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New bifurcations of basin boundaries involving Wada and a smooth Wada basin boundary*

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This paper demonstrates and analyses double heteroclinic tangency in a three-well potential model, which can produce three new types of bifurcations of basin boundaries including from smooth to Wada basin boundaries, from fractal to Wada basin boundaries in which no changes of accessible periodic orbits happen, and from Wada to Wada basin boundaries. In a model of mechanical oscillator, it shows that a Wada basin boundary can be smooth.

Keywords: Wada basin boundary, heteroclinic tangency, smooth Wada basin boundary
PACC: 0547

1. Introduction

For a nonlinear dynamical system, it can possess multiple coexisting attractors. The set of initial points which eventually approach each particular attractor is called its basin of attraction. The basin boundaries can be smooth, fractal,[1] Wada[2−4] or even riddled.[5−7] An understanding of basin boundaries is of great physical significance. For example, the fractal property of basin boundaries can make us difficult to predict the behaviour of dynamical systems.[1,4,8] Besides, the coexisting attractors have much influences on the dynamics of systems, therefore they should be taken into account when analysing dynamical behaviours of systems.[9,10]

We focus on Wada basin boundaries in this paper. If three or more attractors coexist in the phase space, a basin boundary can have Wada property. Namely every basin boundary point is on the boundary of at least two other basins.[2−4] A rigorous criterion guaranteeing this phenomenon was presented.[3] It states that if the unstable manifold of an accessible periodic orbit (its definition will be given later or see Ref.[3]) on the basin boundary intersects at least three basins, then the basin boundary has Wada property. We also use this criterion to guarantee the appearance of a Wada basin boundary in this paper. The existence of Wada basin boundaries has been reported for both Hamiltonian and dissipative systems.[11−14] It has been found that saddle-node bifurcation[15] and fixed point tangent bifurcation[16] can give rise to Wada basin boundaries.

The bifurcations of basin boundaries called metamorphoses in Ref.[17], can be successfully characterized by change of accessible periodic orbits on the basin boundaries when only two attractors are involved.[18] An accessible point is the first boundary point that was hit by a curve starting from the interior of the basin.[18] There can be infinite accessible points, but they usually converge to one accessible periodic orbit. A numerical procedure to find the accessible periodic orbits was presented in Ref.[19].

The geometric property of a Wada basin boundary is also interesting.[3,20] In some cases, the structure of a Wada basin boundary is determined by the basin cell formed by the unstable and stable manifolds of an accessible periodic orbit.[20] All the previous examples of Wada basin boundaries are fractal. Usually there will be a chaotic saddle embedded on a Wada basin boundary.[3,13,21,22] So people think a Wada basin boundary must be a fractal basin boundary, but the converse is false.[14,16]

In this paper, we first study bifurcations of basin boundaries involving Wada in a three-well potential model which describes the dynamics of an elastic ferromagnetic beam.[23] We demonstrate that double heteroclinic tangency can happen in this system, rather than using Melnikov method.[23] The double hetero-

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clinic tangency induces three new types of bifurcations of basin boundaries, from smooth to Wada, from fractal to Wada, and from Wada to Wada. No changes of accessible periodic orbits happen in the bifurcation from fractal to Wada.

Then we show that a Wada basin boundary can be smooth in a mechanical oscillator, which was intensively investigated by Thompson and co-workers (See Ref.[24] and references therein). So the fractal property is not a necessary condition for a Wada basin boundary.

The stable and unstable manifolds of accessible periodic orbit are obtained by the method presented in Ref.[25]. We use method discussed in Ref.[26] for computing eigenvectors of saddle point in the Poincaré map.

2. Double heteroclinic (homoclinic) tangency in the three-well potential model

We investigate a three-well potential model that can be modelled by equations of the form[23]

$$\dot{x} = y,$$
$$\dot{y} = -\delta y - x(x^2 - x_0^2)(x^2 - 1) + r\cos(\omega t), \qquad (1)$$

where δ is the damping ratio, ω and r are equivalent to driving frequency and amplitude. If $r = 0$, the system has three sinks, one at the origin and two at $(\pm 1, 0)$, and two saddles at $(\pm x_0, 0)$. Figure 1 shows the stable manifolds and unstable manifolds of two saddle points when $\delta = 0.27$, $x_0 = 0.5$ and $r = 0.0$.

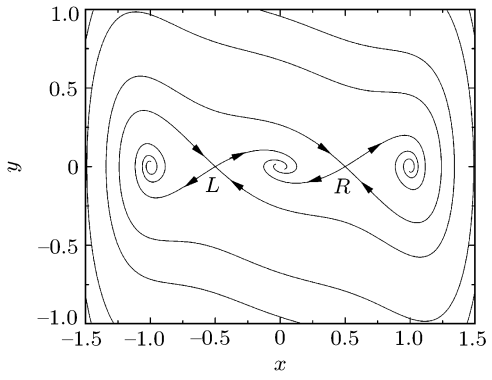


Fig.1. Stable manifold and unstable manifold of system (1) with $\delta = 0.27$, $x_0 = 0.5$ and $r = 0.0$, which are symmetrical about origin.

Poincaré map is useful in periodically driven systems. When $r \neq 0$ and is relatively small, there will be still two saddle points in the Poincaré map of system (1). Double heteroclinic (homoclinic) tangency can happen in this system. Namely two heteroclinic (homoclinic) tangencies happen at the same bifurcation value. With assumptions such that both δ and r are small, double heteroclinic (homoclinic) tangency is analysed by Melnikov method and approximate bifurcation values are obtained.[23]

In fact, the double heteroclinic (homoclinic) tangency is seriously guaranteed in Poincaré maps for system (1) due to symmetry. In the presence of r, the phase portraits of a Poincaré map for system (1) are not symmetric about origin any more (take Fig.4 in this paper for example). However, if we consider two Poincaré maps constructed by two different phases φ_1, φ_2 with $|\varphi_1 - \varphi_2| = \pi$, and note that the right hand sides of system (1) y and $-\delta y - x(x^2 - x_0^2)(x^2 - 1)$ are odd functions of x and y, then the phase portraits of these two Poincaré maps will be symmetric about origin because $\cos(\omega t + \varphi_1) = -\cos(\omega t + \varphi_2)$. So the stable (unstable) manifolds in these two Poincaré maps are symmetric about origin too. The stable (unstable) manifolds of left saddle for one Poincaré map are symmetric about origin to those of another Poincaré map. These symmetries can guarantee the occurrence of double heteroclinic (homoclinic) tangency for a Poincaré map.

3. From smooth to Wada basin boundaries through double heteroclinic bifurcation

From this section to Section 4, bifurcations of basin boundaries induced by double heteroclinic tangency are investigated. We fix parameter values $\delta = 0.27$ and $x_0 = 0.5$, while regard r and ω as two variable parameters.

We choose $\omega = 0.5$. In this case, we will encounter double heteroclinic tangency first when r is increased from a small value, for example 0.01. So before bifurcation, all the basin boundaries are smooth. The critical value $r_{\rm cr}$ for double heteroclinic tangency is about 0.067621. Its calculation will be discussed later in this section.

Before bifurcation, we choose $r = 0.06762$. The Poincaré map with phase $\pi/2$ has two saddle points denoted by $L = (-0.4675, 0.0551)$ and $R = (0.5157, 0.0517)$ respectively. There are three attrac-

tors, denoted by $A_{\rm L}$=(–0.9875, –0.0493) (leftmost attractor), $A_{\rm M}$ =(0.2875, 0.1160) (middle attractor) and $A_{\rm R}$=(1.0175, –0.0253) (rightmost attractor) respectively. Figure 2 shows the attractors, saddles, and their stable (unstable) manifolds involved in the tangency.

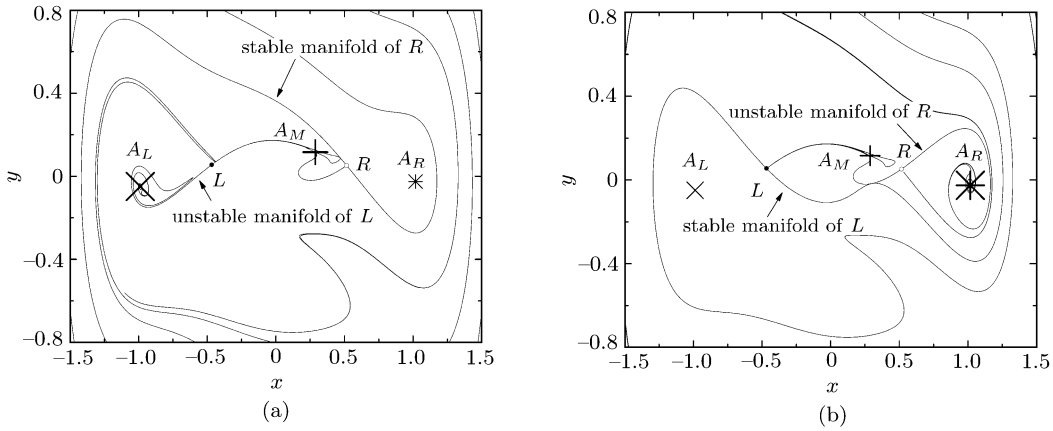


Fig.2. Double heteroclinic tangency of system (1) with $\delta = 0.27$, $x_0 = 0.5$, $\omega = 0.5$ and $r = 0.06762$. Two saddles points are denoted by L and R respectively. Three period-1 attractors are denoted by A_L, A_M and A_R respectively. (a) The tangency of stable manifold of R and the unstable manifold of L. (b) The tangency of stable manifold of L and the unstable manifold of R.

Now we explain how to locate the critical value for the double heteroclinic tangency. Before double heteroclinic tangency, the saddle point L is both accessible from basins of leftmost attractor A_L and middle attractor A_M. After double heteroclinic bifurcation, stable manifold of R will infinitely intersect upper right branch of unstable manifold of L. These heteroclinic manifold crossings will make L inaccessible from basin of middle attractor. There will be a new periodic orbit accessible from basin of middle attractor. This change of the periodic orbit accessible from basin of middle attractor is used to predict the critical value of double heteroclinic tangency. For this model, a new period-4 saddle orbit accessible from basin of middle attractor is created after double heteroclinic tangency.

After bifurcation, we fix $r = 0.06763$. Note that there are no sudden changes of saddle point accessible from basin of leftmost attractor (basin of rightmost attractor). We still use L and R to indicate the two saddle points accessible from basins of leftmost and rightmost attractors respectively. We located L =(–0.4675, 0.0551) and R =(0.5157, 0.0518). We can see that the unstable manifold of L intersects all the three basins in Fig.3(a). So the boundary of basin of leftmost attractor is a Wada basin boundary. In Fig.3(b), the unstable manifold of R intersects all the three basins too. This means that the basin boundary of rightmost attractor is also a Wada basin boundary. For there are only three basins, if two basin boundaries are Wada basin boundaries, the third basin boundary is certainly a Wada basin boundary. Thus after double heteroclinic tangency, all the basin boundaries are Wada basin boundaries.

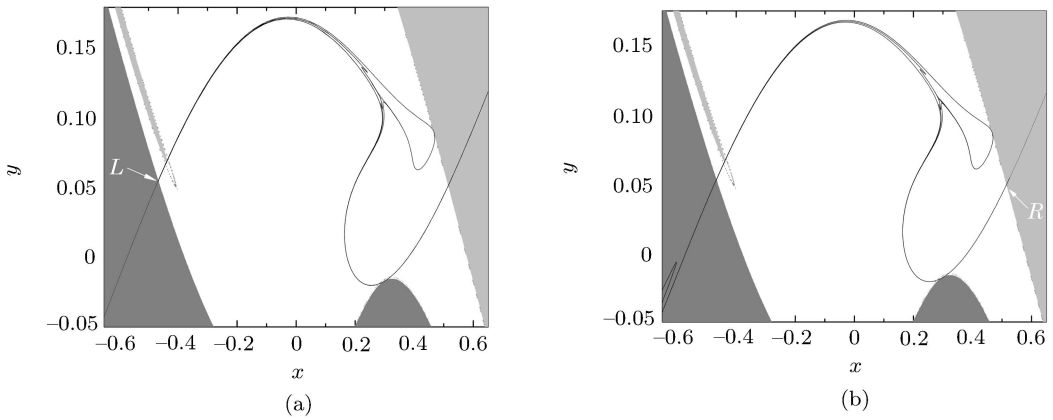


Fig.3. Wada basin boundaries arise just after double heteroclinic tangency ($r = 0.06763$). (a) The unstable manifold of saddle L accessible from basin of leftmost attractor intersects all the three basins. (b) The unstable manifold of saddle R accessible from basin of rightmost attractor intersects all the three basins.

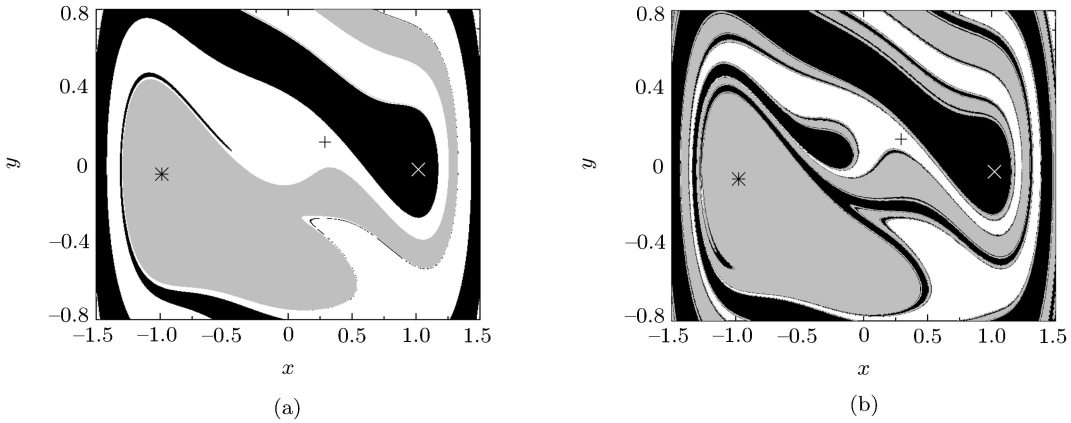


Fig.4. Basins of system (1) are shown before and after bifurcation. The black region is the basin of rightmost attractor denoted by ×. The grey region is the basin of leftmost attractor denoted by ∗. The white region is the basin of middle attractor denoted by +. (a) All the basin boundaries are smooth before bifurcation with $r = 0.06762$. (b) Wada basin boundaries after bifurcation with $r = 0.08$.

Figures 4(a) and 4(b) show three basins of system (1) before and after double heteroclinic tangency respectively.

4. From fractal (not Wada) to Wada basin boundaries and from Wada to Wada basin boundaries through double heteroclinic bifurcation

As shown in Section 3, double heteroclinic tangency can turn smooth basin boundaries into Wada basin boundaries. In this section, we investigate the influences of double heteroclinic tangency to the fractal (not Wada) basin boundaries and Wada basin boundaries.

We fix $\omega = 0.66$, $\delta = 0.27$, and $x_0 = 0.5$. In this case, we will encounter double homoclinic tangency first when r is increased from a small value, for example 0.01. The homoclinic tangency happens at $r \approx 0.075$. So before double heteroclinic tangency, the basin boundaries are fractal. The structure of the fractal basin boundaries before double heteroclinic tangency is shown in Fig.5. The three attractors of the Poincaré map are still distinguished by leftmost attractor, middle attractor and rightmost attractor respectively. There are four accessible saddle points on the basin boundaries. L denotes the saddle point accessible from basin of leftmost attractor. R denotes the saddle point accessible from basin of rightmost attractor. M_1 and M_2 denote two saddle points accessible from basin of middle attractor.

For L and M_1 are on the same fractal basin boundary and each of the stable manifolds of L and M_1 is dense in this boundary,[2] so stable manifold of L and the stable manifold of M_1 can be arbitrary close to each other. Therefore we only need to consider one of them. Similarly, the stable manifold of R and M_2 can also be arbitrary close to each other. Two different double heteroclinic tangencies which give rise to two different metamorphoses are discussed in the following two subsections.

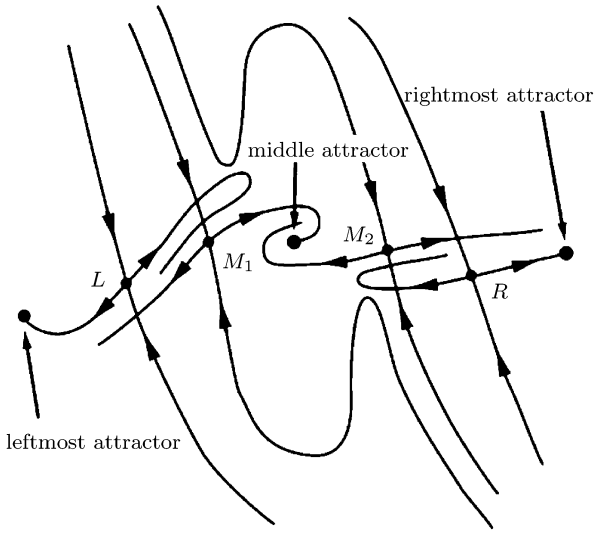


Fig.5. Schematic diagram of four accessible saddle points, L, M_1, M_2 and R, on the fractal basin boundaries of system (1) and their stable and unstable manifolds.

4.1. From fractal to Wada basin boundaries through double heteroclinic bifurcation

When r is increased from a value, for example 0.08, at which all the basin boundaries are fractal but not Wada, we will encounter the first critical value of double heteroclinic tangency $r_{\rm cr} \approx 0.114137$. The tangency between upper left branch of stable manifold of M_2 and upper right branch of unstable manifold of L, and the tangency between lower right branch of stable manifold of M_1 and lower left branch of unstable manifold of R cause this double heteroclinic tangency (see Fig.5).

Before the bifurcation, the saddle L is accessible from the basin of leftmost attractor through its lower left branch of unstable manifold. After the bifurcation, the stable manifold of M_2 infinitely intersect with the upper right branch of unstable manifold of L. So L is still accessible from basin of leftmost attractor through its lower left branch of unstable manifold. Similarly, R is also accessible from basin of rightmost attractor. For the stable manifold of M_2 does not intersect with the upper right branch of unstable manifold of M_1 through which the saddle M_1 is accessible from the basin of the middle attractor. So after bifurcation, M_1 is still accessible from the basin of middle attractor. Similarly, M_2 is also still accessible from the basin of middle attractor.

Therefore there is no change of accessible periodic orbits when r passes this critical value.

The unstable manifold of L and R will intersect all the three basins after the double heteroclinic tangency. Therefore Wada basin boundaries are born through double heteroclinic tangency. This change can be used to predict the critical value of double heteroclinic tangency. We find $r_{\rm cr} \approx 0.114137$.

Figure 6 shows the stable and unstable manifold just before bifurcation with $r = 0.11413$. Figure 6(a) shows the tangency of stable manifold of M_1 (or L) with unstable manifold of R (or M_2). Figure 6(b) shows the tangency of stable manifold of M_2 with unstable manifold of L.

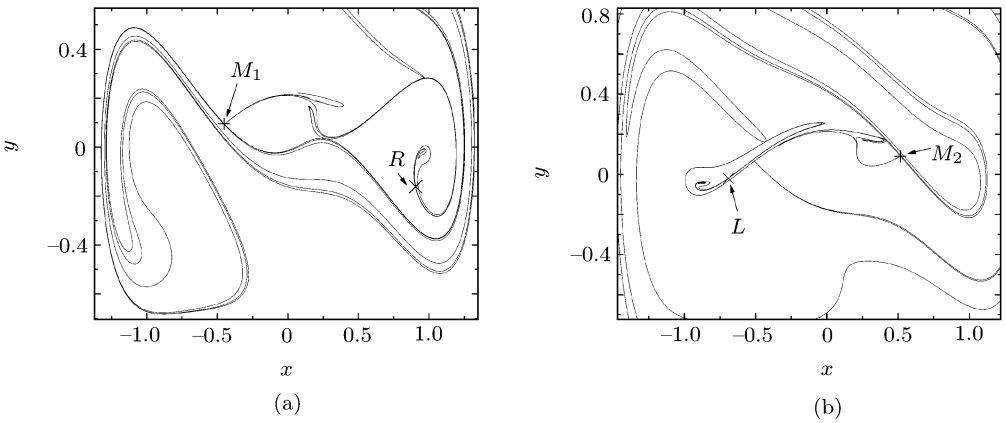


Fig.6. Double heteroclinic tangency of system (1) with $\delta = 0.27$, $x_0 = 0.5$, $\omega = 0.66$ and $r = 0.11413$. (a) The tangency of stable manifold of M_1 and the unstable manifold of R. (b) The tangency of stable manifold of M_2 and the unstable manifold of L.

Figure 7 shows creation of Wada basin boundaries just after bifurcation with $r = 0.11414$. Two saddle points that are accessible from basins of leftmost attractor and rightmost attractor respectively are denoted by L and R. The unstable manifold of saddle point L shown in Fig.7(a) intersects all the three basins, so the basin boundary of leftmost attractor is Wada basin boundary. Also the unstable manifold of R shown in Fig.7(b) intersects all the three basins, so the basin boundary of rightmost attractor is a Wada basin boundary. The third basin boundary, the basin boundary of the middle attractor, is also a Wada basin boundary.

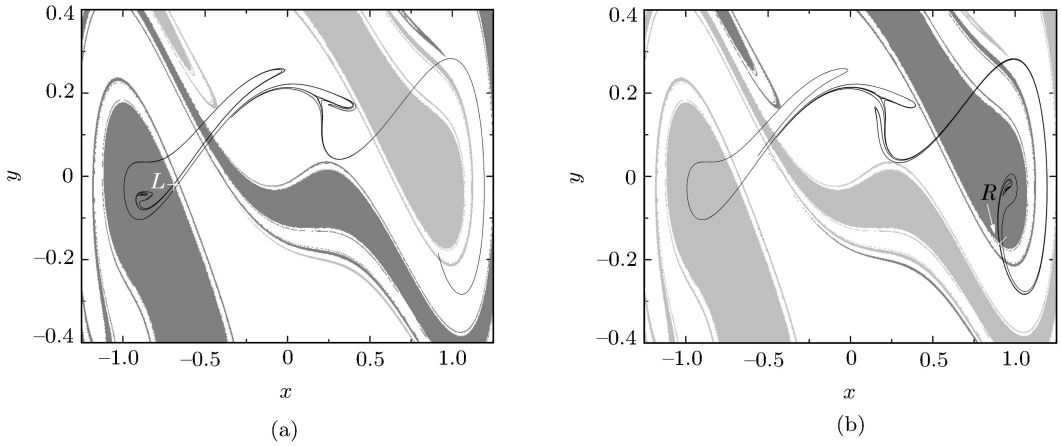


Fig.7. Wada basin boundaries arise just after double heteroclinic tangency ($r = 0.11414$). (a) The unstable manifold of saddle L accessible from basin of leftmost attractor crosses all the three basins. (b) The unstable manifold of saddle R accessible from basin of rightmost attractor crosses all the three basins.

4.2. Bifurcation of Wada basin boundaries through double heteroclinic bifurcation

After the double heteroclinic bifurcation discussed in subsection 4.1, all the basin boundaries are Wada. If we still increase r, we will encounter another double heteroclinic bifurcation. Namely, the stable manifold of M_2 becomes tangent with the unstable manifold of M_1 and the stable manifold of M_1 becomes tangent with the unstable manifold of M_2. After double heteroclinic bifurcation, stable manifold of M_1 will infinitely intersect upper right branch unstable manifold of M_2. This makes M_2 inaccessible from basin of middle attractor. Similarly, M_1 is also inaccessible from basin of middle attractor. A new period-5 saddle orbit accessible from basin of middle attractor will be created. This change could be used to predict the bifurcation. We find the critical value $r_{\rm cr} \approx 0.123857$.

This bifurcation of basin boundaries can be characterized by the change of accessible periodic saddle orbits. Figure 8 shows this result. After the double heteroclinic tangency, the basin boundaries suddenly change their characters.

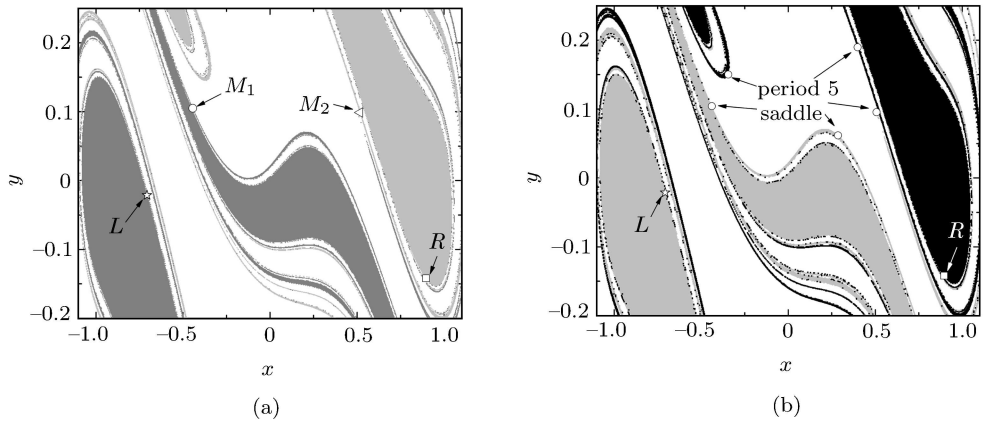


Fig.8. Changes of accessible orbits in the metamorphoses of basin boundaries from Wada to Wada. (a) Four accessible saddle points are marked by L, M_1, M_2 and R respectively just before bifurcation ($r = 0.12385$). (b) A periodic 5 saddle orbit accessible from white basin replaces the two saddle points M_1 and M_2 in (a) after bifurcation ($r = 0.12386$).

5. A smooth Wada basin boundary

In this section, we turn to geometric property of a Wada basin boundary. Suppose that a dynamical system has three basins. Each of the basin boundaries has Wada property, or only one basin boundary is Wada while the other two are not.[4] All the previous examples of Wada basin boundaries are all fractal. People think that the fractal property is a necessary condi-

tion for a Wada basin boundary.[14,16] We find that a Wada basin boundary can be smooth. The model we use is a damped, driven, mechanical oscillator

$$\ddot{x} + \beta\dot{x} + x - x^2 = F\sin(\omega t). \tag{2}$$

This model was intensively investigated by Thompson and co-workers. See Ref.[24] and references therein. Also this model was used to study Wada basin boundaries.[15]

We choose $\beta = 0.1$, $F = 0.04$, and $\omega = 0.91$. In this case, the Poincaré map with phase π has three attractors involving two stable fix points (0.49375, −0.00875) and (0.26375, –0.16375), and the attractor at infinity.

Figure 9(a) shows the three basins. White and black denote basins of two stable fixed points respectively. Grey denotes points diverging to infinity. It seems that the basin boundary of infinity is a Wada basin boundary because the white and black points accumulate on the grey boundary. There is one saddle point (1.0009, 0.01985) accessible from the basin of infinity. Its stable manifold and unstable manifold (denoted by arrow) are shown in Fig.9(b). Comparing Fig.9(a) with Fig.9(b), we can see that the lower left branch unstable manifold intersects with two basins of two stable fixed points respectively, and the upper right branch unstable manifold intersects the basin of infinity. Thus the unstable manifold of the accessible saddle point intersects with three basins. So the basin boundary of infinity is a Wada basin boundary. While there are homoclinic intersections, the Wada basin boundary is smooth.

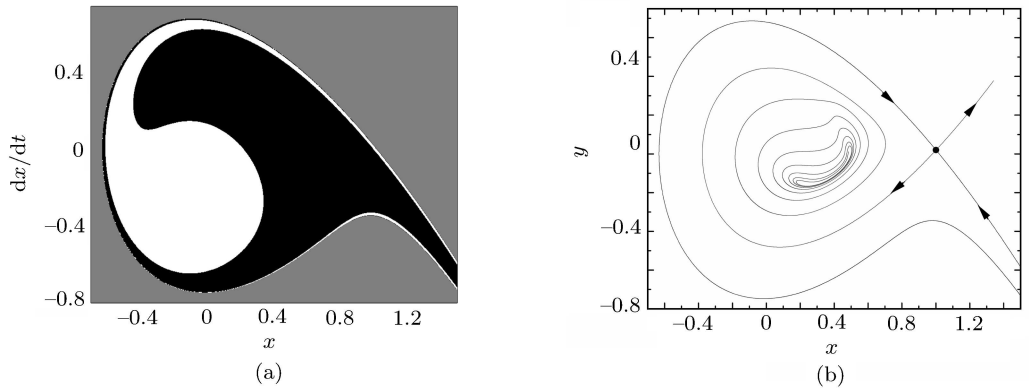


Fig.9. (a) Three basins for system (2) with $\beta = 0.1, F = 0.04$, and $\omega = 0.91$. White and black denote basins of two stable fixed points respectively. Grey denotes points diverging to infinity. (b) Stable and unstable manifold of saddle point accessible from basin of infinity.

We show how this Wada basin boundary occurs. There is a saddle point both accessible from two basins (white and black) of stable fixed points. Its stable manifold composes the basin boundary of two stable fixed points. According to Fig.9, we can see that this stable manifold intersects with the unstable manifold (shown by arrow in Fig.9(b)) of saddle point on the basin boundary of attractor at infinity. There will be infinite heteroclinic crossings. These crossings draw the two basins of stable fixed points to the basin boundary of the attractor at infinity. So the formation of the Wada basin boundary is due to the heteroclinic intersections.

6. Conclusion

We demonstrate and analyse double heteroclinic tangency in a three-well potential model. Its occurrence is guaranteed by the symmetry. Double heteroclinic tangency in this model can produce three new types of bifurcations of basin boundaries involving (i) from smooth to Wada basin boundaries; (ii) from fractal to Wada basin boundaries; (iii) from Wada to Wada basin boundaries. Bifurcation (i) and bifurcation (iii) can be characterized by the changes of accessible periodic orbits, which are consistent with previous results.[18] However, there are no changes of accessible periodic orbits in bifurcation (ii).

In a model of mechanical oscillator, we show that a Wada basin boundary can be smooth. Its formation is due to heteroclinic intersections. So a Wada basin boundary can either be smooth or fractal. The fractal property is not a necessary condition for a Wada basin boundary.

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