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Wavelet–fractional Fourier transforms*

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This paper extends the definition of fractional Fourier transform (FRFT) proposed by Namias V by using other orthonormal bases for $L^2(R)$ instead of Hermite–Gaussian functions. The new orthonormal basis is gained indirectly from multiresolution analysis and orthonormal wavelets. The so defined FRFT is called wavelets-fractional Fourier transform.

Keywords: multiresolution analysis, fractional Fourier transform, wavelets-fractional Fourier transform

PACC: 4230, 4230K, 4200

1. Introduction

The fractional Fourier transform (FRFT) emerged[1,2] as an extension of the Fourier transform, which, here, is referred to as the ordinary Fourier transform (FT) and is one of the most commonly used tools in signal processing. It has recently been demonstrated to be useful in quantum mechanics[3,4] and has also been introduced in bulk optics,[5−7] as well as in fibre optics,[8,9] as a fundamental tool for optical information processing. The FRFT has extended the conventional FT to a more general time–frequency (or space–frequency) joint representation[10] by allowing the concept of the fractional integral to be applied in the Fourier integrals. Along with their challenging applications, their mathematical properties still attract much attentions. Later, Shih[11] reinvented a technique to fractionalize the FT with only the weighted composition of four basic functions. It is a 4-periodic operation not only with respect to the order of the transform α but also to the order of the Hermite–Gaussian (HG) functions m. Recently, Liu and Zhang *et al*[12,13] extend the FRFT proposed by Shih to FRFT operators with $4L$ periodicity by analysing the periodicity of FRFT. Ran *et al* extend the $4L$-periodic FRFT to a more general FRFT called high order generalized permutational fractional Fourier transform (HGPFRFT).[14] We can see that FRFT has the property of multiplicity. In fact, the multiplicity arises for two reasons. One is that there are different eigenfunctions for FRFT, the other is that there are a lot of techniques to fractionalize the eigenvalues in the procedure of constructing FRFT.

Wavelet analysis is a relatively recent development in applied mathematics.[15] It has developed in parallel with FRFT as two separate branches of FT. In fact, the window FT proposed by Gabor in 1946, which is a standard technique for time-frequency localization, led to the emergence of the wavelet transform. The concept of wavelets can be viewed as a synthesis of ideas in engineering, physics, and pure mathematics. As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers in many different fields. Especially, wavelets play a role in optics physics.[16] On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications. In particular, orthonormal wavelets and multiresolution analysis (MRA) as tools have good properties in studying other branches of mathematics. Already they have led to exciting applications in signal analysis and numerical analysis. Many other applications are being studied now.

It is well known that the FT has orthonormal eigenfunctions $\phi_m(x) = 1/\sqrt{2^m m!\sqrt{\pi}} H_m(x) \exp\left(-x^2/2\right), m = 0, 1, 2, \ldots$ (HG functions) which constitute the orthonormal basis (OB) for $L^2(R)$. Since the multiplicity of FRFT is caused by choosing different orthonormal eigenfunctions on one hand, we can obtain many different FRFT by using different orthonormal eigenfunctions which constitute OB for $L^2(R)$. When choosing different eigenfunctions, the FRFT operator will be also different. In fact, the new OB will be obtained by making unitary transforma-

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tions on the known HG functions, moreover, so will be the new FRFT. In this paper the unitary transformations or unitary matrices are gained from orthonormal wavelets and MRA by analysing the properties of scaling function, wavelet function and their coefficients.

The aim of this paper is to introduce an approach to the multiplicity of the FRFT based on the OB of wavelets. We firstly discuss the MRA, orthonormal wavelets and some properties of them. We can explicitly see how to construct unitary matrices by making use of the properties of orthonormal wavelets and MRA. Then we will discuss in detail how to construct new OB by making unitary transformations on the HG functions and see how to define new FRFT by expressing them as the superposition of OB different from HG functions. In order to understand the wavelet–FRFT better, computer simulations are given to compare wavelet–FRFT with several other FRFT.

This paper is arranged as follows. In Section 2, we introduce MRA, orthonormal wavelets and compactly supported orthonormal wavelets for the reason that we need it to construct unitary matrices. In Section 3, the conventional FRFT is introduced briefly. In fact, the basic idea that we use to construct wavelet–FRFT is similar to the idea of conventional FRFT. In Section 4, we can see how to obtain OB by making unitary transformations on the HG functions. We can also see in detail how to define FRFT based on wavelets. And an example is given. Finally, the conclusions are presented.

2. Multiresolution analysis and orthonormal wavelets

2.1. Multiresolution analysis and orthonormal wavelets

$\{V_j; j \in Z\}$ is a sequence of successive approximation closed subspaces of $L^2(R)$, and $\phi(x)$ is a function in $L^2(R)$, they satisfy

i) $V_j \subset V_{j+1}, \forall j \in Z$,

ii) $\bigcap_{j\in Z} V_j = \{0\}$,

iii) $\overline{\left(\bigcup_{j\in Z} V_j\right)} = L^2(R)$,

iv) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad \forall j \in Z$,

v) $\{\phi(x-n); n \in Z\}$ is an OB for V_0. We call $\{\{V_m; m \in Z\}; \phi(x)\}$ is a MRA in $L^2(R)$.

According to the definition above, $\{\phi_{j,n}(x) = 2^{j/2}\,\phi\left(2^j x - n\right); n \in Z\}$ is an OB for V_j for all $j \in Z$.

Now let us see how to use a MRA to construct an orthonormal wavelet $\psi(x)$ so that $\left\{2^{j/2}\psi\left(2^j x - k\right); (j,k) \in Z \times Z\right\}$ is an OB for $L^2(R)$.

For every $j \in Z$, define W_j to be the orthogonal complement of V_j in V_{j+1}. We have

$$W_j \perp V_j, V_{j+1} = W_j \oplus V_j.$$

It is easy to see that

i) $\forall j \neq l, \quad W_j \perp W_l$,

ii) $L^2(R) = \bigoplus_{l\in Z} W_l$,

iii) $\forall j \in Z, \quad g(x) \in W_j \Leftrightarrow g(2x) \in W_{j+1}$.

ii) means that we can construct an OB for $L^2(R)$ just by finding out an OB for subspace W_j. iii) means that the task of construction is reduced to find out the OB for W_0. So the key problem is how to construct a function $\psi(x)$ so that $\{\psi(x-k); k \in Z\}$ is an OB for W_0.

Since $\phi(x) \in V_0 \subseteq V_1$, and $\{\sqrt{2}\phi(2x-n); n \in Z\}$ is an OB in V_1, there must be only and only coefficients $\{h_n; n \in Z\} \in l^2(Z)$, such that

$$\phi(x) = \sqrt{2}\sum_{n\in Z} h_n \phi(2x-n). \tag{1}$$

It is usually called scaling function, where

$$\begin{aligned} h_n &= \left\langle \phi(\cdot), \sqrt{2}\phi(2\cdot - n)\right\rangle \\ &= \sqrt{2}\int_R \phi(x)\,\bar{\phi}(2x-n)\,\mathrm{d}x. \end{aligned} \tag{2}$$

We can rewrite the scaling function as

$$\Phi(\omega) = \frac{1}{\sqrt{2}}\sum_n h_n \Phi\left(\frac{\omega}{2}\right)\exp\left(-\mathrm{i}\frac{\omega}{2}n\right), \tag{3}$$

or

$$\Phi(\omega) = H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right), \tag{4}$$

where

$$H(\omega) = \frac{1}{\sqrt{2}}\sum_{n\in Z} h_n \exp(-\mathrm{i}n\omega). \tag{5}$$

On the other hand, since wavelet function $\psi(x) \in W_0 \subseteq V_1$, there must be a sequence of coefficients $\{g_n; n \in Z\}$, such that

$$\psi(x) = \sqrt{2}\sum_{n\in Z} g_n \phi(2x-n). \tag{6}$$

It is called the wavelet function. So the question reduces to find the corresponding scaling coefficient sequence $\{g_n; n \in Z\}$. We can rewrite the wavelet functions as

$$\Psi(\omega) = \frac{1}{\sqrt{2}}\sum_n g_n \Phi\left(\frac{\omega}{2}\right)\exp\left(-\mathrm{i}\frac{\omega}{2}n\right), \tag{7}$$

or

$$\Psi(\omega)=\Gamma\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right), \tag{8}$$

where

$$\Gamma(\omega)=\frac{1}{\sqrt{2}}\sum_{n\in Z}g_n\exp(-\mathrm{i}n\omega). \tag{9}$$

It is easy to see that $H(\omega)$ and $\Gamma(\omega)$ are both 2π-periodic functions in $L^2(0,2\pi)$. Since $\{\phi(x-n);\ n\in Z\}$ is an OB for V_0, we have

$$\sum_{n\in Z}|\Phi(\omega+2n\pi)|^2=1,\ \text{a.e.}\ \omega\in R. \tag{10}$$

Substituting Eq.(4) into Eq.(10) we have

$$\begin{aligned}
&\sum_{n\in Z}|\Phi(\omega+2n\pi)|^2\\
=&\sum_{n\in Z}\left|H\left(\frac{\omega}{2}+n\pi\right)\Phi\left(\frac{\omega}{2}+n\pi\right)\right|^2\\
=&\sum_{n=2l}\left|H\left(\frac{\omega}{2}+2l\pi\right)\Phi\left(\frac{\omega}{2}+2l\pi\right)\right|^2\\
&+\sum_{n=2l+1}\left|H\left(\frac{\omega}{2}+\pi+2l\pi\right)\Phi\left(\frac{\omega}{2}+\pi+2l\pi\right)\right|^2\\
=&\left|H\left(\frac{\omega}{2}\right)\right|^2\sum_{n=2l}\left|\Phi\left(\frac{\omega}{2}+2l\pi\right)\right|^2\\
&+\left|H\left(\frac{\omega}{2}+\pi\right)\right|^2\sum_{n=2l+1}\left|\Phi\left(\frac{\omega}{2}+\pi+2l\pi\right)\right|^2\\
=&\left|H\left(\frac{\omega}{2}\right)\right|^2+\left|H\left(\frac{\omega}{2}+\pi\right)\right|^2.
\end{aligned}$$

So Eq.(10) implies

$$|H(\omega)|^2+|H(\omega+\pi)|^2=1,\ \text{a.e.}\ \omega\in R. \tag{11}$$

On the other hand, since $\{\psi(x-k);k\in Z\}$ is an OB for W_0, we have

$$\sum_{n\in Z}|\Psi(\omega+2n\pi)|^2=1,\ \text{a.e.}\ \omega\in R. \tag{12}$$

W_0 and V_0 are orthogonal in V_1, so $\{\psi(x-k);k\in Z\}$ and $\{\phi(x-n);n\in Z\}$ are orthogonal, i.e.

$$\langle\phi(\cdot-n),\psi(\cdot-k)\rangle=0,\ \forall(n,k)\in Z\times Z.$$

Substituting Eqs.(4) and (8) leads to

$$\begin{aligned}
&\langle\phi(\cdot-n),\psi(\cdot-k)\rangle\\
=&\frac{1}{2\pi}\int_R\Phi(\omega)\exp(-\mathrm{i}n\omega)\overline{(\Psi(\omega)\exp(-\mathrm{i}k\omega))}\mathrm{d}\omega\\
=&\frac{1}{2\pi}\sum_{l\in Z}\int_{4l\pi}^{4(l+1)\pi}H\left(\frac{\omega}{2}\right)\bar{\Gamma}\left(\frac{\omega}{2}\right)\left|\Phi\left(\frac{\omega}{2}\right)\right|^2\\
&\times\exp(-\mathrm{i}(n-k)\omega)\,\mathrm{d}\omega\\
=&\frac{1}{2\pi}\int_0^{4\pi}H\left(\frac{\omega}{2}\right)\bar{\Gamma}\left(\frac{\omega}{2}\right)\exp(-\mathrm{i}(n-k)\omega)\,\mathrm{d}\omega\\
=&\frac{1}{2\pi}\int_0^{2\pi}\left[H\left(\frac{\omega}{2}\right)\bar{\Gamma}\left(\frac{\omega}{2}\right)+H\left(\frac{\omega}{2}+\pi\right)\bar{\Gamma}\left(\frac{\omega}{2}+\pi\right)\right]\\
&\times\exp(-\mathrm{i}(n-k)\omega)\,\mathrm{d}\omega.
\end{aligned}$$

As $\left\{1/\sqrt{2}\exp(\mathrm{i}nx);\ \ n\in Z\right\}$ is an OB for $L^2(0,\ 2\pi)$, $\langle\phi(\cdot-n),\ \psi(\cdot-k)\rangle=0,\ \forall(n,k)\in Z\times Z$ is equal to

$$\begin{aligned}
&H(\omega)\bar{\Gamma}(\omega)+H(\omega+\pi)\bar{\Gamma}(\omega+\pi)=0,\\
&\text{a.e.}\ \ \omega\in R.
\end{aligned} \tag{13}$$

Theorem If $\psi(x)=\sqrt{2}\sum_{n\in Z}g_n\phi(2x-n)$, $\{\psi(x-k);k\in Z\}$ is an OB for W_0 only and only if $\boldsymbol{M}(\omega)$ is unitary matrix, or

$$\boldsymbol{M}^*(\omega)\boldsymbol{M}(\omega)=I,\ \ \text{a.e.}\ \omega\in R, \tag{14}$$

where

$$\boldsymbol{M}(\omega)=\begin{pmatrix}H(\omega) & \Gamma(\omega)\\ H(\omega+\pi) & \Gamma(\omega+\pi)\end{pmatrix}. \tag{15}$$

We choose

$$\Gamma(\omega)=\exp(-\mathrm{i}\omega)\bar{H}(\omega+\pi). \tag{16}$$

Then $\boldsymbol{M}(\omega)$ that we defined above is a unitary matrix. So we can obtain the wavelet function $\psi(x)$ in the form of frequency

$$\Psi(\omega)=\exp(-\mathrm{i}\omega/2)\bar{H}(\pi+\omega/2)\Phi(\omega/2). \tag{17}$$

Substituting Eqs.(5) and (9), we have

$$g_n=(-1)^{n-1}\bar{h}_{1-n},\ \ n\in Z. \tag{18}$$

So the form of the wavelet function $\psi(x)$ in time domain is

$$\psi(x)=\sqrt{2}\sum_{n\in Z}(-1)^{n-1}\bar{h}_{1-n}\phi(2x-n). \tag{19}$$

Now rewrite Eq.(6). By using sequences $\{h_n\}_{n\in Z}$ and $\{g_n\}_{n\in Z}$, we can rewrite

$$\begin{aligned}
&|H(\omega)|^2+|H(\omega+\pi)|^2=1,\\
&H(\omega)\bar{\Gamma}(\omega)+H(\omega+\pi)\bar{\Gamma}(\omega+\pi)=0,\ \ \text{a.e.}\ \omega\in R,\\
&|\Gamma(\omega)|^2+|\Gamma(\omega+\pi)|^2=1,
\end{aligned}$$

as

$$\begin{aligned}
&\sum_n h_n\bar{h}_{n-2m}=\delta(m),\\
&\sum_n h_n\bar{g}_{n-2m}=0,\ \ m\in Z,\\
&\sum_n g_n\bar{g}_{n-2m}=\delta(m).
\end{aligned} \tag{20}$$

Define matrices

$$\begin{aligned}
&\boldsymbol{H}=(h_{n,m})_{+\infty\times-\infty},\ \ h_{n,m}=\bar{h}_{m-2n},\\
&\boldsymbol{\Gamma}=(g_{n,m})_{+\infty\times-\infty},\ \ g_{n,m}=\bar{g}_{m-2n},
\end{aligned}\quad m,n\in Z,$$

where $h_{m,n},g_{m,n}$ are respectively the elements in row n and column m of matrices $\boldsymbol{H}$ and $\boldsymbol{\Gamma}$, $m,n\in Z$. We have

$$\boldsymbol{H}=\begin{pmatrix}\cdots & \cdots & & & & & & & & \\ \cdots & \bar{h}_{-2} & \bar{h}_{-1} & \bar{h}_0 & \bar{h}_1 & \bar{h}_2 & \cdots & & & \\ & & \cdots & \bar{h}_{-2} & \bar{h}_{-1} & \bar{h}_0 & \bar{h}_1 & \bar{h}_2 & \cdots & \\ & & & & \cdots & \bar{h}_{-2} & \bar{h}_{-1} & \bar{h}_0 & \bar{h}_1 & \bar{h}_2 & \cdots \\ & & & & & & & & \cdots & \cdots \end{pmatrix},$$

$$\boldsymbol{\Gamma}=\begin{pmatrix}\cdots & \cdots & & & & & & & & \\ \cdots & \bar{g}_{-2} & \bar{g}_{-1} & \bar{g}_0 & \bar{g}_1 & \bar{g}_2 & \cdots & & & \\ & & \cdots & \bar{g}_{-2} & \bar{g}_{-1} & \bar{g}_0 & \bar{g}_1 & \bar{g}_2 & \cdots & \\ & & & & \cdots & \bar{g}_{-2} & \bar{g}_{-1} & \bar{g}_0 & \bar{g}_1 & \bar{g}_2 & \cdots \\ & & & & & & & & \cdots & \cdots \end{pmatrix}.$$

Then it is easy to see that

$$\tilde{\boldsymbol{M}}^*\tilde{\boldsymbol{M}}=\tilde{\boldsymbol{M}}\tilde{\boldsymbol{M}}^*=\boldsymbol{E}(\infty),$$

where

$$\tilde{\boldsymbol{M}}=\begin{pmatrix}\boldsymbol{H}\\ \boldsymbol{\Gamma}\end{pmatrix},\quad\text{and}\quad \boldsymbol{E}(\infty)=\begin{pmatrix}\cdots & & & & & & \\ \cdots & 0 & 0 & 1 & 0 & 0 & \\ & & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 1 & 0 & 0 & \cdots \\ & & & & & & & & \cdots \end{pmatrix}.$$

Generally, we are interested in compactly supported orthonormal wavelets ψ. While the easiest way to ensure compact support for the wavelet ψ is to choose the scaling function ϕ with compact support. It then follows from the definition of the h_n, that only finitely many h_n are nonzero, so that ψ reduces to a finite linear combination of compactly supported functions, and therefore automatically has compact support itself. Therefore in every column of the matrix $\tilde{\boldsymbol{M}}$ corresponding to the orthonormal wavelets, there only finite h_n are nonzero.

2.2. An example of compactly supported orthonormal wavelets—Haar wavelet

Define a function $\varphi(t)$

$$\varphi(t)=\begin{cases}0, & t\bar{\in}[0,1),\\ -1, & t\in[0,1)\end{cases}\tag{21}$$

and construct a subspace $V_m=\text{Closespan}\{2^{m/2}\varphi(2^m t-n);n\in Z\}$ so that it is closed subspace in $L^2(R)$ for every $m\in Z$. It is easy to see that $\{\{V_m;m\in Z\};\varphi(t)\}$ is a MRA in $L^2(R)$ and $\{\varphi(t-k);k\in Z\}$ is an OB for subspace V_0. Under this condition, the scaling function is

$$\varphi(t)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\varphi(2t)+\frac{1}{\sqrt{2}}\varphi(2t-1)\right),\tag{22}$$

so $h_0=h_1=1/\sqrt{2}$, the other scaling coefficients are zero. Moreover, we know that $g_0=-1/\sqrt{2}$, $g_1=1/\sqrt{2}$ by knowing the relationship between scaling coefficients and wavelet coefficients. So we have the wavelet function

$$\psi(t)=\sqrt{2}\left(-\frac{1}{\sqrt{2}}\varphi(2t)+\frac{1}{\sqrt{2}}\varphi(2t-1)\right).\tag{23}$$

In fact, the wavelet function above is actually the Haar wavelet $h(x)$,

$$h(x)=\begin{cases}1 & 0\le x\le 2^{-1},\\ -1 & 2^{-1}\le x\le 1,\\ 0 & x\notin(0,1).\end{cases}$$

From above, we know that $\{h_{j,k}(x)=2^{j/2}h(2^j x-k;(j,k)\in Z\times Z\}$ constitutes an OB for $L^2(R)$ and Haar wavelet is a compactly orthonormal wavelet. So the corresponding infinite matrix

$$\tilde{\boldsymbol{M}}\,(\text{Haar}) = \begin{pmatrix} \cdots & \cdots & & & & & & & & \\ \cdots & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & & & \\ & & \cdots & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & \\ & & & & \cdots & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots \\ & & & & & & & & & \cdots & \cdots \\ \cdots & \cdots & & & & & & & & & \\ \cdots & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & & & & \\ & & \cdots & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & & \\ & & & & \cdots & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots \\ & & & & & & & & & \cdots & \cdots \end{pmatrix}$$

is a unitary matrix.

3. Conventional fractional Fourier transforms

Namias defined FRFT as the supposition of HG functions

$$\left(F^{\alpha} f\right)(x) = \sum_{m=0}^{\infty} d_m \lambda_m^{\alpha} \phi_m(x),$$

where

$$f(x) = \sum_{m=0}^{\infty} d_m \phi_m(x), \quad d_m = \int_R f(x) \bar{\phi}_m(x)\,\mathrm{d}x.$$

$\phi_m(x) = 1/\sqrt{2^m m! \sqrt{\pi}} H_m(x) \exp\left(-x^2/2\right)$, $m = 0, 1, \ldots$ are the standard eigenfunctions. The corresponding eigenvalue to the standard eigenfunction ϕ_m is $\lambda_m = \exp(-m\pi \mathrm{i}/2)$. The eigenfunction here referred to means that its FT is multiplication of the corresponding eigenvalue and itself. And its norm is 1. In fact, since the integer-order FT is a 4-periodic operation, $\{\phi_m, m = 0, 1, 2, \ldots\}$ can be divided into four $\{\phi_k, \phi_{4+k}, \phi_{8+k}, \ldots, \phi_{4m+k}, \ldots; m = 0, 1, 2, \ldots\}$ for $k = 0, 1, 2, 3$. Each of them are the OB for the corresponding subspace which has eigenvalue $\exp(-ki\pi/2)$, $k = 0, 1, 2, 3$. In that way, $L^2(R)$ can be rewritten as the direct sum of four subspaces $V_k\,(k = 0, 1, 2, 3)$ which are orthogonal to each other i.e.

$$L^2(R) = \oplus V_k\,,\ V_j \perp V_k,\ j \neq k, j,\ k = 0, 1, 2, 3\,.$$

4. Wavelets-fractional Fourier transforms

We know that an OB is still an OB after making a unitary transformation on it. Based on this theorem, we can obtain another OB for $L^2(R)$ by making unitary transformation on the OB of four subspaces respectively and join them together. The result is a new OB for $L^2(R)$. We can reconstruct a FRFT by using the new OB. While the basis is infinite, so we must choose infinite unitary matrices. From Section 2, we know that matrices corresponding to orthonormal wavelets are infinite unitary matrices. Here we use the orthonormal wavelets as the tool to construct different FRFT.

4.1. Haar wavelet-fractional Fourier transforms

First, we choose the Haar wavelet as the tool for unitary transformation here. The original OB for V_0 is $\{\phi_k, \phi_{4+k}, \phi_{8+k}, \ldots, \phi_{4m+k}, \ldots; m = 0, 1, 2, \ldots\}$ and the unitary matrix is $\tilde{\boldsymbol{M}}\,(\text{Haar})$. Making unitary transformation

$$\begin{pmatrix} \gamma_0^{(0)}(t) \\ \gamma_1^{(0)}(t) \\ \vdots \\ \mu_0^{(0)}(t) \\ \mu_1^{(0)}(t) \\ \vdots \end{pmatrix} = \tilde{\boldsymbol{M}}\,(\text{Haar}) \begin{pmatrix} \phi_0 \\ \phi_4 \\ \phi_8 \\ \vdots \\ \phi_{4m} \\ \vdots \end{pmatrix},$$

where $\left\{\gamma_m^{(0)}(t), \mu_m^{(0)}(t); m=0,1,2,\ldots\right\}$ is the new orthonormal basis for V_0, we have

$$\begin{cases} \gamma_m^{(0)}(t) = \dfrac{1}{\sqrt{2}}\left(\phi_{8m}(t) + \phi_{8m+4}(t)\right), \\ \mu_m^{(0)}(t) = \dfrac{1}{\sqrt{2}}\left(\phi_{8m+4}(t) - \phi_{8m}(t)\right), \end{cases} \quad m = 0,1,2,\ldots. \tag{24}$$

We can obtain the other OB for V_1, V_2, V_3 through the same procedure as above. Let the notation $\{\gamma_m^{(k)}(t), \mu_m^{(k)}(t); m=0,1,2,\ldots\}$ denote the OB for V_k for $k=0,1,2,3$, we have

$$\begin{cases} \gamma_m^{(k)}(t) = \dfrac{1}{\sqrt{2}}\left(\phi_{8m+k}(t) + \phi_{8m+4+k}(t)\right), \\ \mu_m^{(k)}(t) = \dfrac{1}{\sqrt{2}}\left(\phi_{8m+4+k}(t) - \phi_{8m+k}(t)\right), \end{cases} \quad m = 0,1,2,\ldots. \tag{25}$$

It is easy to see that $\left\{\gamma_m^{(k)}(t), \mu_m^{(k)}(t); m=0,1,2,\ldots, k=0,1,2,3\right\}$ is a new OB for $L^2(R)$. We arrange the basis as below

$$\tilde{\phi}_{m=8n+k}(t) = \begin{cases} \dfrac{1}{\sqrt{2}}\left(\phi_{8n+k}(t) + \phi_{8n+4+k}(t)\right) & k=0,1,2,3, \\ \dfrac{1}{\sqrt{2}}\left(\phi_{8n+k}(t) - \phi_{8n+k-4}(t)\right) & k=4,5,6,7, \end{cases} \quad n = 0,1,2,\ldots. \tag{26}$$

The notation $\left\{\tilde{\phi}_m(t); m=0,1,2,\ldots\right\}$ denotes the result after arrangement. We define FRFT as

$$\left(F_{\mathrm{H}}^{\alpha} f\right)(x) = \sum_{m=0}^{\infty} \tilde{d}_m \lambda_m^{\alpha} \tilde{\phi}_m(x),$$

where

$$\tilde{d}_m = \int_R f(x)\bar{\tilde{\phi}}_m(x)\,\mathrm{d}x, \quad \lambda_m^{\alpha} = \exp\left(-m\alpha\pi\mathrm{i}/2\right).$$

From above, we can easy to see that we will obtain different operators when arranging the new OB in other way.

4.2. Wavelets-fractional Fourier transforms

During the construction above, we make the same unitary transform on bases for the four different subspaces. Generally we may make different unitary transforms on different bases so that four new OB will be obtained. Then by arranging them in some way, we can construct new FRFT. Suppose four compactly supported orthonormal wavelets (they can be the same) are given here, and the corresponding unitary matrix is

$$\tilde{\boldsymbol{M}}(k) = \begin{pmatrix} \tilde{\boldsymbol{H}}(k) \\ \tilde{\boldsymbol{\Gamma}}(k) \end{pmatrix},$$

where $\tilde{\boldsymbol{H}}(k)$ and $\tilde{\boldsymbol{\Gamma}}(k)$ are respectively

$$\tilde{\boldsymbol{H}}(k) = \begin{pmatrix} \cdots & & & & & & & & \\ \cdots & \alpha_0^{(k)} & \alpha_1^{(k)} & \alpha_2^{(k)} & \alpha_3^{(k)} & \cdots & \alpha_{n(k)}^{(k)} & \cdots & \\ & & & \alpha_0^{(k)} & \alpha_1^{(k)} & \alpha_2^{(k)} & \alpha_3^{(k)} & \cdots & \alpha_{n(k)}^{(k)} & \cdots \\ & & & & & & & & & \cdots \end{pmatrix},$$

$$\tilde{\boldsymbol{\Gamma}}(k) = \begin{pmatrix} \cdots & & & & & & & & \\ \cdots & \beta_0^{(k)} & \beta_1^{(k)} & \beta_2^{(k)} & \beta_3^{(k)} & \cdots & \beta_{n(k)}^{(k)} & \cdots & \\ & & & \beta_0^{(k)} & \beta_1^{(k)} & \beta_2^{(k)} & \beta_3^{(k)} & \cdots & \beta_{n(k)}^{(k)} & \cdots \\ & & & & & & & & & \cdots \end{pmatrix}$$

for $k = 0, 1, 2, 3$. Making unitary transformations on OB for four subspaces respectively, we will have four new OB

$$\begin{cases} \gamma_m^{(k)}(t) = \sum\limits_{l=0}^{n(k)} \alpha_l^{(k)} \phi_{8m+k+4l}, \\ \mu_m^{(k)}(t) = \sum\limits_{l=0}^{n(k)} \beta_l^{(k)} \phi_{8m+k+4l}, \end{cases} \quad m = 0, 1, 2, \ldots \tag{27}$$

for $k = 0, 1, 2, 3$. Arrange them as below:

$$\tilde{\phi}_{m=8p+q}(t) = \begin{cases} \gamma_p^{(q)}(t) \ q = 0, 1, 2, 3, \\ \mu_p^{(q-4)}(t) \ q = 4, 5, 6, 7, \end{cases} \quad p = 0, 1, 2, 3 \ , \ldots . \tag{28}$$

So we define the new FRFT as

$$(F_{\mathrm{W}}^{\alpha} f)(x) = \sum_{m=0}^{\infty} \tilde{d}_m \lambda_m^{\alpha} \tilde{\phi}_m(x), \tag{29}$$

where

$$d_m = \int_R f(x) \bar{\tilde{\phi}}_m(x) \, \mathrm{d}x, \quad \lambda_m^{\alpha} = \exp(-m\alpha\pi \mathrm{i}/2).$$

We can know that when choosing other arrangement, the FRFT operator will be different from the above one.

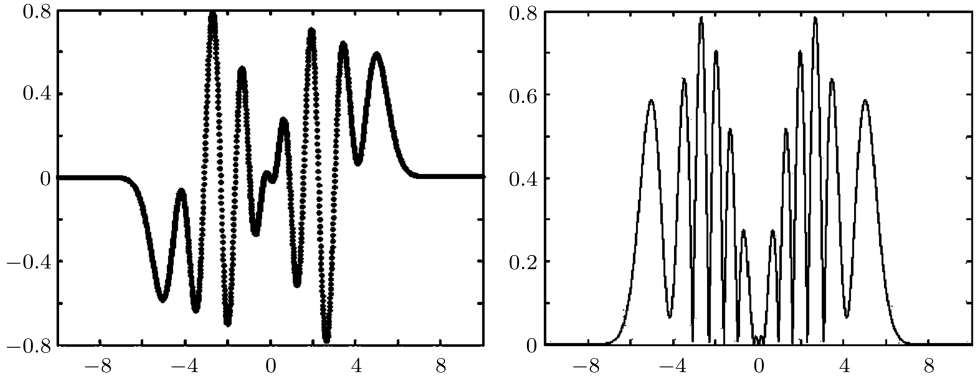
4.3. Computer simulation of wavelet–FRFT

In order to understand the difference between wavelet–FRFT and conventional FRFT, we give out computer simulation of some special functions. Here we choose the linear combination of HG functions with integral coefficients.

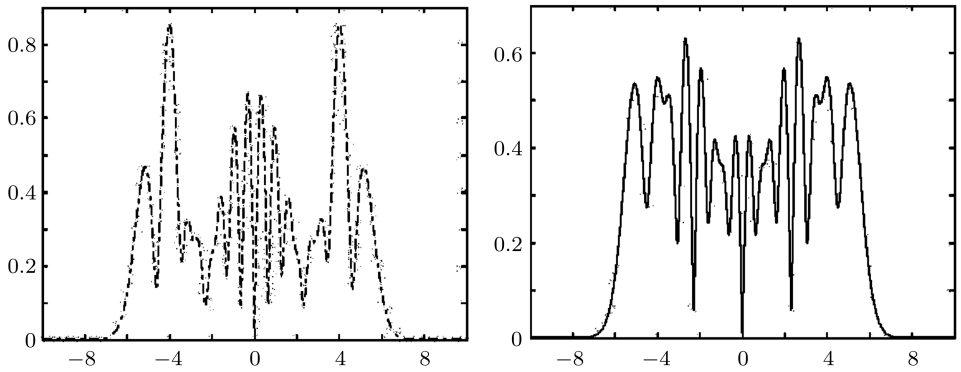
In Fig.1, the fractional order of FRFT is $\alpha = 2\sqrt{5}/11$. There are eight pictures all together. In this figure, the original signal is sum of 9th and 15th HG functions. Figures 1(a) and 1(b) figure out the original signal and amplitude of the original signal respectively. Figure 1(c) figures out the conventional FRFT of original signal. Figure 1(d) is the figure of the standard weighted FRFT of the original signal. Figures 1(e) and 1(f) figure out the first and the second 3-periodic multi-FRFT of the original signal respectively. Figure 1(g) is the figure of Haar wavelet–FRFT of original signal. Figure 1(h) compares all of them together and shows the difference between them.

In Fig.2, the sequence of every picture is as the same as Fig.1. Here, the original signal is minus of 13th and 9th HG functions. And the fractional order of wavelet–FRFT is $\alpha = \sqrt{83}/5$.

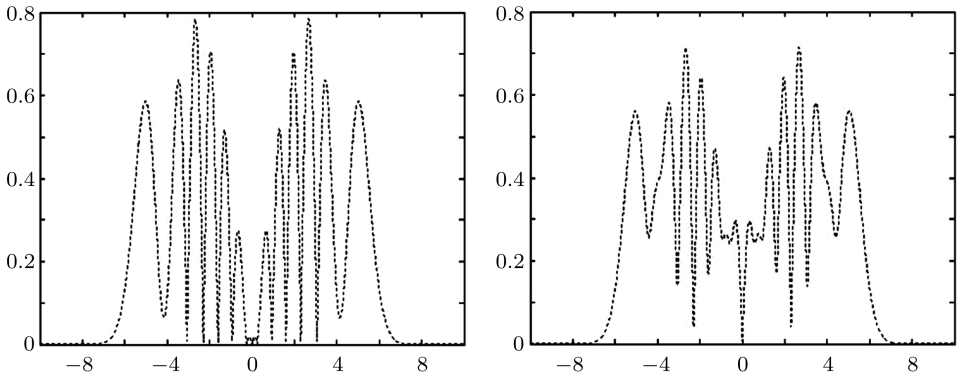
We can see from the simulations above, the wavelet FRFT is a different operator from the conventional FRFT. Since each function in $L^2(R)$ can be expressed as combination of HG functions, the wavelet–FRFT can be used to analyse any signal in $L^2(R)$.



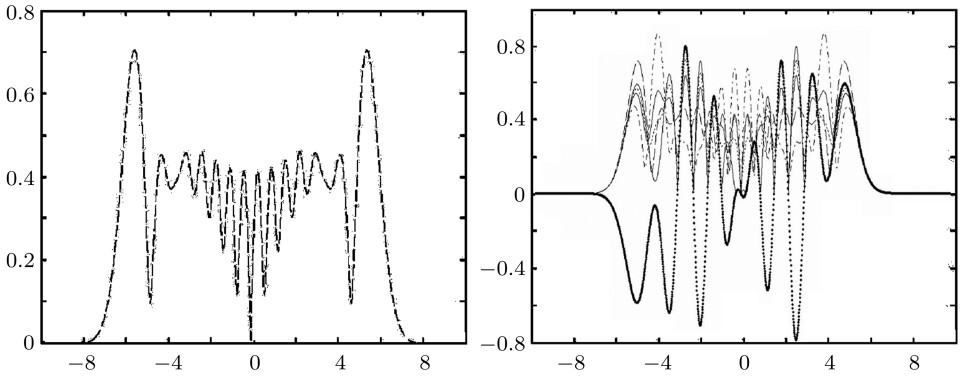
(a) Sum of the 9th and 15th Hermite–Gauss functions. (b) Amplitude of the original signal.



(c) Conventional FRFT of the original signal. (d) Standard weighted FRFT of the original signal.

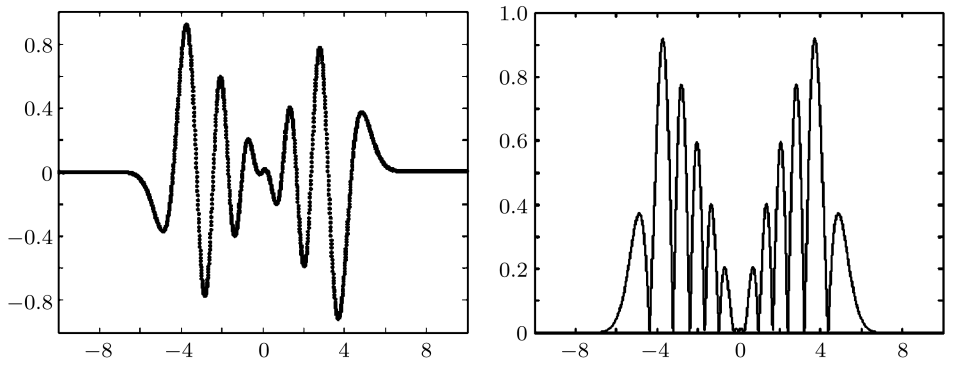


(e) First 3-periodic MFRFT of the original signal. (f) Second 3-periodic MFRFT of the original signal.



(g) Wavelet-FRFR of the original signal. (h) Comparison of several FRFT of the original signal.

Fig.1. The original signal as sum of the 9th and 15th HG functions and comparison of several FRFT at these signals with fraction $\alpha = 2\sqrt{5}/11$.



(a') Difference of the 13th and 9th HG functions. (b') Amplitude of the original signal.

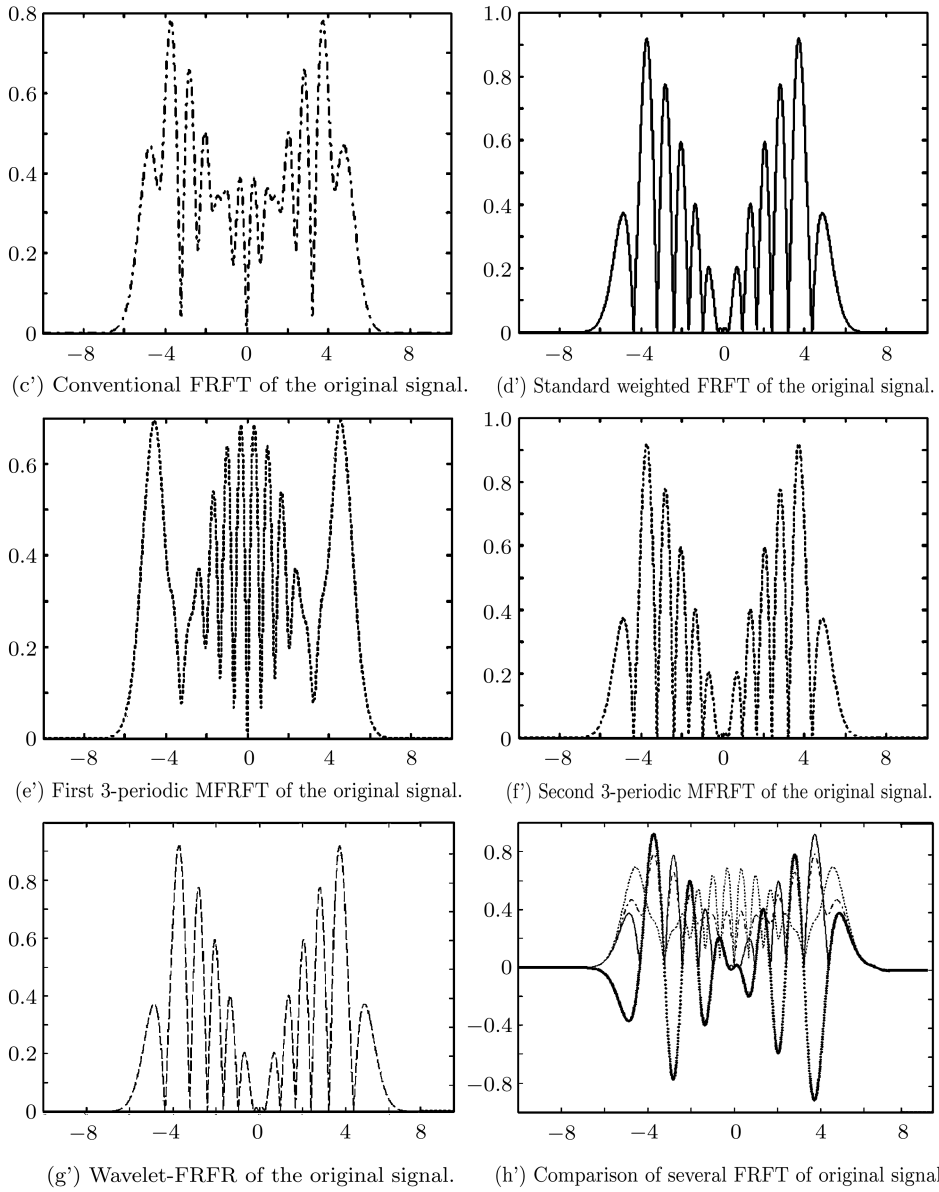


Fig.2. The original signal as minus of the 13th and 9th HG functions and comparison of several FRFT at these signals with fraction $\alpha = \sqrt{83}/5$.

5. Conclusions

We have generalized the definition of FRFT by using different OB which are obtained indirectly from MRA and orthonormal wavelets and expressed the FRFT as the superposition of the orthonormal bases or orthonormal eigenfunctions. When choosing different wavelets, different wavelet–FRFT will be obtained. In fact, wavelet–FRFT is more extensive than the original FRFT. And the original FRFT is just a particular case of wavelet–FRFT whose OB is HG functions. In other words, the original FRFT is just the transform that HG functions are transformed by unitary transformation corresponding to matrix $\boldsymbol{E}(\infty)$. In this paper, we only use the compactly supported orthonormal wavelets as tool to reconstruct FRFT. We can assume that other kinds of discrete wavelets maybe also work in construction of FRFT. In fact, they do. Transformations on HG functions are different and relevant to the corresponding wavelets transforms. The new OB which are transformed from HG function can be arrange in several ways so that different FRFT can be gained.

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