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More relaxed condition for dynamics of discrete time delayed Hopfield neural networks*

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The dynamics of discrete time delayed Hopfield neural networks is investigated. By using a difference inequality combining with the linear matrix inequality, a sufficient condition ensuring global exponential stability of the unique equilibrium point of the networks is found. The result obtained holds not only for constant delay but also for time-varying delays.

Keywords: discrete time, delayed Hopfield neural networks, difference inequality
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1. Introduction

In recent years, the dynamics of continuous-time delayed Hopfield neural networks has been extensively studied due to its important role in the applications such as pattern recognition, associative memories, optimization problems and image processing. The readers are referred to Refs.[1–16] and references cited therein for more details. However, when simulating the continuous-time Hopfield neural networks on digital computers, it is essential to formulate a discrete-time analogue of the continuous-time neural network. Recently, the dynamics of discrete-time delayed Hopfield neural networks has received much attention, see, for example, Refs.[17–19]. In this paper, global exponential stability of discrete-time Hopfield neural network with time-varying delays is considered and a sufficient condition is presented. The condition removes some restriction on the delays. For this reason, the result in this paper is less conservative than those provided in the earlier literature.

2. Stability Analysis

The dynamic behaviour of discrete-time Hopfield neural networks with time-varying delays is described as follows:[18,19]

$$x_i(n+1) = a_i x_i(n) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n-\kappa(n))) + J_i, \quad (1)$$

for $i \in \{1, 2, \ldots, m\}, n \in \{0, 1, 2, \ldots\}$, where m is the number of units in a neural network; $\boldsymbol{x}(n) = [x_1(n), \ldots, x_m(n)]^{\mathrm{T}} \in R^m$ are the state vectors; $f(x(n)) = [f_1(x_1(n)), \ldots, f_m(x_m(n))]^{\mathrm{T}} \in R^m$ are the activation function of the neurons; $f(x(n-\kappa(n))) = [f_1(x_1(n-\kappa(n))), \ldots, f_m(x_m(n-\kappa(n)))]^{\mathrm{T}} \in R^m$; $A = \mathrm{diag}(a_i)$ $(a_i \in (0,1))$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $B = \{b_{ij}\}$ is the delayed feedback matrix, while $\boldsymbol{J} = [J_1, \ldots, J_n]^{\mathrm{T}}$ is an external bias vector, $\kappa(n)$ is the transmission delay along the axon of the jth unit and satisfies $0 \le \kappa(n) \le \kappa$ (κ is a constant).

The initial conditions associated with system (1) are of the form

$$x_i(l) = \varphi_i(l), \ i \in \{1, 2, \ldots, m\}, \quad (2)$$

where l is an integer with $l \in [-\kappa, 0]$.

Definition 1 A vector $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^{\mathrm{T}}$ is said to be an equilibrium point of system (1), if the following equation holds

$$x_i^* = a_i x_i^* + \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + J_i. \quad (3)$$

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In this paper, we will assume that the activation functions $f_i, i = 1, 2, \ldots, m$ is bounded and satisfies the following condition

$$|f_i(\xi_1) - f_i(\xi_2)| \leq L_i|\xi_1 - \xi_2| \ , \forall \xi_1, \xi_2 \in R. \quad (4)$$

This type of activation functions is clearly more general than both the usual sigmoid activation functions and the piecewise linear function (PWL): $f_i(x) = \frac{1}{2}(|x+1| - |x-1|)$ which is used in Ref.[20].

Under the assumption of the activation functions, we can easily know that there must exist an equilibrium point of system (1). Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ be an equilibrium point of Eq.(1), one can derive from (1) that the transformation $y_i(n) = x_i(n) - x_i^*$ transforms system (1) into the following system:

$$y_i(n+1) = a_i y_i(n) + \sum_{j=1}^{m} b_{ij} g_j(y_j(n - \kappa(n))), \quad (5)$$

or equivalently,

$$y(n+1) = Ay(n) + BG(y(n - \kappa(n))), \quad (6)$$

where

$$y(n) = (y_1(n), y_2(n), \ldots, y_m(n))^{\mathrm{T}},$$
$$A = \mathrm{diag}(a_1, a_2, \ldots, a_m), \ B = (b_{ij})_{m\times m},$$
$$G(y(n)) = (g_1(y_1(n)), g_2(y_2(n)), \ldots, g_m(y_m(n)))^{\mathrm{T}}$$

and $g_j(y_j(n)) = f_j(y_j(n) + x_j^*) - f_j(x_j^*)$.

Since each function $f_j(\cdot)$ satisfies the hypothesis (4), hence, each $g_j(\cdot)$ satisfies

$$|g_j(\xi_j)| \leq L_j|\xi_j|. \quad (7)$$

To prove the stability of x^* of the Eq.(1), it is sufficient to prove the stability of the trivial solution of Eq.(5).

Definition 2 The trivial solution of Eq.(5) is said to be globally exponentially stable if for any solution $x_i(n, \varphi_i)$ with the initial condition $x_i(l) = \varphi_i(l)$ for $l \in [-\kappa, 0]$, there exist constants $\varepsilon \in (0, 1)$ and $M \geq 1$ such that

$$||y(n)|| \leq M||\phi||\varepsilon^n, \ n \geq 0, \quad (8)$$

where

$$||y(n)|| = \left[\sum_{i=1}^{m} y_i^2(n)\right]^{1/2},$$
$$||\phi|| = \max_{l\in[-\kappa,0]} \left\{ \left[\sum_{i=1}^{m} \varphi_i^2(l)\right]^{1/2} \right\}.$$

Lemma 1[21] Let $r > 0$ be a natural number, and let $\{z(n)\}_{n\geq -r}$ be a sequence of real numbers satisfying the inequality (9)

$$\Delta z(n) \leq -pz(n) + q\max\{z(n), z(n-1), \ldots, z(n-r)\}, \quad n \geq 0. \quad (9)$$

If $0 < q < p \leq 1$, then there exists a constant $\lambda_0 \in (0, 1)$ such that

$$z(n) \leq \max\{0, z(0), z(-1) \ldots, z(-r)\}\lambda_0^n, \quad n \geq 0. \quad (10)$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0, 1)$ of equation

$$\lambda^{r+1} + (p-1)\lambda^r - q = 0. \quad (11)$$

Now we will present a sufficient condition for ensuring global exponential stability of Eq.(1).

Theorem 1 The equilibrium point of Eq.(1) is globally exponentially stable if there exists a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_m) > 0$ such that

$$a_{\max}^2 + d_{\max}L_{\max}^2\lambda_{\max}\left(D^{-1}B^{\mathrm{T}}DBD^{-1}\right) < 0.5,$$

where $a_{\max} = \max\{a_i\}, d_{\max} = \max\{d_i\}, L_{\max} = \max\{L_i\}$ and $\lambda_{\max}(A)$ denotes the largest eigenvalue of the matrix A.

Proof Consider the following function:

$$z(n) = y^{\mathrm{T}}(n)Dy(n), \quad (12)$$

then, we obtain

$$\begin{aligned} & z(n+1) \\ =& y^{\mathrm{T}}(n+1)Dy(n+1) \\ =& [Ay(n) + BG(y(n-\kappa(n)))]^{\mathrm{T}} \times D \\ & \times [Ay(n) + BG(y(n-\kappa(n)))] \\ =& \left[y^{\mathrm{T}}(n)A^TD + G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}D\right] \\ & \times [Ay(n) + BG(y(n-\kappa(n)))] \\ =& y^{\mathrm{T}}(n)A^{\mathrm{T}}DAy(n) + 2y^{\mathrm{T}}(n)A^{\mathrm{T}}DBG(y(n-\kappa(n))) \\ & + G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}DBG(y(n-\kappa(n))). \end{aligned}$$

Calculating the difference $\Delta z(n) = z(n+1) - z(n)$ along (5), we obtain

$$\begin{aligned} & \Delta z(n) \\ =& z(n+1) - z(n) \\ =& y^{\mathrm{T}}(n)A^{\mathrm{T}}DAy(n) + 2y^{\mathrm{T}}(n)A^{\mathrm{T}}DBG(y(n-\kappa(n))) \\ & + G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}DBG(y(n-\kappa(n))) \\ & - y^{\mathrm{T}}(n)Dy(n) \\ =& y^{\mathrm{T}}(n)\left(A^TDA - D\right)y(n) \\ & + 2y^{\mathrm{T}}(n)A^{\mathrm{T}}DBG(y(n-\kappa(n))) \\ & + G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}DBG(y(n-\kappa(n))). \end{aligned} \quad (13)$$

Recall that the inequality $2a^{\mathrm{T}}b \le a^{\mathrm{T}}Qa + b^{\mathrm{T}}Q^{-1}b$ holds for any $a, b \in R^m$ and any positive definite matrix Q. By using this inequality, we have

$$\begin{aligned} 2y^{\mathrm{T}}(n)A^{\mathrm{T}}DBG(y(n-\kappa(n))) &= 2\left(DAy(n)\right)^{\mathrm{T}} BG(y(n-\kappa(n))) \\ &\le y^{\mathrm{T}}(n)A^{\mathrm{T}}DAy(n) + G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}DBG(y(n-\kappa(n))). \end{aligned} \tag{14}$$

Substituting (14) into (13) yields

$$\begin{aligned} \Delta z(n) &\le y^{\mathrm{T}}(n)\left(2A^{\mathrm{T}}DA - D\right)y(n) + 2G^{\mathrm{T}}(y(n-\kappa(n)))B^{\mathrm{T}}DBG(y(n-\kappa(n))) \\ &\le \left(2a_{\max}^2 - 1\right)y^{\mathrm{T}}(n)Dy(n) + 2\lambda_{\max}\left(D^{-1}B^{\mathrm{T}}DBD^{-1}\right)G^{\mathrm{T}}(y(n-\kappa(n)))D^2G(y(n-\kappa(n))) \\ &\le \left(2a_{\max}^2 - 1\right)y^{\mathrm{T}}(n)Dy(n) + 2\lambda_{\max}\left(D^{-1}B^{\mathrm{T}}DBD^{-1}\right)d_{\max}L_{\max}^2 y^{\mathrm{T}}(n-\kappa(n))Dy(n-\kappa(n)) \\ &= -pz(n) + qz(n-\kappa(n)) \\ &\le -pz(n) + q\max\left\{z(n), z(n-1), \ldots, z(n-\kappa)\right\}, \end{aligned}$$

where

$$\begin{aligned} p &= 1 - 2a_{\max}^2, \\ q &= 2d_{\max}L_{\max}^2\lambda_{\max}\left(D^{-1}B^{\mathrm{T}}DBD^{-1}\right). \end{aligned} \tag{15}$$

By Lemma 1 above we know if

$$0 < q < p \le 1, \tag{16}$$

then there exists a constant $\lambda_0 \in (0,1)$ such that

$$\begin{aligned} z(n) = y^{\mathrm{T}}(n)Dy(n) &\le \max\left\{0, z(0), z(-1)\ldots, z(-\kappa)\right\}\lambda_0^n \\ &= \max\left\{0, \sum_{i=1}^m d_i y_i^2(0)\ \sum_{i=1}^m d_i y_i^2(-1)\ldots, \sum_{i=1}^m d_i y_i^2(-\kappa)\right\}\lambda_0^n \\ &\le \max\left\{0, d_{\max}\sum_{i=1}^m y_i^2(0)\ d_{\max}\sum_{i=1}^m y_i^2(-1)\ldots, d_{\max}\sum_{i=1}^m y_i^2(-\kappa)\right\}\lambda_0^n \le d_{\max}||\phi||^2\lambda_0^n. \end{aligned} \tag{17}$$

On the other hand, we obtain

$$z(n) \ge d_{\min}\sum_{i=1}^m y_i^2(n). \tag{18}$$

It follows from (16) and (17) that

$$\sum_{i=1}^m y_i^2(n) \le \frac{d_{\max}}{d_{\min}}||\phi||^2\lambda_0^n,$$

that is

$$||y(n)|| \le \left(\frac{d_{\max}}{d_{\min}}\right)^{1/2}||\phi||\left(\lambda_0^{1/2}\right)^n.$$

This completes the proof.

Corollary 1 The equilibrium point of Eq.(1) is globally exponentially stable if

$$a_{\max}^2 + L_{\max}^2\lambda_{\max}\left(B^{\mathrm{T}}B\right) < 0.5.$$

Proof By taking $d_i = 1$ $(i = 1, 2, \ldots, m)$ in Theorem 1, it can be easily obtained.

Remark 1 In Ref.[18], the authors discussed the global exponential stability of discrete-time cellular neural networks with constant delays and presented a sufficient condition. However, the result in Ref.[18] does not apply to the time-varying delays.

Remark 2 In Ref.[19], under the assumption that the delay satisfies $1 < \kappa(n+1) < 1 + \kappa(n)$, the authors obtained some global exponential stability criteria for system (1). In fact, this restriction on the delay function is rigorous. In this paper, our Theorem 1 drops off this restriction and only requires the delays to be bounded. Hence, our result is less restrictive than that given in Ref.[19].

3. An Example

In this section, we will give an example to show the condition given in this paper is milder than that given in some earlier literature.

Example 1 Consider the following discrete-time delayed Hopfield neural networks with two neurons

$$\begin{aligned} x_1(n+1) &= 0.4x_1(n) + 0.3f(x_1(n-\kappa(n))) \\ &\quad - 0.1f(x_2(n-\kappa(n))) - 2, \\ x_2(n+1) &= 0.3x_2(n) + 0.2f(x_1(n-\kappa(n))) \\ &\quad + 0.5f(x_2(n-\kappa(n))) + 1, \end{aligned} \tag{19}$$

i.e.,

$$C = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 0.3 & -0.1 \\ 0.2 & 0.5 \end{bmatrix}, \tag{20}$$

and the nonlinear input–output function is chosen as $f(x) = \frac{1}{2}(|x-1| - |x+1|)$. It can be verified that this function satisfies the hypothesis (4) with $L_1 = L_2 = 1$, but does not satisfy the following hypothesis used in Ref.[17]:

$$\begin{aligned} 0 &\le (f_i(\xi_1) - f_i(\xi_2))(\xi_1 - \xi_2) \\ &\le L_i(\xi_1 - \xi_2)^2, \forall \xi_1, \xi_2 \in R. \end{aligned}$$

Hence, the results in Ref.[17] cannot be employed in this example. The unique equilibrium point of system (18) can be obtained as

$$x^* = (x_1^*, x_2^*)^{\mathrm{T}} = (-2.6667, 1.0000)^{\mathrm{T}}. \tag{21}$$

Furthermore, one can easily check that

$$\begin{aligned} a_{\max}^2 + L_{\max}^2 \lambda_{\max}\left(B^{\mathrm{T}}B\right) &= 0.16 + 0.2905 \\ &= 0.4505 < 0.5. \end{aligned}$$

Therefore, from our Theorem 1, it follows that the equilibrium point of system (18) is globally exponentially stable.

In the numerical simulation, let $\kappa(n) = [2.1|\cos(2n)|]$ in (18), where $[r]$ denotes the integer part of the real number r. It is obvious that $0 \le \kappa(n) \le 2$ and does not satisfy the condition $1 < \kappa(n+1) < 1 + \kappa(n)$ which is needed by Ref.[19]. Therefore, the results in Ref.[19] cannot be applied to this example.

The initial state $(\varphi_1(l), \varphi_2(l))^{\mathrm{T}} = (-5, 2)^{\mathrm{T}}$ for $l \in [-1, 0]$. The numerical simulation is illustrated in Fig.1.

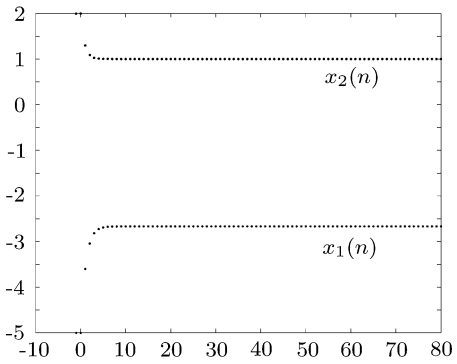


Fig.1. The solution trajectories of Eq.(19).

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