Vol 17 No 1, January 2008
1674-1056/2008/17(01)/0135-05

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Equilibrium points and bifurcation control of a chaotic system*

Liang Cui-Xiang(梁翠香)† and Tang Jia-Shi(唐驾时)

College of Mechanics and Aerospace, Hunan University, Changsha 410082*, China*

(Received 25 January 2007; revised manuscript received 17 April 2007)

Based on the Routh–Hurwitz criterion, this paper investigates the stability of a new chaotic system. State feedback controllers are designed to control the chaotic system to the unsteady equilibrium points and limit cycle. Theoretical analyses give the range of value of control parameters to stabilize the unsteady equilibrium points of the chaotic system and its critical parameter for generating Hopf bifurcation. Certain nP periodic orbits can be stabilized by parameter adjustment. Numerical simulations indicate that the method can effectively guide the system trajectories to unsteady equilibrium points and periodic orbits.

Keywords: chaotic system, Routh–Hurwitz criterion, Hopf bifurcation, feedback control
PACC: 0547, 0545

1. Introduction

Since the first chaotic attractor was found, there has been much attention paid to the study of chaos. Recently, the trend of analysing nonlinear dynamic system has been expanded to new phases: control chaos that has been extensively used in engineering and control bifurcation[1] which has attracted increasing attention. The existing control methods can be classified, mainly, into two categories. The first one is based on a feedback control strategy, such as by Ott, Grebogi, Yorke (OGY),[2,3] state space feedback control.[4] The second one is based on non-feedback control strategy,[5,6] for instance, the entrainment and migration control method.[7,8] However, it is difficult to guarantee the stability in the control process via the second method. Thus, in this paper, we adopt the first method, which has been extensively studied and used in many fields of engineering.[9−21]

We investigate the chaotic system in Ref.[22], which is described by

$$\begin{aligned} \dot{x} &= a(z-x), \\ \dot{y} &= bx - dx^2, \\ \dot{z} &= kxy - cy - gz, \end{aligned} \tag{1}$$

where x, y, z are the state variables, a, b, c, k, g are real positive constant parameters. In this paper, state feedback controllers are designed to control the chaotic system to the unsteady equilibrium points and periodic orbits. Numerical simulations show the effectiveness of the method.

2. Stability analysis

The system (1) has a chaotic attractor when $a = 8$, $b = 40$, $k = 1$, $d = 1$, $c = 10/3$, $g = 4$ as shown in Fig.1. We notice that the chaotic system (1)

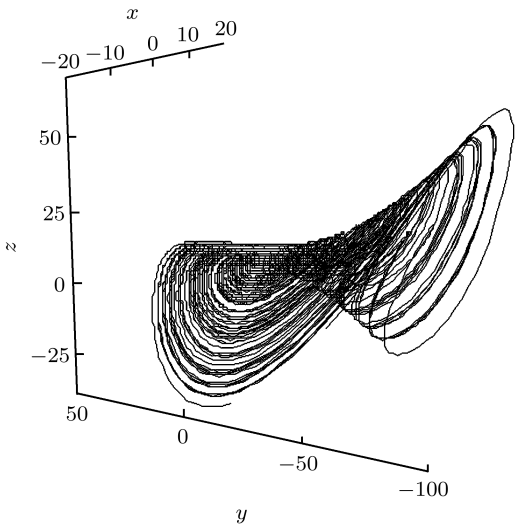


Fig.1. The strange attractor of system(1).

*Project supported by the National Natural Science Foundation of China (Grant No 10672053).
†E-mail: liangcuix@163.com

has two equilibrium points which are $E_0 = (0, 0, 0)$ and $E_1 = (b/d, bg/bk - cd, b/d)$. The Jacobi matrix of the system (1) is $J_0 = \begin{bmatrix} -a & 0 & a \\ b - 2dz & 0 & 0 \\ ky & -c + kx & -g \end{bmatrix}$. If the equilibrium point is $E_0 = (0, 0, 0)$, then the characteristic equation is $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$. When $a = 8$, $k = 1$, $d = 1$, $g = 4$, $c = 10/3$, then $a_1 = a + g > 0$, $a_2 = ag > 0$, $a_3 = abc > 0$, $a_1a_2 < a_3$, According to the Routh–Hurwitz criterion, the equilibrium point E_0 is unstable. The same principle may prove that the equilibrium point E_1 is unstable.

3. Equilibrium points control

According to the control theory, the nonlinear system with state feedback controller is described by

$$\dot{X} = f(X) + u(X), \tag{2}$$

where $X \in R^n$, $f(X)$ is an n-dimensional vector field, $u(X)$ is an n-dimensional control vector. Consider the controller as

$$u(x, y, z) = [k_1(\hat{x} - x), -k_2\dot{x}, 0]^{\mathrm{T}}, \tag{3}$$

then the controlled system is

$$\begin{aligned} \dot{x} &= a(z - x) - k_1(x - \hat{x}), \\ \dot{y} &= bx - dx^2 - k_2a(z - x), \\ \dot{z} &= kxy - cy - gz, \end{aligned} \tag{4}$$

where k_1, k_2 are positive feedback coefficients. When $\hat{x} = 0$, if the feedback coefficients satisfy the following condition:

$$4k_1^2 + 80k_1 - 320k_2 - 682.667 > 0, \tag{5}$$

then the system (4) will gradually converge to unsteady equilibrium point $E_0(0, 0, 0)$. When $\hat{x} = b/d$, if the feedback coefficients satisfy the following condition

$$4k_1^2 + 45.091k_1 + 1466.667k_2 - 11768.24 > 0, \tag{6}$$

then the system (4) will gradually converge to unsteady equilibrium point $E_1 = (b/d, bg/bk - cd, b/d)$.

Proof When $\hat{x} = 0$, the Jacobi matrix of system (4) is

$$\hat{J}_0 = \begin{bmatrix} -a - k_1 & 0 & a \\ b + ak_2 & 0 & -ak_2 \\ 0 & -c & -g \end{bmatrix},$$

then the characteristic matrix is

$$\left[\hat{J}_0 - I\lambda\right] = \begin{bmatrix} -a - k_1 - \lambda & 0 & a \\ b + ak_2 & -\lambda & -ak_2 \\ 0 & -c & -g - \lambda \end{bmatrix},$$

hence the characteristic equation is $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ where

$$\begin{aligned} b_1 &= a + g + k_1, \\ b_2 &= ag + gk_1 - ack_2, \\ b_3 &= abc + ack_1k_2. \end{aligned}$$

For k_1, k_2 are positive, obviously $b_1 > 0$, $b_3 > 0$. If $b_1b_2 - b_3 = 4k_1^2 + 80k_1 - 320k_2 - 682.667 > 0$, according to the Routh–Hurwitz theorem, the system (4) will gradually converge to equilibrium $E_0(0, 0, 0)$. When $\hat{x} = b/d$, the Jacobi matrix of system (4) is

$$\hat{J}_1 = \begin{bmatrix} -a - k_1 & 0 & a \\ -b + k_2 & 0 & ak_2 \\ kbg/kb - cd & -c + kb/d & -g \end{bmatrix}.$$

Then the characteristic equation is $\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$, where

$$\begin{aligned} c_1 &= a + g + k_1, \\ c_2 &= ag - abgk/bk + gk_1 - ack_2 + k_2abk/d, \\ c_3 &= ab^2k/d - abc + (ac - a^2c - abk/d \\ &\quad + a^2bk/d)k_2 + (abk/d - ac)k_1k_2. \end{aligned}$$

When $a = 8, b = 40, k = 1, d = 1, c = 10/3, g = 4$, obviously, $c_1 > 0$, $c_2 > 0$. When

$$c_1c_2 - c_3 = 4k_1^2 + 45.091k_1 + 1466.66k_2 - 11768.24 > 0,$$

then the system (4) will gradually converge to the equilibrium point $E_1 = (b/d, bg/bk - cd, b/d)$.

4. Hopf bifurcation and period-doubling bifurcation control

Consider the controller as

$$u(x, y, z) = [0, 0, k_1\dot{x}]^{\mathrm{T}}, \tag{7}$$

then the controlled system is

$$\begin{aligned} \dot{x} &= a(z - x), \\ \dot{y} &= bx - dx^2, \\ \dot{z} &= kxy - cy - gz - k_1\dot{x}. \end{aligned} \tag{8}$$

Theoretical analysis indicates that when $k_0 = 2.66666667$ the system (8) undergoes a Hopf bifurcation.

Proof Liu[23] derived an equivalent implicit algorithm criterion of Hopf bifurcation on the basis of the Routh–Hurwitz stability criterion, which is stated in terms of the coefficients of the characteristic equations instead of the traditional Hopf bifurcation criterion which is stated in terms of the properties of eigenvalues. The Jacobi matrix of the system (8) is

$$\hat{J}_{\mathrm{h}} = \begin{bmatrix} -a & 0 & a \\ -b & 0 & 0 \\ ak_1 & -c & -g - ak_1 \end{bmatrix}.$$

Then the eigenvalue matrix is

$$\left[\hat{J}_{\mathrm{h}} - I\lambda\right] = \begin{bmatrix} -a-\lambda & 0 & a \\ b & -\lambda & 0 \\ ak_1 & -c & -g - ak_1 - \lambda \end{bmatrix},$$

hence the characteristic equation is $\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0$, where

$$\begin{aligned} d_1 &= a + g + ak_1, \\ d_2 &= ag, \\ d_3 &= abc. \end{aligned} \tag{9}$$

According to Hopf bifurcation criterion, when the following conditions are satisfied,

$$\begin{aligned} &d_1 > 0, \quad d_3 > 0, \quad d_1 d_2 = d_3, \\ &\left.\frac{\mathrm{d}(d_1 d_2 - d_3)}{\mathrm{d}k_1}\right|_{k_1 = k_0} \neq 0, \end{aligned} \tag{10}$$

then there is Hopf bifurcation at $k_1 = k_0$. Thus, the critical parameter is $k_0 = 2.66666667$. Therefore, we can numerically compute all eigenvalues and eigenvectors such as $\lambda_{1,2}(k_0) = \pm 5.65685$, $\lambda_3 = -33.3333$. By adjusting the control parameter, we find that the $2P$, $4P$, $8P$, $16P$ periodic orbits can be stabilized with appropriate k_1.

5. Numerical simulation results

The above theoretical analyses are verified and illustrated via numerical simulation in this section. The proposed controllers are applied to the chaotic system (1) in simulation by using the fourth-Runge–Kutta method. Doing calculations, we find that the parameter values are: $a = 8,\ b = 40,\ d = 1,\ g = 4,\ c = 10/3$. And initial value is $(0.1, 10, 4.5)$.

Applying the controller (3), when the feedback coefficients satisfy the condition (5), we find that the controlled system (4) will gradually converge to equilibrium point $E_0(0, 0, 0)$. Figure 2 shows the trajectories of the system (4) under the feedback coefficients settings $k_1 = 8.00,\ k_2 = 0.1$. When the coefficients satisfy the condition (6), the system (4) will gradually converge to equilibrium point $E_1 = (40, 48/11, 40)$. Figure 3 shows the trajectories of the system (4) under the feedback coefficients settings $k_1 = 0.5,\ k_2 = 17.5$. Applying the controller (7), when $k_1 = 2.66666667$, we find that the system (8) undergoes Hopf bifurcation as shown in Fig.4. As k_1 decreased further, there is a period-doubling bifurcation. In Fig.5 $k_1 = 0.5$, the controlled system (8) works in a period-2 state. In Fig.6 $k_1 = 0.4375$, the controlled system (8) works in a period-4 state. In Fig.7 $k_1 = 0.408125$, the controlled system works in a period-8 state. In Fig.8 $k_1 = 0.39375$, the controlled system (8) works in a period-16 state.

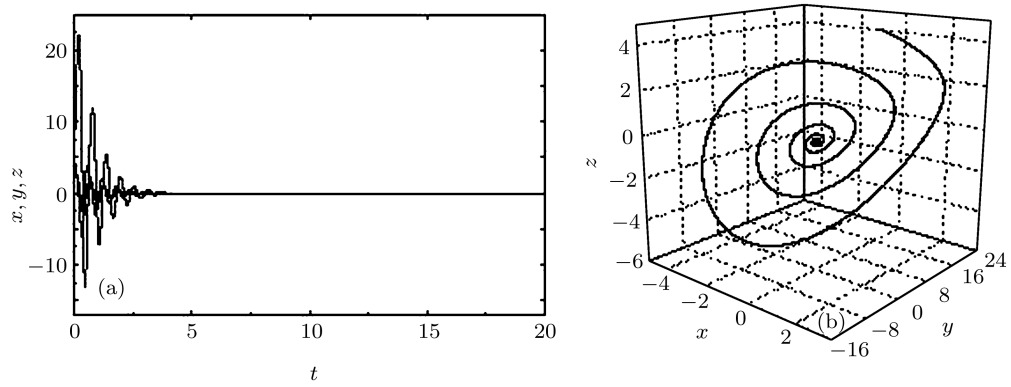


Fig.2. The state of system (4) converges to $E_0(0, 0, 0)$: (a) curve of $t - x$ (b) trajectories of $x - y - z$.

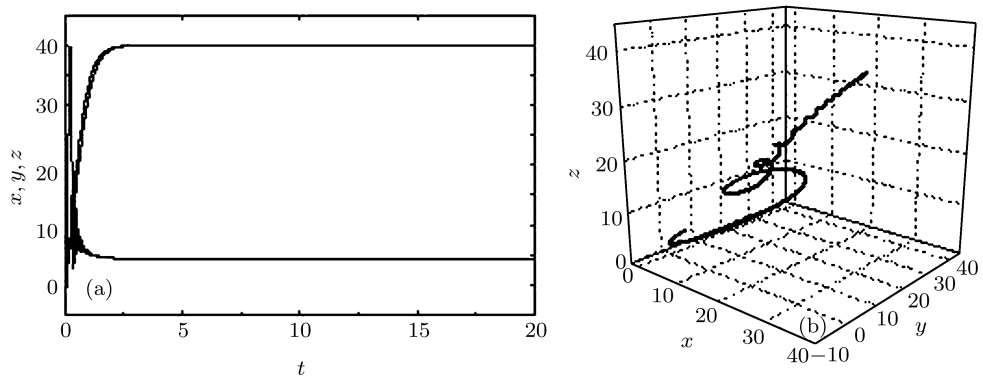


Fig.3. The state of system (4) converges to $E_1 = (40, 48/11, 40)$: (a) curve of $t - x$, (b) trajectories of $x - y - z$.

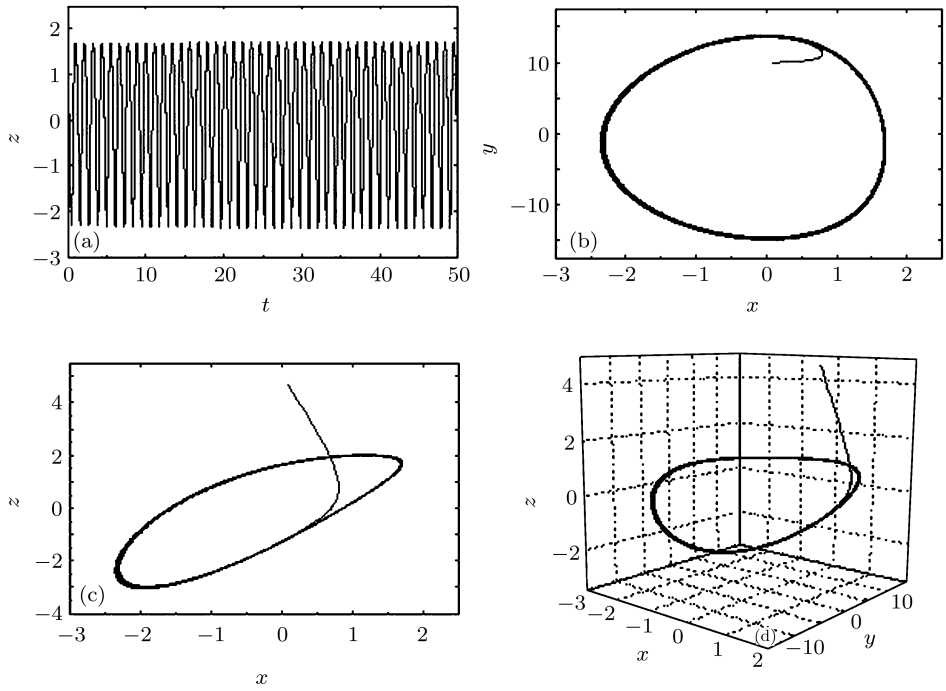


Fig.4. The state of system (8) converges to a limit cycle: (a) curve of $t - x$, (b) trajectories of $x - y$, (c) trajectories of $x - z$, (d) trajectories of $x - y - z$.

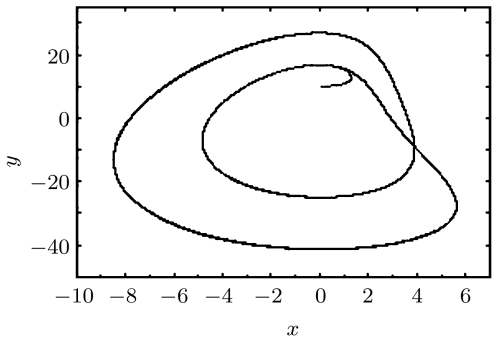


Fig.5. The state of system (8) converges to period-2 trajectory.

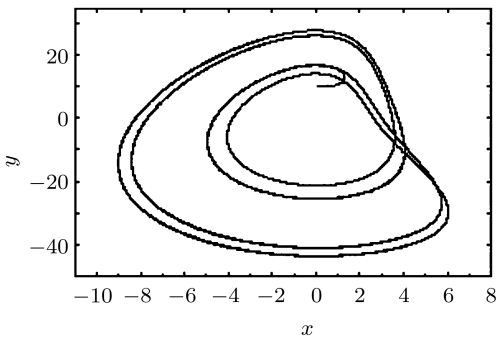


Fig.6. The state of system (8) converges to period-4 trajectory.

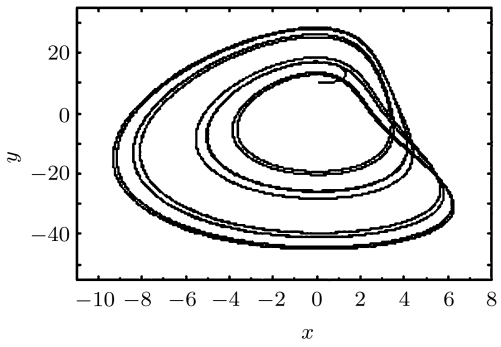


Fig.7. The state of system (8) converges to period-8 trajectory.

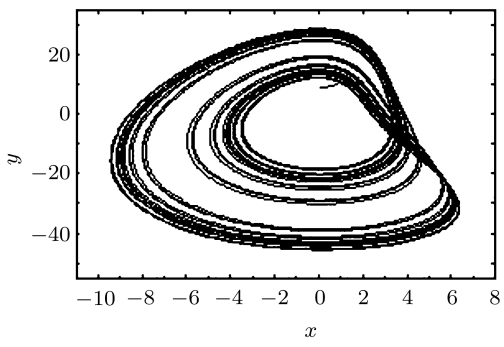


Fig.8. The state of system (8) converges to period-16 trajectory.

6. Conclusions

This paper is concerned about the equilibrium points and bifurcation control of a chaotic system. Applying Routh–Hurwitz criterion, we investigate the stability of the equilibrium points. State feedback controllers are designed to control the equilibrium points and bifurcation, and detailed controllability conditions are also presented. Strict theoretical proof is put forward and numerical simulation results indicate that the controllers can effectively guide the system trajectories to equilibrium points and periodic orbits.

References

[1] Chen G R, Moiola J L and Wang H O 2000 *Int. J. Bifur. Chaos* **10** 511
[2] Ott E, Grebogi C and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196
[3] Chen G and Dong X 1993 *Int. J. Bifur. Chaos* **3** 1343
[4] Luo X S, Fang J Q, Kong L J and Weng J Q 2000 *Acta Phys. Sin.* **49** 2108 (in Chinese)
[5] Luo X S, Fang J Q, Wang L H, Kong L J and Weng J Q 1999 *Acta Phys. Sin.* **48** 895 (in Chinese)
[6] Chen L and Wang D S 2007 *Acta Phys. Sin.* **56** 91 (in Chinese)
[7] Hubler A 1989 *Phys. Acta* **62** 343
[8] Jackson E A and Kodogeorgiou A 1992 *Physica* **54** 253
[9] Tang J S, Fu W B and Li K A 2002 *Chin. Phys.* **11** 1004
[10] Yassen M T 2005 *Chaos, Solitons and Fractals* **26** 913
[11] Tang J S and Qian C Z 2003 *J. Sound Vib.* **268** 1036
[12] Fu W B and Tang J S 2004 *J. Vib. Eng.* **17** 365
[13] Tang J S and Chen Z L 2006 *Int. J. Bifur. Chaos* **16** 487
[14] Qian C Z and Tang J S 2006 *Nonlinear Dynamics* **45** 15
[15] Tang J S and Xiao H 2007 *Acta Phys. Sin.* **56** 101 (in Chinese)
[16] Luo X S, Chen G R, Wang B H, Fang J Q, Zou Y L and Quan H J 2003 *Acta Phys. Sin.* **52** 0790 (in Chinese)
[17] Tang J S and Ouyang K J 2006 *Acta Phys. Sin.* **55** 4437 (in Chinese)
[18] Lu L, Du Z and Luan L 2004 *Acta Photo. Sin.* **33** 14 (in Chinese)
[19] Li R H, Xu W and Li S 2006 *Acta Phys. Sin.* **55** 1000 (in Chinese)
[20] Song Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266
[21] Zhang R and Xu Z Y 2006 *Acta Phys. Sin.* **55** 5070 (in Chinese)
[22] Liu L, Su Y C and Liu C X 2006 *Acta Phys. Sin.* **55** 3933 (in Chinese)
[23] Liu W M 1994 *J. Math. Anal. Appl.* **182** 250