimum quickly. Some transfer trajectories of the spacecraft in the earth–moon system are successfully simulated which verify the desired efficiency and robustness of the method of this paper.

Keywords: celestial mechanics, dynamical astronomy, orbit
PACC: 9510C, 9575P, 9635F

1. Introduction

Up to now, a number of literatures have been devoted to the aspects in designing low energy transfer trajectories in three-body problems (see Refs.[1]–[5]). Their works dramatically illustrate the need for a new technique: methodology using the theories from dynamical systems. As in Ref.[1], Conley proved the existence of transit trajectories from one Hill's region to another, while Koon *et al* proposed a systematic way of designing those transfers in Refs.[2,3]. In particular, the stable and unstable invariant manifold tubes associated to libration point orbits are the phase space conduits transporting material between primary bodies for separate three-body systems. More recent work (see Ref.[4]) by Villac *et al* focused on obtaining practical characterizations of the escaping trajectories via the Poincaré map. As proposed in Ref.[5], the dynamical structures of the three-body problem, such as stable and unstable centre manifolds, reveal much about the morphology and transport of particles (such as asteroids, dust grains or spacecrafts) within the solar system. Meanwhile, in Refs.[6,7], the effect of gravitational radiation damping on the variation of the orbital elements of binary system were studied. In order to guarantee the flight safety of the spacecraft in the space, as proposed in Ref.[8], it is necessary to predict the sunspot number effectively.

In this paper, dynamics of the earth–moon system is investigated and low energy transfer trajectories are solved numerically using a two-stage strategy. First, the genetic hide and seek method which is extended from Ref.[9] localizes the global minimum $f(\eta)$ region. Then, the Nelder–Mead method (for instance Ref.[10]) is employed to get the minimum accurately. In order to get closer to some realistic situations, the underlying earth–moon model in this paper is circular restricted three-body problem (from now on, CRTBP) since the model gives a good insight of the dynamics of planetary orbiters and co-orbital satellites.

2. Dynamics of the CRTBP

The so-called CRTBP (for example Ref.[11]) describes the motion of an infinitesimal particle under the gravitational attraction (Newton's law) of two points like masses called primaries, denoted as M_1 and M_2 where M_1 is larger than M_2. The primaries are describing Keplerian orbits around their common centre of masses. In this section, we give some dynamical properties of the CRTBP in brief, which are used in the subsequent analysis.

*Project supported by the National Natural Science Foundation of China (Grant Nos 10472091 and 10332030).

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2.1. Equations of motion and equilibrium points

In the context, M_1 is the mass of the earth while M_2 is the mass of the moon. The planar CRTBP model is considered. The earth and the moon move in coplanar circular Keplerian orbits about their centre of mass. Denote (x, y) as the rectangular coordinates of a rotating coordinate system such that M_1 is at $(-\mu, 0)$ and M_2 at $(1-\mu, 0)$, where $\mu = M_2/(M_1+M_2) \approx 0.0123$. A standard canonical system of units associated with this model is used. More precisely, the mass unit (MU)=$M_1 + M_2 \approx 6.0471 \times 10^{24}$ kg, the distance unit (DU)$\approx 3.844 \times 10^5$ km and the time unit (TU)$\approx$ 104 h. With these units, the total mass, the gravitational constant and the angular velocity of the system are normalized to one. The resulting equations of motion are

$$\begin{cases} x'' - 2y' = \Omega_x, \\ y'' + 2x' = \Omega_y, \end{cases} \tag{1}$$

with $\Omega = (x^2+y^2)/2 + (1-\mu)/\rho_1 + \mu/\rho_2$, where $\rho_1 = \sqrt{(x+\mu)^2+y^2}$ and $\rho_2 = \sqrt{(x+\mu-1)^2+y^2}$ are the distances of the spacecraft from the earth and the moon, respectively.

Return to Eq.(1). Suppose that the first and second derivatives of x and y all vanish initially. Equation (1) shows that all the higher derivatives of x, y vanish initially and hence vanish for all values of time by Taylor's theorem. Thus, if x', y', x'', y'' all vanish initially, then we can obtain the following five equilibrium points L_j $(j = 1, \cdots, 5)$ (see Table 1). Two of them can be found as the third vertex of the two equilateral triangles that can be formed using the two primaries as vertices (they are also called triangular points, Lagrangian points or simply $L_{4,5}$). The collinear points lie on the x-axis and are called $L_{1,2,3}$ or Eulerian points.

Table 1. Coordinates and Jacobi energies of equilibrium points in the earth–moon system.

coordinate and Jacobi energy	L_1	L_2	L_3	L_4	L_5
x/DU	0.8369	1.1557	–1.0051	0.4879	0.4879
y/DU	0	0	0	0.8660	–0.8660
J (Jacobi energy)	–3.1897	–3.1733	–3.0123	–2.9829	–2.9829

2.2. Jacobi integral, Hill's region and zero velocity surface

Now multiply Eqs.(1), respectively, by x' and y', then form the resulting sum:

$$x'x'' + y'y'' = xx' + yy' - (1-\mu)[(x+\mu)x' + yy']/\rho_1^3 - \mu[(x+\mu-1)x' + yy']/\rho_2^3.$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[0.5((x')^2+(y')^2)\big] = \frac{\mathrm{d}}{\mathrm{d}t}\big[0.5(x^2+y^2)\big] - \frac{\mathrm{d}}{\mathrm{d}t}\big[(1-\mu)/\rho_1\big] - \frac{\mathrm{d}}{\mathrm{d}t}\big(\mu/\rho_2\big).$$

Then we obtain

$$0.5((x')^2+(y')^2) - 0.5(x^2+y^2) - (1-\mu)/\rho_1 - \mu/\rho_2 = -2C = \text{const}. \tag{2}$$

Here C is called the Jacobi constant and Eq.(2) is called Jacobi integral. Hence we defined the Jacobi energy as

$$J = (x')^2 + (y')^2 - (x^2+y^2) - 2(1-\mu)/\rho_1 - 2\mu/\rho_2. \tag{3}$$

The Jacobi energy at five equilibrium points L_j $(j = 1, \cdots, 5)$ are shown in Table 1.

The level surface of the Jacobi constant, which is also called energy surface, is written as $\Pi = \big\{(x, y, x', y')|C = \text{const}\big\}$. The projection of this surface onto position space is called a Hill's region $\Pi_0 = \big\{(x, y)|J > J_{L_j}\big\}$. The boundary of Π_0 is the zero velocity surface and J_{L_j} is the critical value of J for which the zero velocity surface contains a neck about the Eulerian points $L_{1,2,3}$, whereas the Hill' region is almost closed as $J = J_{L_{1,2,3}}$ (except three points $L_{1,2,3}$), as shown in Fig.1. Meanwhile, Fig.1

describes the forbidden region restricted by Eq.(3) at some given J.

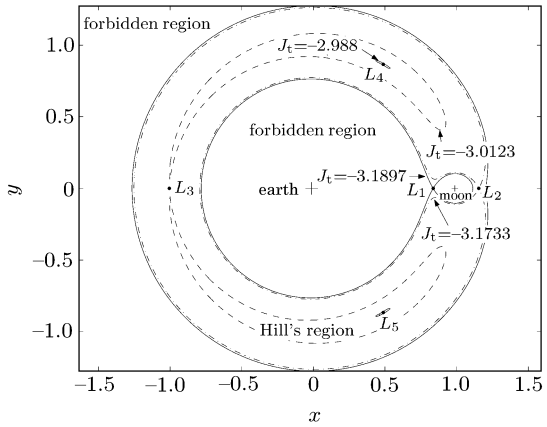


Fig.1. Hill's region and forbidden region.

3. Numerical methods and simulations

3.1. Preliminary analysis

As in Ref.[12], we consider the transfer of a spacecraft from an initial (subscript i) circular orbit around the earth to a final (subscript f) circular orbit around the moon by means of a bi-impulsive transfer (subscript t) trajectory. The minimum velocity variation $\Delta V_{\rm s}$ needed to place the spacecraft into the transfer trajectory is

$$\Delta V_{\rm s}^{\min} = v_{\rm s}^{\min} - v_{\rm i} = \sqrt{v_{\rm i}^2 - J_{\rm i} + J_{\rm t}} - v_{\rm i}. \quad (4)$$

Likewise, the minimum velocity variation $\Delta V_{\rm e}$ needed to place the spacecraft into the final circular orbit is

$$\Delta V_{\rm e}^{\min} = v_{\rm e}^{\min} - v_{\rm f} = \sqrt{v_{\rm f}^2 - J_{\rm f} + J_{\rm t}} - v_{\rm f}. \quad (5)$$

Therefore, the minimum total velocity variation $\Delta V_{\rm total}^{\min} = \Delta V_{\rm s}^{\min} + \Delta V_{\rm e}^{\min}$ is given by

$$\Delta V_{\rm total}^{\min} = \sqrt{v_{\rm i}^2 - J_{\rm i} + J_{\rm t}} + \sqrt{v_{\rm f}^2 - J_{\rm f} + J_{\rm t}} - (v_{\rm i} + v_{\rm f}). \quad (6)$$

A remarkable simplification is possible under the hypothesis of low-height initial and final orbits, i.e., $R_{\rm i}$, $R_{\rm f} \ll 1$ (DU). Here $R_{\rm i}$ and $R_{\rm f}$ are the radii of the initial and final orbits respectively. Similarly to Ref.[12], substituting $v_{\rm i} \approx \sqrt{(1-\mu)/R_{\rm i}} - R_{\rm i}$ and $v_{\rm f} \approx \sqrt{\mu/R_{\rm f}}$ into Eq.(3) we obtain

$$v_{\rm i}^2 - J_{\rm i} \approx (x_{\rm i}^2 + y_{\rm i}^2) + 2\left[(1-\mu)/R_{\rm i} + \mu/(\rho_2)_{\rm i}\right],$$
$$v_{\rm f}^2 - J_{\rm f} \approx (x_{\rm f}^2 + y_{\rm f}^2) + 2\left[(1-\mu)/(\rho_1)_{\rm f} + \mu/R_{\rm f}\right],$$

where $(\rho_2)_{\rm i}$ and $(\rho_1)_{\rm f}$ are, respectively, the distances of the spacecraft from the moon at the beginning of the transfer and from the earth at the end of the transfer. Noting that $x_{\rm i}^2 + y_{\rm i}^2 \geq (R_{\rm i} - \mu)^2$, $x_{\rm f}^2 + y_{\rm f}^2 \geq [R_{\rm f} - (1-\mu)]^2$, $(\rho_2)_{\rm i} \leq 1 + R_{\rm i}$ and $(\rho_1)_{\rm f} \leq 1 + R_{\rm f}$, we deduce the following simplified total velocity variation

$$\begin{aligned}\Delta\widetilde{V}_{\rm total}^{\min} \approx & \sqrt{(R_{\rm i}-\mu)^2 + 2(1-\mu)/R_{\rm i} + 2\mu/(1+R_{\rm i}) + J_{\rm t}} \\ & + \sqrt{[R_{\rm f} - (1-\mu)]^2 + 2(1-\mu)/(1+R_{\rm f}) + 2\mu/R_{\rm f} + J_{\rm t}} \\ & - \left(\sqrt{(1-\mu)/R_{\rm i}} - R_{\rm i} + \sqrt{\mu/R_{\rm f}}\right). \end{aligned} \quad (7)$$

Denote S and E as the spacecraft positions at the beginning and at the end of the ballistic transfer respectively. For a given value of $J_{\rm t}$, we look for the bi-impulsive trajectory connecting S and E that minimizes $\Delta V_{\rm total}$ with the mission time constraint. Since the solutions of the system (1) are sensitive to the initial conditions and the spacecraft must pass through the interface $x = x_{L_1} = 0.8369$, we select a particular point M of the trajectory whose abscissa is x_{L_1}. Thus, the transfer trajectory consists of the two arcs SM and ME corresponding to a terrestrial and a lunar phase trajectory. The initial state vector is

$$\boldsymbol{x}_m = [(x_{L_1}, y_m), (v_m\cos(\delta), v_m\sin(\delta))], \quad (8)$$

where δ is the angle between velocity v_m and the x axis, the first two components are position components and the remains are velocity components. Now

we achieve that

$$\Delta V_{\rm s} = \sqrt{v_{\rm s}^2 + v_{\rm i}^2 - 2v_{\rm s}v_{\rm i}\cos(\gamma_{\rm s})} \tag{9}$$

and

$$\Delta V_{\rm e} = \sqrt{v_{\rm e}^2 + v_{\rm i}^2 - 2v_{\rm e}v_{\rm i}\cos(\gamma_{\rm e})}, \tag{10}$$

where $\gamma_{\rm s}$ and $\gamma_{\rm e}$ are the flight-path angles at points S and E, respectively. Obviously, having fixed $R_{\rm i}$, $R_{\rm f}$ and $J_{\rm t}$, we find that the minimum $\Delta V_{\rm total} = \Delta V_{\rm s} + \Delta V_{\rm e}$ ballistic trajectory is determined by y_m and δ.

3.2. Two-stage method

The minimum $\Delta V_{\rm total}$ ballistic trajectory can be found by solving the problem

$$\Delta V_{\rm total}^{\min}(\eta^*) = \min_{\eta\in F} \Delta V_{\rm total}(\eta), \tag{11}$$

where $\eta = (y_m, \delta)$ and F, collection of feasible solutions, is a compact body in $\Re^2$. In order to impose the mission time in our optimization problem, we propose the following objective function

$$f(\eta) = \Delta V_{\rm total}(\eta) + f_{\rm t}(\eta). \tag{12}$$

Here

$$f_{\rm t}(\eta) = \begin{cases} t_{\rm f}, & \text{if } |t_{\rm f} - \Delta V_{\rm total}(\eta)| < |t_{\rm f}/t^{\max} - \Delta V_{\rm total}(\eta)| \\ t_{\rm f}/t^{\max}, & \text{else} \end{cases},$$

where $t_{\rm f}$ is a final mission time and $t^{\max}$ is a maximum mission time. Once the initial state is fixed, the corresponding mission time $t_{\rm f}$ can be calculated by solving the earth–moon model (1) via a 8-order Runge–Kutta–Fehlberg integration method. Hence we obtain the optimization problem as follows

$$\min_{\eta\in F} f(\eta), \tag{13}$$

where objective function $f(\eta)$ is given by function (12).

3.2.1. . Genetic hide and seek algorithm

At any given temperature τ, the probability of a system being in a state r is given by the Boltzmann's distribution $P(r) \sim {\rm e}^{-E(r)/(k_{\rm b}\tau)}$, where $E(r)$ is the energy associated with state r and $k_{\rm b}$ is Boltzmann's constant. Hence, the most probable state is that of the lowest energy. However, as in Ref.[9], the Boltzmann's distribution was replaced by Metropolis criterion $\beta_\tau(x_1, x_2) = \min(1, {\rm e}^{(f(x_2)-f(x_1))/\tau})$. Here x_1, x_2 are two different design points. For more details about the hide and seek algorithm, we refer readers to Ref.[9].

Here we propose a genetic hide and seek algorithm to confine the global minimum $f(\eta)$ region. Obviously, Step 0) of the optimization problem (13) consists mainly of finding a feasible starting point in F. We find this feasible starting point by using the random walk algorithm in Ref.[13] to solve the following auxiliary optimization problem

$$\min_{\eta}\{\sqrt{(x_{\rm s}-x_{\rm i})^2 + (y_{\rm s}-y_{\rm i})^2} + \sqrt{(x_{\rm e}-x_{\rm f})^2 + (y_{\rm e}-y_{\rm f})^2}\} \tag{14}$$

with a constraint $t_{\rm f} < t^{\max}$. When a starting point is found, choose a high enough starting temperature τ_0 and set $j = 0$.

1) Choose searching direction $\theta_j \in \Re^2$ on the surface of a unit circle with uniform distribution.

2) Choose step λ_j from the uniform distribution such that $\Lambda_j = \{\lambda_j \in \Re \mid v_{j+1} = \eta_j + \lambda_j\theta_j \in F\}$.

3) Choose Z_j from the [0,1] uniform distribution. Determine the next searching point η_{j+1} from

$$\eta_{j+1} = \begin{cases} v_{j+1}, & \text{if } Z_j \in [0, \beta_\tau(\eta_j, v_{j+1})], \\ \eta_j, & \text{if } Z_j \in [\beta_\tau(\eta_j, v_{j+1}), 1]. \end{cases} \tag{15}$$

Here $\beta_\tau(\eta_j, v_{j+1})$ stems from the Metropolis criterion.

4) If $f(\eta_{j+1})$ is less than all previous function values, use $\tau = 2\left\{\Delta\widetilde{V}_{\rm total}^{\min} + \left[f_{\rm t}(\eta_{j+1}) + \frac{f_{\rm t}(\eta_{j+1}) - f_{\rm t}(\eta_j)}{(1-p)^{-n/2} - 1}\right]\right\}/\chi_{1-p}^2(n)$ to update the temperature. In this step, $0 < p < 1$ and $\chi_{1-p}^2(n)$ is the $100(1-p)$ percentile point of the chi-square distribution with n degree-of-freedom. This temperature update schedule gives a decrease in function value over the current iteration point with a probability at least p. The value of p and n for numerical simulations reported in this paper is set to 0.05 and 1, respectively.

5) Produce random numbers R_j from the $[0,1]$ uniform distribution and set a genetic probability $p_G = 0.3$. If $R_j < p_G$, choose $\theta_j = \dfrac{v_{j+1} - \eta_j}{|v_{j+1} - \eta_j|}$ and goto step 2); otherwise, goto step 1).

3.2.2. . Nelder–Mead method

The Nelder–Mead method is a simplex method for finding a local minimum of a function of several variables. It's discovery is attributed to Nelder and Mead (refer to Ref.[10]). For two variables, a simplex is a triangle and the method is a pattern search that compares function values at the three vertices of a triangle. At first, we should choose three noncollinear vertices in the collection of feasible solutions F to construct the initial triangle. A new vertex can be obtained by selecting the midpoint of the good side, reflecting the triangle through the good side, shrinking the triangle toward the best vertex and etc. The worst vertex, where $f(\eta)$ is largest, is rejected and replaced with the new vertex. A new triangle is formed and the search is continued. The process generates a sequence of triangles (which might have different shapes), for which the function values at the vertices get smaller and smaller. The size of the triangles is reduced and the coordinates of the minimum point are found.

3.2.3. . Two-stage strategy

As we know, the hide and seek algorithm is a simulated annealing method and the calculations are performed largely and slowly. Therefore, when the objective function values are in a steady state, we select the Nelder–Mead method to solve problem (13) to make the optimization procedure converge more quickly and accurately.

To summarize, we propose the following two-stage strategy: a simulated annealing method, the genetic hide and seek method, explores a large search domain to confine the global minimum $f(\eta)$ region; the Nelder–Mead simplex method is employed to get the minimum quickly and accurately.

3.3. Numerical results

We use the mission data taken from Ref.[12]. The problem is to find the minimum $f(\eta)$ ballistic trajectories that transfer a spacecraft from a low earth circular orbit at 167km altitude ($R_{\rm i} = 1.7026 \times 10^{-2}$DU) to a low circular orbit around the moon of 100km altitude ($R_{\rm f} = 4.7866 \times 10^{-3}$DU). The maximum mission time $t^{\max} = 32.31$TU and $J_{\rm t} \in [-2.757, -1.556]$. The rationale for such a small variation range of the Jacobi energy is that values of $J_{\rm t}$ outside the chosen interval are impractical because they correspond to too long mission times (for $J_{\rm t} < -2.757$) or too large values of $\Delta V_{\rm total}(\eta)$ (for $J_{\rm t} > -1.556$). In our simulations, we discuss two cases in which $J_{\rm t} = -1.556$ and $J_{\rm t} = -2.67$, respectively.

Table 2. Performances of optimized transfer trajectories.

	$J_{\rm t} = -1.556$		$J_{\rm t} = -2.67$	
	$S \to M$	$M \to E$	$S \to M$	$M \to E$
x/DU	–0.0279	0.9879	–0.0251	0.9860
y/DU	–0.0098	–0.0058	–0.0124	–0.0055
v_x/(DU/TU)	6.3891	2.3767	6.3693	2.0648
v_y/(DU/TU)	–8.0551	–0.0802	–8.2112	–0.5421
γ/Rad	0.1071	0.0724	0.1095	0.0472
$\Delta V_{\rm total}$/(DU/TU)	2.8439	0.7878	2.9570	0.5389
t/TU	0.4670	0.1290	19.4400	0.3060

Numerical results are shown in Table 2 and Fig.2. The effectiveness of the two-stage method is confirmed by the fact that the trajectories are tangent to both the earth and the moon circular orbits, as expected. See the flight-path angle γ in Table 2. At the same time, our numerical simulations coincide with the results in Ref.[12].

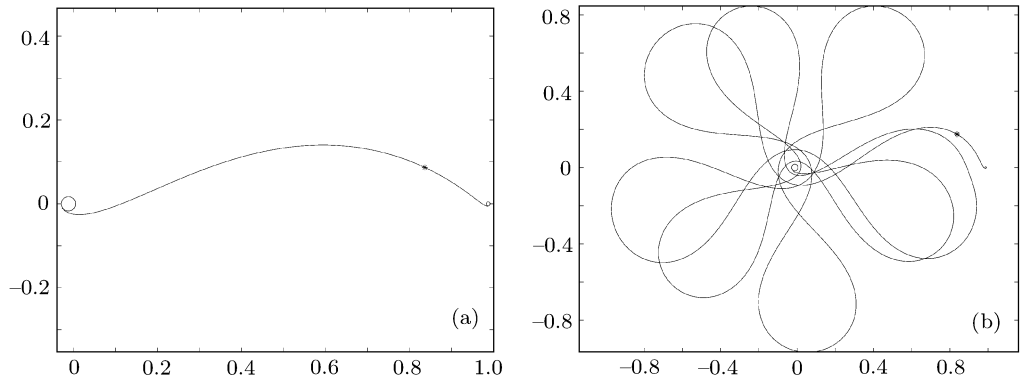


Fig.2. Optimized transfer trajectories in rotating frame:(a) $J_t = -1.556$; (b) $J_t = -2.67$.

4. Conclusions

Firstly, dynamics of the earth–moon system is investigated. We then present a new way to investigate the transfer trajectories in Hill's region of the CRTBP via a hybrid methodology: the genetic hide and seek algorithm examines a large searching domain to localize the global minimum $f(\eta)$ region; the deterministic Nelder–Mead method is imposed to obtain the minimum quickly and accurately. Efficiency and stability issues are also considered in the design of the method and are verified numerically.

Acknowledgment

The author gratefully acknowledges the support of Youth for Teachers Scientific and Technological Innovation Foundation of Northwestern Polytechnical University.

References

[1] Conley C 1968 *SIAM J. Appl. Math.* **16** 732
[2] Koon W S, Lo M W, Marsden J E and Ross S D 2001 *Celest. Mech. Dynam. Astron.* **81** 63
[3] Koon W S, Lo M W, Marsden J E and Ross S D 2002 *Contemp. Math., AMS* **292** 129
[4] Villac B F and Scheeres D J 2003 *J. Guidance, Control, and Dynamics* **26** 224
[5] Jorba À and Masdemont J 1999 *Physica* D **132** 189
[6] Li L S 1989 *Acta Phys. Sin.* **38** 1877 (in Chinese)
[7] Li L S 1994 *Acta Phys. Sin.* **43** 694 (in Chinese)
[8] Ding G and Zhong S S 2007 *Acta Phys. Sin.* **56** 1224 (in Chinese)
[9] Teckinalp O and Bingol M 2004 *J. Guidance, Control, and Dynamics* **27** 616
[10] Lagarias J C, Reeds J A, Wright M H and Wright P E 1998 *SIAM J. Opt.* **9** 112
[11] Vinti J P 1998 *Orbital and Celestial Mechanics* (Washington: AIAA)
[12] Mengali G and Quarta A 2005 *J. Guidance, Control, and Dynamics* **28** 209
[13] Romeijn H E and Smith R L 1994 *J. Global Opt.* **5** 101