Vol 17 No 1, January 2008
1674-1056/2008/17(01)/0129-06

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Analysis of two-torus in a new four-dimensional autonomous system*

Wu Wen-Juan(吴文娟)†, Chen Zeng-Qiang(陈增强)‡, and Yuan Zhu-Zhi(袁著祉)

Department of Automation, Nankai University, Tianjin 300071, *China*

(Received 27 April 2007; revised manuscript received 7 May 2007)

In this paper, we report the dynamical behaviours of a four-dimensional autonomous continuous dissipative system analysed when the parameter is varied in the range we are interested in. The system changes its dynamical modes between periodic motion and quasiperiodic motion. Furthermore, the existence of two-torus is investigated numerically by means of Lyapunov exponents. By taking advantage of phase portraits and Poincaré sections, two types of the two-torus are observed and proved to have the structure of ring torus and horn torus, both of which are known to be the standard tori.

Keywords: two-torus, two-frequency quasiperiodicity, standard torus, Poincaré section, Lyapunov exponent

PACC: 0547

1. Introduction

Quasiperiodic motion has always been an interesting and attractive research topic. Quasiperiodicity means a motion that is regular but never exactly repeating and is composed of multiple but simple periodic motions. If there exists a length of time that evenly divide the frequencies of the underlying motions, the composite motion will also be periodic. However, no such a length of time exists. Assume that the trajectory of a system is quasiperiodic and consists of two incommensurate frequencies, then the system is called a two-frequency quasiperiodic system. The attractor that the two-frequency quasiperiodic system has is a two-torus.

Lyapunov exponents,[1] which are related to the exponentially fast divergence or convergence of nearby orbits in the phase space, provide a qualitative and quantitative characterization of dynamical behaviours. Therefore, we could decide whether the system has a two-torus attractor or not by calculating its Lyapunov exponents. In a three-dimensional continuous dissipative system, the spectrum that a two-torus has is as follows: (0, 0, −). If we extend this result to four-dimensional continuous dissipative systems, the Lyapunov exponents spectrum for a two-torus should be (0, 0, −, −), and for a limit circle it has only one null and three negative Lyapunov exponents. Besides, we could use Poincaré section to analyse them. For a two-torus attractor and a limit circle, their Poincaré section should be a limit circle and a fixed point respectively.

In previous studies, quasiperiodic behaviour has attracted researcher's attention in many fields, such as phononic crystal,[2] superlattice film,[3] chaotic systems,[4,5] and atmosphere analysis.[6] Especially, much attention has been paid to two-frequency quasiperiodic attractors, i.e. two-torus, in electronic circuits,[7] laser,[8] power system[9] and so forth. Furthermore, two-torus are easily observed in periodically forced dynamical systems.[10−13] For example, in Ref.[13] Zheng investigated a two-torus, and even the torus doubling route to chaos in a model of N sinusoidally coupled phase oscillators with different natural frequencies and nearest couplings. There are also a few papers about two-torus in some autonomous systems and autonomous circuits. In 1987, Matsumoto *et al* reported a torus in three-dimensional autonomous circuits.[14] Nishiuchi *et al* observed a stable torus and thoroughly investigated the bifurcation

*Project supported by the National Natural Science Foundation of China (Grant Nos 60774088 and 60574036), the Specialized Research Fund for the Doctoral Program of China (Grant No 20050055013), the Program for New Century Excellent Talents in University of China (NCET) and the Science and Technology Research Key Project of Education Ministry of China (Grant No 107024).

†E-mail: aywwj@mail.nankai.edu.cn

‡E-mail: chenzq@nankai.edu.cn

phenomena of various limit cycles and tori in a simpler three-dimensional autonomous circuit.[15] More recently, there has been a great deal of interest in chaos or hyperchaos generation,[16−20] control[21−24] and synchronization[25−28] in autonomous systems. In some of these systems, researchers have reported some two-frequency quasiperiodic behaviours (the first and second largest Lyapunov exponents are null), but these attractors do not have the standard torus structure.

In this paper, we will show a four-dimensional autonomous system, which is also continuous and dissipative, in order to utilize the Lyapunov exponents to verify the existence of a two-torus. In this system, we observe two types of two-torus. As we all know, the three types of two-torus, known as standard tori, are as follows: ring torus, horn torus and spindle torus. The two types of tori studied in this paper are ring torus and horn torus. We verify these results by taking advantage of phase portraits along with the proper Poincaré sections, and through observing the phase projection on the Poincaré section we decide whether the two-torus is a ring torus or a horn torus.

This paper is organized as follows: In Section 2, we present the model of this four-dimensional autonomous continuous system and show that it is dissipative. Detailed simulations and analysis of two-torus in this system when the parameter is varied in the region we are interested, including Lyapunov exponents spectrum, phase portraits and Poincaré sections, are presented in Section 3. We verify that the two-torus in this system has two types of standard torus structure, i.e. ring torus and horn torus. Finally, conclusion is given in Section 4.

2. The model

The system studied in this paper is a four-dimensional autonomous continuous dynamical system. The model description is given as follows:

$$\begin{aligned} \dot{x} &= 56(y-x) + 30yz - 8w, \\ \dot{y} &= 49x - 9y - xz, \\ \dot{z} &= xy - 16z, \\ \dot{w} &= ay + 40yz, \end{aligned} \tag{1}$$

where $(x, y, z, w) \in R^4$, a is a real constant parameter determining the dynamical behaviours of system (1).

To further study the system, it is better to consider the dissipation of system (1) first. We have

$$\begin{aligned} \nabla V =& \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} \\ =& -(56 + 16 + 9) = -81 < 0. \end{aligned} \tag{2}$$

Obviously, system (1) is dissipative, therefore we have the equation $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{e}^{-81}$, and for initial volume V_0, we have $V(t) = V_0\mathrm{e}^{-81t}$. Thus, the system converges to a measure zero set exponentially, and there exists an attractor in system (1). In this paper, we mainly analyse the quasiperiodic attractors in the system by numerical simulations.

3. Simulations

In this section, utilizing Lyapunov exponents spectrum, phase portraits and Poincaré sections, we analyse the different dynamical behaviours when the parameter a varies in the region considered in this work. The simulation results are exciting, for we find two types of standard tori, which have been seldom reported in previous studies.

In Fig.1, the Lyapunov exponents spectrum corresponding to the three largest Lyapunov exponents is presented. The exponents are calculated based upon Wolf's work in 1985. As is well known, Lyapunov exponents are the average divergence or convergence of nearby orbits in a phase space. The sum of the Lyapunov exponents indicates the average divergence rate of the phase space. In system (1), the sum of four Lyapunov exponents almost equals -81 , which is consistent with the theoretical results in formula (2). From Fig.1, we can conclude that system (1) changes its dynamical modes between periodic attractor and two-frequency quasiperiodic attractor when the parameter a varies from -1950 to -1940. A part of Fig.1(a) is also magnified to show the bifurcation point more clearly, as in Fig.1(b).

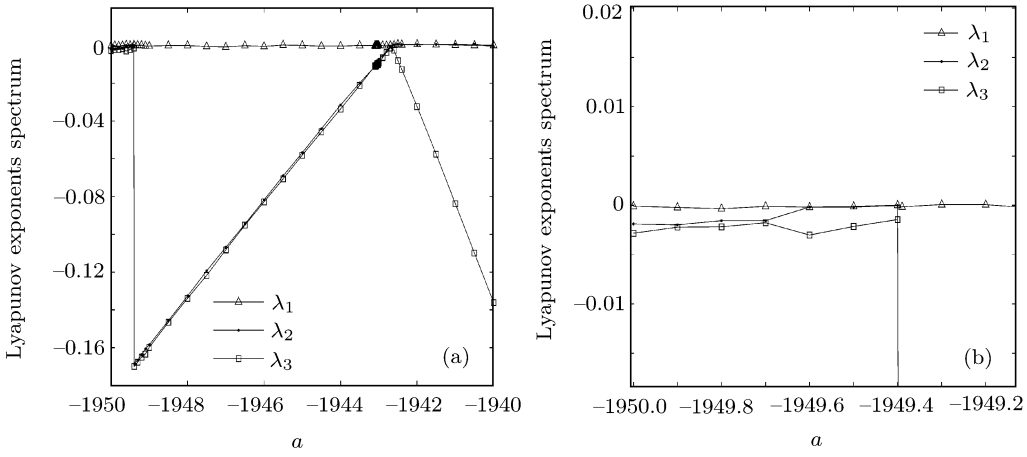


Fig.1. Representation of the three largest Lyapunov exponents of system (1) when a is varied in the range shown in the abscissa.

System (1) is periodic when a is increased from -1950 until $a = -1949.6$, there after the system evolves into a quasiperiodic attractor, which is a type of standard tori illustrated in the following. After passing the short quasiperiodic window of $[-1949.6, -1949.4]$, there is a phase locking and then the system evolves into periodic motion again. When $a = -1942.7$, λ_2 and λ_3 touch zero simultaneously and then only λ_3 decreases, so system (1) is quasiperiodic as a varies between -1942.7 and -1940. However, in the quasiperiodic window $[-1942.7, -1940]$, not all values of parameter a can give birth to a standard torus, and we will discuss this later.

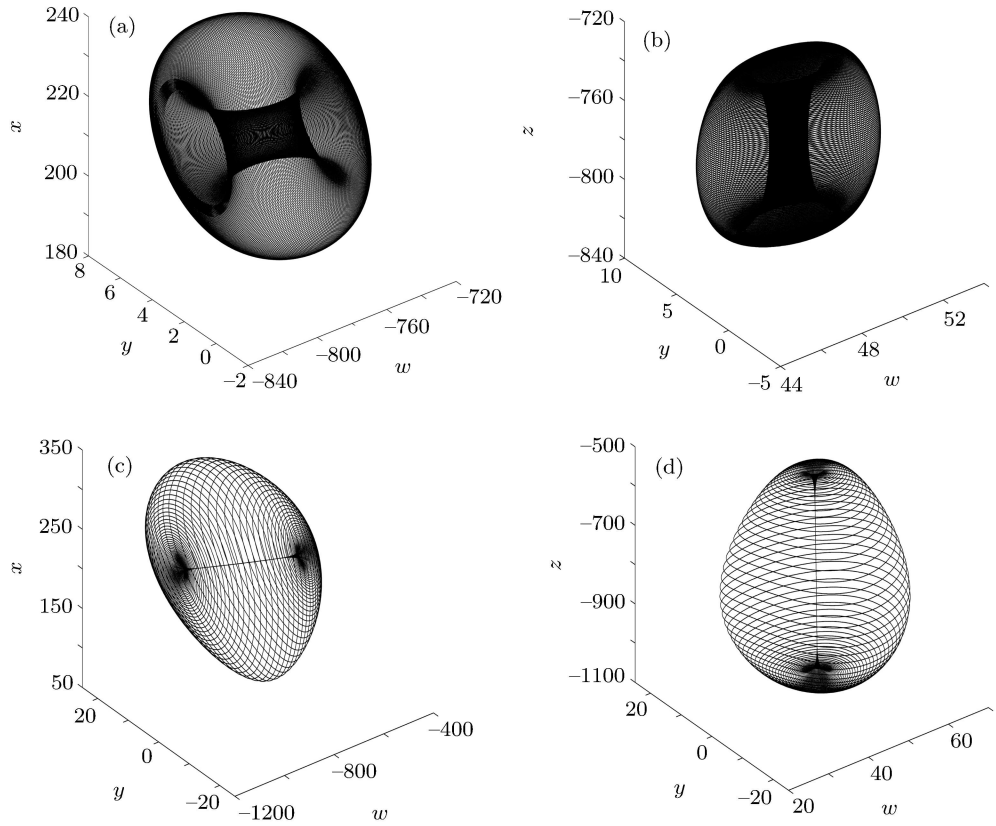


Fig.2. Projection of the phase portrait of the two-torus onto three-dimensional subspace. (a) the x –y –w space when $a = -1949.6$, (b) the y –z –w space when $a = -1949.6$, (c) the x –y –w space when $a = -1942.6$, and (d) the y –z –w space when $a = -1942.6$.

The more global techniques for the analysis of dynamical systems to display the quasiperiodic attractors, i.e. two-torus, are Poincaré sections, which are obtained by intersecting the attractor with a proper hyperplane.[29,30] Poincaré sections can result in the elimination of at least one dimension, so that we can study a lower dimensional system instead of the original higher-dimensional system. The Poincaré sections of a two-torus and a periodic attractor should be a limit cycle and a fixed point, respectively. Because constructing a Poincaré section requires some knowledge of the geometry of the flow generated by system (1), before giving the Poincaré sections, we depict some phase portraits of two-torus in Fig.2.

Obviously, the shape of the quasiperiodic attractor when $a = -1949.6$ is different from that when $a = -1942.6$. Figures 2(a) and 2(b) look like a tyre structurally, but Figs.2(c) and 2(d) appear more like a balloon. We figure from their shape that both the two tori have the standard two-torus structures, and testify them by finding the proper hyperplane to gain the Poincaré sections. Let us review the three types of standard two-torus at first, as depicted in Fig.3.

Ring torus, horn torus and spindle torus are illustrated in Fig.3, where the first column of the pictures shows the full torus, the second the cut-away of the bottom half and the third the cross section by a plane passing through z-axis. For ring torus, the cross-section, namely the Poincaré section, yields two non-intersectant circles. But the Poincaré section of the horn torus shows that the two circles are tangent to each other. Combining the knowledge containing in Fig.3 and the phase portraits depicted in Fig.2, we select an appropriate hyperplane to intersect the flow generated by system (1).

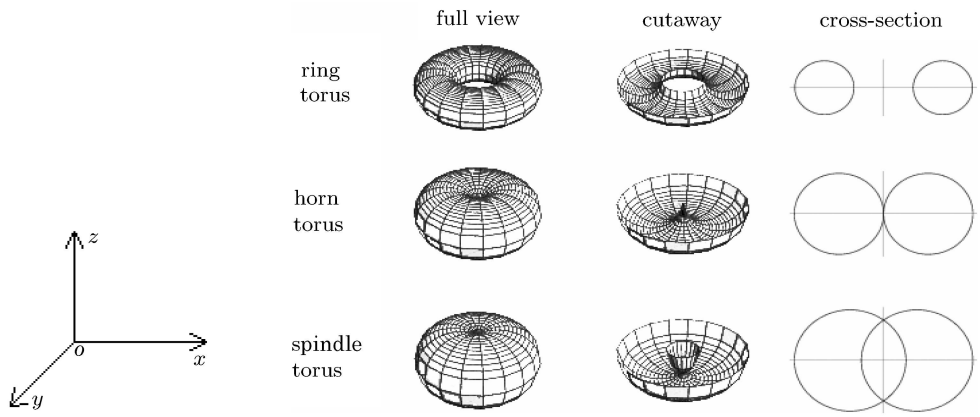


Fig.3. Three standard two-tori.

In Fig.4(a), the Poincaré section on y–w plane is obtained by intersecting the two-torus with the hyperplane $x = 210$ when $a = -1949.6$, and there are two limit circles in this section. As expected, the attractor should be a torus and its dimensionality is reduced. Besides, the Poincaré section is in accordance with that of ring torus, namely, the two circles are not intersectant, therefore, the two-torus for $a = -1949.6$ is verified to be a ring torus. Another hyperplane $y = 3.683$ intersects this two-torus and we obtain a Poincaré section on z–w plane, as shown in Fig.4(b), which also yields two non-intersectant circles. For $a = -1942.6$, by observing the phase portraits we select a hyperplane $z = 48.855$ to intersect the two-torus and the Poincaré section is shown in Fig.4(c). If we select another proper hyperplane, we could have a Poincaré section looking just like that in Fig.4(c), which we will not discuss here. It is clearly seen that there are two limit circles tangent to each other in the plot, which indicate definitely that the attractor is a two-torus and in fact a horn torus.

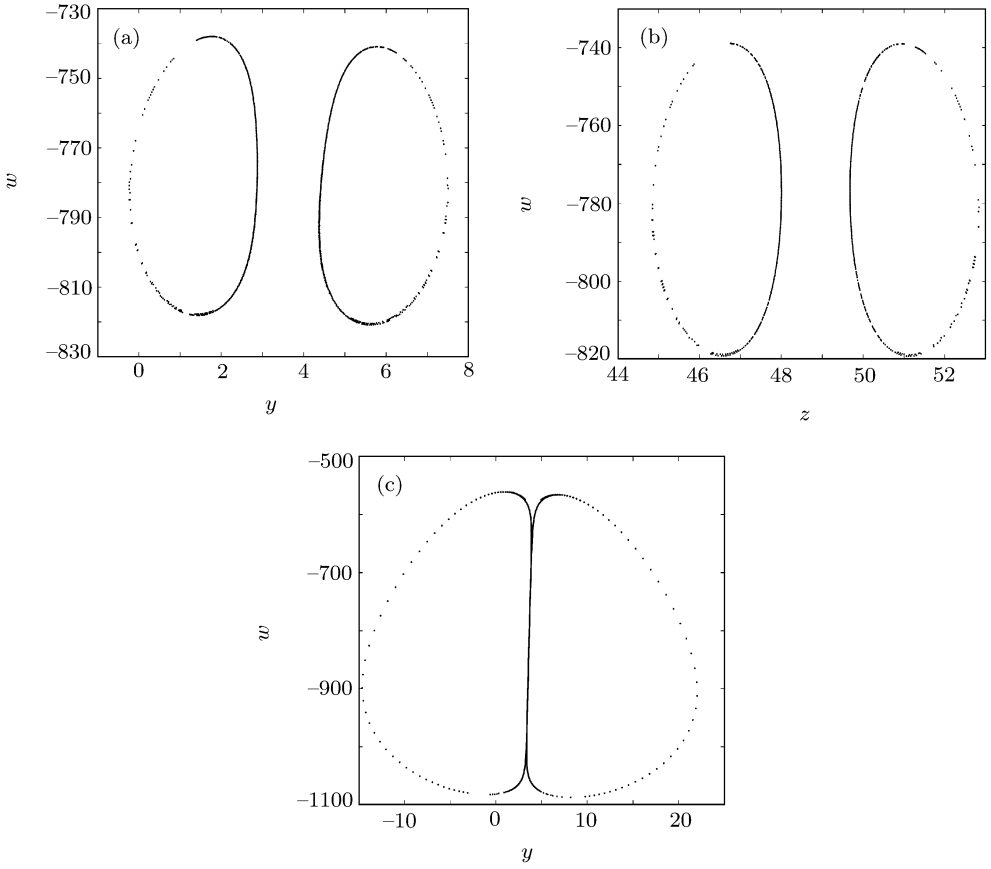


Fig.4. Plot of a Poincaré section of the two-torus for system (1). The system exhibits two-frequency quasiperiodic dynamical behaviour (one-frequency in the Poincaré section). (a) projection on y–w plane when $a = -1949.6$; (b) projection on z–w plane when $a = -1949.6$; (c) projection on y–w plane when $a = -1942.6$.

As a increases, the standard torus will disappear at the point $a = -1940.69$. Although system (1) has two null Lyapunov exponents, it loses the standard torus structure and evolves into a quasiperiodic attractor of a different shape. When $a = -1940$, we depict a phase portrait of system (1) along with a Poincaré section in Fig.5. The Poincaré section does not exhibit two non-intersectant circles or two tangent circles, but shows many little circles. Accordingly, no matter what is the phase portrait or the corresponding Poincaré section, this quasiperiodic attractor is proved not to be a standard torus.

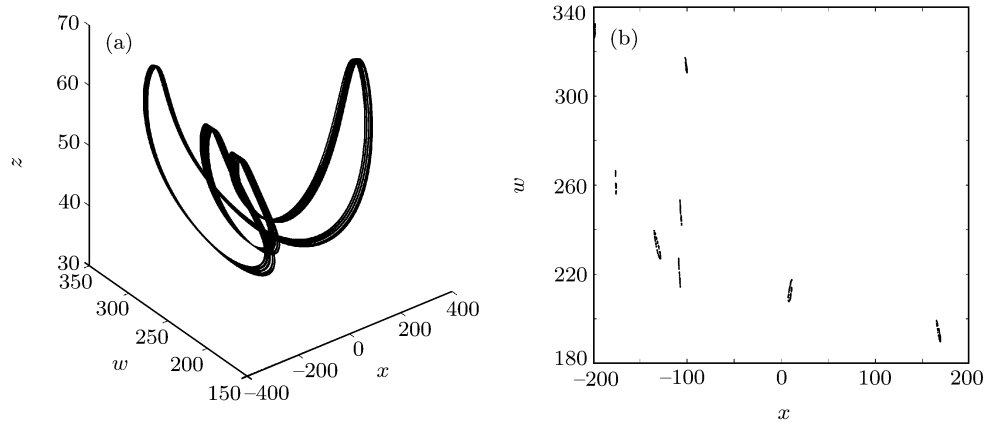


Fig.5. Plot of a phase portrait and the corresponding Poincaré section when $a = -1940$. (a) x–z–w space; (b) Poincaré section.

4. Conclusion

In this paper, we have studied the quasiperiodic behaviours in a four-dimensional autonomous continuous dissipative system, which has just one parameter a. When a is varied in our interesting range, by computing the Lyapunov exponents, we have shown that the system changes its dynamical mode between periodic behaviour and two-frequency quasiperiodic behaviour. More interestingly, in the quasiperiodic windows, there are two kinds of quasiperiodic attractors, one is ring torus, and the other one is horn torus, both of which are known as standard tori. We verify them by taking advantage of Poincaré sections. The two-torus having the standard torus structure reported in this paper has been seldom reported in previous studies, and our phenomenological study may provide a trivial contribution to the understanding and study of the complicated quasiperiodicity in dynamical systems. There are still some other dynamical behaviours in this system, such as new tori and even the chaotic properties, which deserve to be further investigated in detail in the future.

References

[1] Wolf A, Swift J B, Swinney H L and Vastano J A 1985 *Physica* D **16** 285

[2] Cao Y J, Dong C H and Zhou P Q 2006 *Acta Phys. Sin.* **55** 6470 (in Chinese)

[3] Li X S and Gong C D 1988 *Acta Phys. Sin.* **37** 618 (in Chinese)

[4] He W P, Feng G L, Gao X Q and Chou J F 2006 *Acta Phys. Sin.* **55** 3175 (in Chinese)

[5] Wang G R and Hao B L 1984 *Acta Phys. Sin.* **33** 1321 (in Chinese)

[6] Wang P and Dai X G 2005 *Acta Phys. Sin.* **54** 4961 (in Chinese)

[7] Ashwin P 1997 *Chaos, Solitons Fractals* **9** 1279

[8] Green K, Krauskopf B and Engelborghs K 2002 *Physica* D **173** 114

[9] Chakravarthy S K and Nayar C V 1996 *Elect. Power Energy Syst.* **18** 483

[10] Ding W C and Xie J H 2006 *Phys. Lett.* A **349** 324

[11] Grebogi, Ott E and Yorke J A 1985 *Physica* D **15** 354

[12] Seimenis J 1995 *Rep. Math. Phys.* **36** 433

[13] Zheng Z G 2001 *Chin. Phys.* **10** 703

[14] Matsumoto T, Chua L O and Tokunaga R 1987 *IEEE Trans. Circ. Syst.* **CAS-34** 240s

[15] Nishiuchi Y, Ueta T and Kawakami H 2006 *Chaos, Solitons Fractals* **27** 941

[16] Wang J Z, Chen Z Q and Yuan Z Z 2006 *Chin. Phys.* **15** 1216

[17] Wang F Q and Liu C X 2006 *Chin. Phys.* **15** 963

[18] Chen Z Q, Yang Y, Qi G Y and Yuan Z Z 2007 *Phys. Lett.* A **360** 696

[19] Zhang Y H, Qi G Y, Liu W L and Yan Y 2006 *Acta Phys. Sin.* **55** 3307 (in Chinese)

[20] Qi G Y, Du S Z, Chen G R, Chen Z Q and Yuan Z Z 2005 *Chaos, Solitons and Fractals* **23** 1671

[21] Guan X P, Fan Z P, Peng H P and Wang Y Q 2001 *Acta Phys. Sin.* **50** 2108 (in Chinese)

[22] Song Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266

[23] Wu C L, Ma S J, Sun Z K and Fang T 2006 *Acta Phys. Sin.* **55** 6253 (in Chinese)

[24] Sima W X, Liu F, Sun C X, Liao R J and Yang Q 2006 *Acta Phys. Sin.* **55** 5714 (in Chinese)

[25] Hua C C and Guan X P 2004 *Chin. Phys.* **13** 1391

[26] Guan X P, He Y H and Wu J 2003 *Acta Phys. Sin.* **52** 2718 (in Chinese)

[27] Wang M S, Hou Z H and Xin H W 2006 *Chin. Phys.* **15** 2553

[28] Wang Q Y, Lu Q S and Wang H X 2005 *Chin. Phys.* **14** 2189

[29] Sprott J C 2003 *Chaos and Time-Series Analysis* (New York: Oxford)

[30] Wiggins S 1990 *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (New York: Springer)