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An ergodic algorithm for long-term coverage of elliptical orbits*

Xu Ming(徐 明)† and Xu Shi-Jie(徐世杰)

Beijing University of Aeronautics and Astronautics, Beijing 100083, *China*

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This paper deals with the coverage analysis problem of elliptical orbits. An algorithm based on ergodic theory, for long-term coverage of elliptical orbits, is proposed. The differential form of the invariant measure is constructed via the perturbation on mean orbital elements resulted from the J_2 term of non-spherical shape of the earth. A rigorous proof for this is then given. Different from the case of circular orbits, here the flow and its space of the dynamical system are defined on a physical space, and the real-value function is defined as the characteristic function on station mask. Therefore, the long-term coverage is reduced to a double integral via Birkhoff–Khinchin theorem. The numerical implementation indicates that the ergodic algorithm developed is available for a wide range of eccentricities.

Keywords: ergodic algorithm, coverage analysis, elliptical orbit
PACC: 9480, 9510C

1. Introduction

The coverage rate is an important performance index for a selected orbit. The coverage analysis of a nominal orbit, especially long-term coverage, is expected to give the frequency of the satellite relative to some ground station, or the ground station's visible frequency relative to the satellite.

Generally, the satellite's subpoint traces are not repetitive except some special orbit types. It is impossible to give the analytical solutions for the long-term coverage, especially with the effect of perturbations. Thus the numerical method is required to propagate the satellite trajectory for a long term (about 5–10 years), which will engross a lot of computing resources.

In certain cases, however, some solutions on the time mean can be transformed into that on the space mean. Based on the ergodic theory, Lo[1−4] proposed an approach to realize the transformations by a definite integral over the circular region of the elevation mask of a point on the earth's surface. The effects of J_2 perturbation due to the non-spherical shape of the earth are included in the definite integral. However, the method is applicable only for the case of circular orbits. The coverage errors with the method are no more than 0.2% for the circular orbits, but beyond 16% for the elliptical orbits (with its eccentricity 0.05).[1] So the algorithm via the ergodic theory for circular orbits can not extend simply to the elliptical orbits.[1−3]

Lo[4] also considered the case for the elliptical orbits. He indicated that the dimensions for the elliptical orbits are larger than for the circular orbits, and gave a possible method to construct the invariant measure. However, he did not implement his idea. Moreover, the method for constructing the invariant measure is suitable only for an abstract dynamical system, and it needs to calculate the divergence of a vector field which leads to solve a partial differential equation numerically. The solving procedure requires lots of computations, which makes Lo's method difficult to apply to our problems.

Furthermore, Lo gave the original idea on the application of ergodic theory to coverage analysis of circular orbit. However, his statements are still requiring the rigorous mathematical proofs.[1−4]

In this paper, the mean orbital elements of elliptical orbits under the effect of J_2 perturbation are used to construct the invariant measure, and then to transform integrating trajectories into solving a double integral. The invariant measure here is simpler than that in Ref.[4], and the visible region defined here is quite different from the taper defined in Ref.[4]. The proposed theory is verified numerically. The ergodic

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†E-mail: xumingle@gmail.com

algorithm developed is available for a wide range of eccentricities.

2. Ergodic theory

The ergodic theory comes of the Boltzmann ergodic hypothesis in statistical mechanics, which proposes a statistical approach to study a deterministic dynamical system. Abstract ergodic theory deals with the measurable actions of groups and semigroups of transformations, and the fundamental and important theorem of this theory is Birkhoff–Khinchin theorem.[5,6]

Let (M, μ, ϕ_t) be an abstract dynamical system, M a smooth manifold called flow space, $\phi_t : M \to M$ a one-parameter group of measure preserving diffeomorphism, μ an invariant measure with $\mu(\phi_t(x)) = \mu(x), \forall x \in M$ and $\mu(M) = 1$. (M, μ, ϕ_t) is ergodic, if and only if any invariant set of ϕ_t has its measure 1 or 0.

Let $h \in L^1(M, \mu)$ be a real valued and μ-summable function on M. The time mean of h is defined as

$$\langle h \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T h(\phi_t(x)) \mathrm{d}t, x \in M, t \in R \quad (1)$$

and the space mean of h is defined as

$$\bar{h} = \int_M h(x) \mathrm{d}\mu, x \in M. \quad (2)$$

Theorem 1 (Birkhoff–Khinchin) Let (M, μ, ϕ_t) be an abstract dynamical system, $h \in L^1(M, \mu)$ a complex valued μ-summable function on M, then

$$\langle h \rangle = \bar{h}. \quad (3)$$

For further discussions and proof see Refs.[5] and [6].

3. Ergodic analysis on coverage of elliptical orbits

3.1. Elliptical orbits

A real trajectory of a satellite under perturbation (only the J_2 perturbation is considered in this paper) is not a perfect Kepler orbit, so its orbital elements are no longer constants. The mean orbital elements can be used to calculate the long-term coverage approximately,[1−4] so all the orbital elements are considered as the mean elements throughout the paper. Thus the semi-major axis a, eccentricity e, and orbit inclination i under the J_2 effect are constants; the right ascension of ascending node Ω, argument of perigee ω, and mean anomaly M_a are as follows

$$\begin{cases} \Omega = \Omega_0 + C_\Omega(t - t_0), \\ \omega = \omega_0 + C_\omega(t - t_0), \\ M_a = M_{a0} + C_{M_a}(t - t_0), \end{cases} \quad (4)$$

where C_Ω, C_ω, C_{M_a} are respectively mean change rates in a period, which are the functions of J_2.[7] So the satellite's state can be defined just by Ω, ω, M_a.

Note that $\Omega, \omega, M_a \in R/2\pi Z$, so

$$(\Omega, \omega, M_a) \in T^3,$$

where $T^3 = S^1 \times S^1 \times S^1$ is a torus of 3 dimensions.

If $C_\Omega, C_\omega, C_{M_a}$ are rationally independent, then the trajectory determined by Eq.(4) is compact on the torus T^3 and also ergodic. If $C_\Omega, C_\omega, C_{M_a}$ are rationally dependent, then the trajectory has resonant surfaces and no longer ergodic.[5]

Define

$$S_1 := \{(C_\Omega, C_\omega, C_{M_a}) \in R^3 | C_\Omega, C_\omega, C_{M_a} \text{ are rationally independent}\},$$
$$S_2 := \{(C_\Omega, C_\omega, C_{M_a}) \in R^3 | C_\Omega, C_\omega, C_{M_a} \text{ are rationally dependent}\} = S_1^C.$$

Obviously, S_1 is the total measure (Lebesgue Measure) set, and S_2 is the null set. From the viewpoint of probability, $\forall (C_\Omega, C_\omega, C_{M_a}) \in R^3$ there is the property

$$\begin{cases} P[(C_\Omega, C_\omega, C_{M_a}) \in S_1] = 1, \\ P[(C_\Omega, C_\omega, C_{M_a}) \in S_2] = 0, \end{cases} \quad (5)$$

where $P[A]$ is the occurring probability of event A.

Therefore, only the case that $C_\Omega, C_\omega, C_{M_a}$ are rationally independent is analysed for the ergodic algorithm of long-term coverage in this paper.

The latitude φ and longitude θ of satellite's subpoint in a rotating ideal earth have the relationship

with the orbital elements:[7]

$$\begin{cases} \sin(\varphi) = \sin(i)\sin(\omega + f), \\ \theta = \Omega + \arctan[\cos(i)\tan(\omega + f)] - \omega_{\mathrm{e}} t, \end{cases} \tag{6}$$

and then the transformation between the radius r of the elliptical orbit and the orbital elements is as follows[7,8]

$$r = a\left[1 + \frac{1}{2}e^2 - e\left(1 - \frac{3}{8}e^2 + \frac{5}{192}e^4\right)\cos(M_a) + \cdots\right], \tag{7}$$

where f is true anomaly of the orbit, ω_{e} the angle velocity of axial rotation of the earth.

f can be obtained from M_a, namely

$$f = M_a + 2e\sin(M_a) + \frac{5}{4}e^2\sin(2M_a) + \cdots.$$

So the Eqs.(6) and (7) present the non-canonical transformation F_1:$(\Omega, \omega, M_a) \to (r, \varphi, \theta)$ (under spherical coordinates).

3.2. The flow and its space

Different from the constant radius of the circular orbit, the variable radius of the elliptical orbit

$$r \in [r_{\min}, r_{\max}] = [a(1-e), a(1+e)]$$

make elliptical coverage more complicated than circular coverage due to the extra-dimension.[4]

The flow (satellite's subpoint) and its space (the surface of the earth) of the circular orbit are not suitable for those of the elliptical orbit.

The flow ϕ_t for elliptical orbit is defined as the real trajectory of the satellite in R^3.

Let $S_{\min}$ represent the sphere with radius $r_{\min}$, $S_{\max}$ the sphere with radius $r_{\max}$, and Ξ the spherical zone with its latitude inside $-L_i$ and L_i, where

$$L_i = \begin{cases} i, & i \le \pi/2 \\ \pi - i, & i \ge \pi/2 \end{cases}.$$

Then the flow space M is defined as

$$M = \Xi \cap S_{\max} \cap S_{\min}^c.$$

3.3. The invariant measure

The differential form of the measure μ is constructed as

$$\mathrm{d}\mu = \kappa \mathrm{d}\Omega \wedge \mathrm{d}\omega \wedge \mathrm{d}M_a, \tag{8}$$

where κ is undetermined coefficient and $\wedge$ represents the exterior product.

Lemma 1 Let Σ represent the power set of M, then $\{M, \Sigma, \mu\}$ is probability space when κ is assigned to some appropriate value.

Proof Σ is the power set of M, thus Σ is also σ-algebra on M. So $N \in \Sigma$ will be true for $\forall N \subset M$.

According to Eq.(8), taking the Lebesgue integral on N yields

$$\mu(N) = \int_N \mathrm{d}\mu, \tag{9}$$

transforming this by F_1^{-1} yields

$$\mu(N) = \int_N \frac{\kappa}{|\det(\Phi)|}\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta, \tag{10}$$

where $\Phi = DF_1 = \dfrac{\partial(r, \varphi, \theta)}{\partial(\Omega, \omega, M_a)}$, D is the differential operator.

Equation (10) indicates that the value of $\mu(N)$ can be achieved just by Lebesgue integral on N. μ is countably additive which can be derived just from the fact that Lebesgue integral is countably additive. So μ is a measure on Σ.

Set $N = M$. It is easy to get $\mu(M) = 1$ if $\kappa = 1 \Big/ \displaystyle\int_M \frac{1}{|\det(\Phi)|}\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$ (the fact that κ is a finite positive value will be presented in Section 4.1).

Theorem 2 (Measure Preserving Transformation)[9] Let S be a measurable transformation on the finite measure space (X, Π, ν), then ν is S-invariant if and only if

$$\int_X l\mathrm{d}\nu = \int_X l \circ S\mathrm{d}\nu,$$

where $l \in L^\infty(X)$, $\circ$ is denoted as the composition of two transformations.

For the proof of the theorem see Ref.[9].

Lemma 2 The one-parameter group $\phi_t : M \to M$ is μ-measure preserving.

Proof Σ is the power set of M, then $\phi_t^{-1}(A) \in \Sigma$ is always true for $\forall A \in \Sigma$. So ϕ_t is measurable.

As $\forall g(\Omega, \omega, M_a) \in L^\infty(M)$, then

$$\int_M g \circ \phi_t \mathrm{d}\mu = \int_M (g \circ \phi_t)(\Omega, \omega, M_a)\mathrm{d}\Omega\mathrm{d}\omega\mathrm{d}M_a$$

is true. Processing the transformation ϕ_t yields

$$\begin{aligned}\int_M g\circ\phi_t \mathrm{d}\Omega\mathrm{d}\omega\mathrm{d}M_a &= \int_M g(\Omega+C_\Omega t,\omega+C_\omega t,M_a+C_{M_a}t)\mathrm{d}\Omega\mathrm{d}\omega\mathrm{d}M_a\\ &= \int_M g(\Omega+C_\Omega t,\omega+C_\omega t,M_a+C_{M_a}t)\mathrm{d}(\Omega+C_\Omega t)\mathrm{d}(\omega+C_\omega t)\mathrm{d}(M_a+C_{M_a}t)\\ &= \int_M g\mathrm{d}\Omega\mathrm{d}\omega\mathrm{d}M_a.\end{aligned}$$

Associated with theorem 2, it is easy to derive that ϕ_t is μ-measure preserving.

3.4. The real-value function

The relationship among the station mask (SM) half-cone angle λ, the radius r, and the minimum elevation angle γ is as the follows,[7]

$$\lambda = \arccos\left(\frac{R_\mathrm{e}\cos(\gamma)}{r}\right)-\gamma. \tag{11}$$

where R_e is the equatorial radius. So the half-cone angle varied with orbit radius can determine a spherical crown. The visible region of the station is defined as the collection of spherical crowns with their radiuses belonging to the set $[r_{\min}, r_{\max}]$, which is noted as SM and shown in Fig.1.

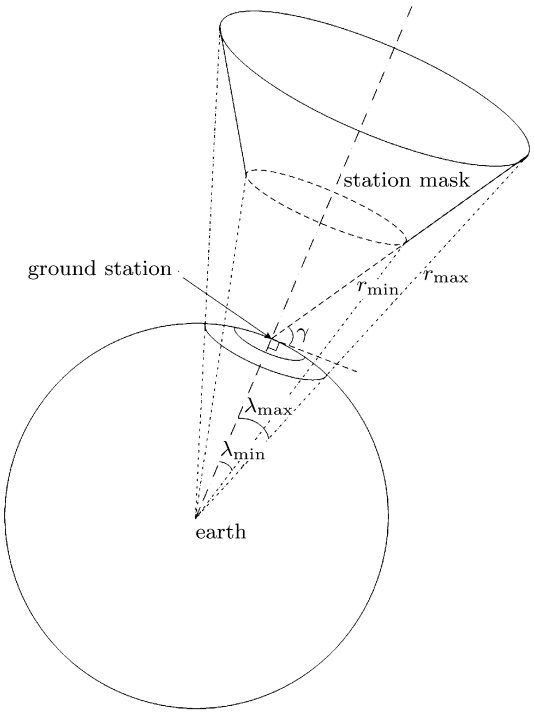


Fig.1. Geometry of the station mask for elliptical orbit.

The ground station's visible frequency ρ is defined as

$$\rho = \lim_{T\to+\infty}\frac{P(T)}{T}, \tag{12}$$

where $P(T)$ is the total time in which the satellite is inside the SM during the interval T.

According to Eq.(1) and the definition of ρ, the real-value function h is defined as the characteristic function

$$h(x) = \begin{cases}1, x\in SM,\\ 0, x\notin SM,\end{cases} \tag{13}$$

then

$$\rho = \lim_{T\to+\infty}\frac{P(T)}{T} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T h(\phi_t(x))\mathrm{d}t. \tag{14}$$

It is easy to verify that h is a real valued and μ-summable function on M, and $h\in L^1(M,\mu)$ from the fact $SM\subset M$.

4. Ergodic algorithm for coverage of elliptical orbits

4.1. The value of κ

Obviously, on the integral domain of M and N (=SM) involved in this paper, the Lebesgue integral is equal to Riemannian integral.

Differentiating F_1 yields

$$|\det(\Phi)| = \frac{\sqrt{\sin^2(i)-\sin^2(\varphi)}}{|\cos(\varphi)\partial r/\partial M_a|}. \tag{15}$$

As shown in Eq.(7), the analytic expression of $\dfrac{\partial r}{\partial M_a}$ is too complicated, so the variable r will be transformed into M_a by the ellipse series (note as the Transformation F_2) to attain

$$\begin{aligned}\kappa =& 1\Big/\int_M |\partial r/\partial M_a|/|\det(\Phi)|\mathrm{d}M_a\mathrm{d}\varphi\mathrm{d}\theta\\ =& 1\Big/\int_M \frac{|\cos(\varphi)|}{\sqrt{\sin^2(i)-\sin^2(\varphi)}}\mathrm{d}M_a\mathrm{d}\varphi\mathrm{d}\theta,\end{aligned} \tag{16}$$

expanding it on M yields

$$\kappa = 1 \Big/ \int_0^{\pi} \mathrm{d}M_a \int_{-L_i}^{L_i} \mathrm{d}\varphi \int_0^{2\pi} \frac{\cos(\varphi)}{\sqrt{\sin^2(i) - \sin^2(\varphi)}} \mathrm{d}\theta, \tag{17}$$

and then

$$\kappa = 1/2\pi^3. \tag{18}$$

4.2. Ergodic algorithm for long-term coverage

The dynamical system (M, μ, ϕ_t) and real-value function f defined in Section 3.2–3.4 have achieved the conditions of using Birkhoff–Khinchin theorem. So the ground station's visible frequency ρ can be obtained as follows

$$\begin{aligned} \rho &= \lim_{T\to+\infty} \frac{P(T)}{T} = \lim_{T\to+\infty} \frac{1}{T} \int_0^T h(\phi_t(x)) \mathrm{d}t \\ &= \int_M h(x) \mathrm{d}\mu = \int_{\mathrm{SM}} \mathrm{d}\mu. \end{aligned} \tag{19}$$

Applied the Transformation F_1 and F_2 to simplify the calculus, a triple integral is then obtained

$$\rho = \frac{1}{2\pi^3} \int_{\mathrm{SM}} \frac{\cos(\varphi)}{\sqrt{\sin^2(i) - \sin^2(\varphi)}} \mathrm{d}M_a \mathrm{d}\varphi \mathrm{d}\theta, \tag{20}$$

expanding it yields

$$\rho = \int_0^{\pi} \mathrm{d}M_a \int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi \int_{-g(\varphi)}^{g(\varphi)} \frac{\kappa \cos(\varphi)}{\sqrt{\sin^2(i) - \sin^2(\varphi)}} \mathrm{d}\theta, \tag{21}$$

and φ_1, φ_2, $g(\varphi)$ are as follows:[1,3]

$$\varphi_1 = \max(\varphi_0 - \lambda, -L_i), \tag{22}$$

$$\varphi_2 = \min(\varphi_0 + \lambda, L_i), \tag{23}$$

$$g(\varphi) = \arccos\left(\frac{\cos(\lambda) - \sin(\varphi)\sin(\varphi_0)}{\cos(\varphi)\cos(\varphi_0)}\right), \tag{24}$$

where φ_0 is the station's latitude, and the SM half-cone angle λ is shown in Eq.(11), the orbit radius is formularized in Eq.(7).

Further simplifying the triple integral (21) yields a double integral

$$\rho = \int_0^{\pi} \mathrm{d}M_a \int_{\varphi_1}^{\varphi_2} \frac{\kappa \cos(\varphi) 2g(\varphi)}{\sqrt{\sin^2(i) - \sin^2(\varphi)}} \mathrm{d}\varphi. \tag{25}$$

Then the above results are extracted as the following theorem.

Theorem 3 The ground station's visible frequency ρ can be obtained by the double integral (25), which is almost true everywhere.

For the resonance, Eq.(25) is not applicable for the null set S_2. The ergodic algorithm just against S_2 will be presented in the next paper.

4.3. Degradation from elliptical orbit to circular orbit

When the elliptical orbit degrades to the circular orbit, the coverage frequency becomes

$$\rho_C = \lim_{e\to 0} \rho, \tag{26}$$

the orbital radius r degrades to the constant a (semi-major axis), and the SM half-cone angle λ degrades to the constant

$$\lambda = \arccos\left(\frac{R_{\mathrm{e}} \cos(\gamma)}{a}\right) - \gamma$$

so φ_1 and φ_2 will degrade to the constants φ_1^C and φ_2^C. Further simplifying Eq.(25) yields

$$\rho_C = \frac{1}{\pi^2} \int_{\varphi_1^C}^{\varphi_2^C} \frac{\cos(\varphi) g(\varphi)}{\sqrt{\sin^2(i) - \sin^2(\varphi)}} \mathrm{d}\varphi. \tag{27}$$

Obviously, the expression derived from Eq.(27) is the same as the one in Refs.[1–4]. The flow space in this paper is on the sphere surface with orbit radius, and the flow is the real trajectory in the space. However, the flow space in Refs.[1–4] is on the earth surface, and the flow is the satellite subpoint, but it can be proved that they are topologically equivalent.

5. Numerical implementation

The circular orbit is a special case of elliptical orbit, so only the cases with large orbit forecast error need to be validated. The calculation adopts same example used in Lo's paper, i.e., the eccentricity is 0.05, and the minimum elevation angle γ is 0°. The compound Simpson algorithm is applied to calculate the theoretical result from Eq.(19). The numeric result is obtained by integrating the trajectory from 0 to 5 years by the Runge–Kutta integrator of 7(8) orders. The comparison of numerical result versus theoretical results is shown in Table 1.

The results show that all the errors are no more than 0.2%, so Lo's results is modified prominently in this paper.

Table 1. Comparison between numerical and theoretical results for elliptical orbits (eccentricity=0.05).

case	semi-major axis/km	inclination /(°)	station latitude/(°)	numeric result	ergodic algorithm	RE/(%)	Lo's result[1] /(%)
1	7714.14	61.0	40.4	0.1019	0.1021	0.196	16.1085
2	7714.14	61.0	–35.4	0.0993	0.0995	0.2	11.5733
3	10000.14	61.0	40.4	0.1845	0.1843	0.11	5.56719
4	10000.14	61.0	–35.4	0.1837	0.1840	0.16	4.65991

*relative error(RE) is defined as $p = |N - E|/N$, where p is denoted as the percentage, N is denoted as numeric result, and E is denoted as the result from the ergodic algorithm.

The ergodic algorithm is available for a wide range of eccentricities, because it bases on the secular term of J_2 perturbation, except for the case that the eccentricity is near 1 (for the analytic perturbation theory of elliptical orbit will fail in the case). The numerical results to demonstrate the relationship between eccentricities and errors brought by the ergodic algorithm are shown in Fig.2, where the orbital semi-axis is set as 7714.14km, the orbital inclination is 61.0°, and the station latitude is 40.4°.

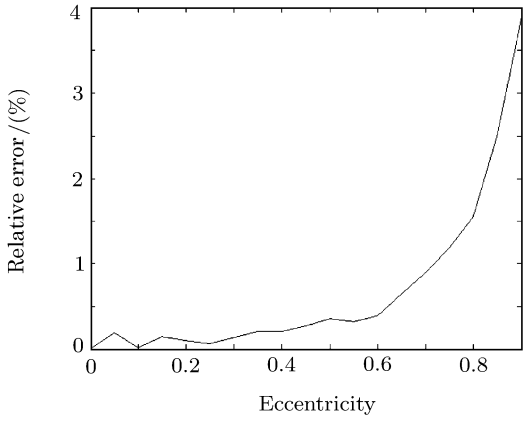


Fig.2. The graph of relative error vs eccentricity.

Figure 2 shows, when the eccentricities increase from 0 to 0.6, that the relative errors are no more than 0.4%, and that the error grows rapidly as the eccentricity approaches 1.

The ergodic algorithm is successful, because it is derived from the secular term of J_2 perturbation, which is essential for the statistic of coverage. For the practical application in engineering, the higher order perturbations ignored will bring errors. However, Lo[1] concluded, from many numerical executions, that the ergodic algorithm works well for the high orbit (with the orbital altitude larger than 800km) instead of low orbit during the life of satellite (5–10 years). It means that the higher order perturbations will bring the error less than 1%[1] (with the orbital altitude between 1336km and 3622km), but the ergodic algorithm will fail in the low orbit due to the atmospheric perturbation.

6. Conclusions and perspective

In this paper, the ergodic theory is applied to analyse the long-term coverage of elliptical orbits. The mean orbital elements of elliptical orbits under the effect of J_2 perturbation are applied to construct the invariant measure, and to define the flow as well as its space. Birkhoff–Khinchin theorem is then applied to transform integrating trajectories into solving a double integral. Finally, the numerical executions have validated that our theory may greatly improve Lo's results. The error brought by our theory, for the elliptical orbit with its eccentricity 0.05, is only 0.2%, compared with 16% by Lo's theory. Furthermore, the numerical implementations indicate that the ergodic algorithm is available for a wide range of eccentricities.

It should be emphasized that many problems in astrodynamics involving in the time mean can be transformed to the problems in the space mean, which can lessen the computation and shorten the design periods. These problems at least include the follows:

1) the frequency of the orbits near the earth being shaded by the earth;

2) the coverage areas to the earth's surface of the libration point orbits (Halo and Lissajous orbits);

3) the frequency of the communication satellite running on a Lissajous orbit round L_2 point of earth-moon system being shaded by the moon.

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