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Design of output feedback controller for a unified chaotic system*

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In this paper, the synchronization of a unified chaotic system is investigated by the use of output feedback controllers; a two-input single-output feedback controller and single-input single-output feedback controller are presented to synchronize the unified chaotic system when the states are not all measurable. Compared with the existing results, the controllers designed in this paper have some advantages such as small feedback gain, simple structure and less conservation. Finally, numerical simulations results are provided to demonstrate the validity and effectiveness of the proposed method.

Keywords: chaotic system, sliding mode, Lorenz system, exponentially stable
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1. Introduction

Chaos is a very interesting nonlinear phenomenon and chaos problem has been intensively studied since Lorenz's first research work in 1963, due to its potential applications in physics，automatic control，biological systems and communication engineering etc.[1−4] A fundamental character of the chaotic systems is their sensitivity to initial conditions of the state. The trajectories of the chaotic systems starting from any pair of arbitrary close positions will diverge exponentially and will become more and more uncorrected with the time. Therefore the study of chaotic secure communication has become a new research direction. In 1990, Ott *et al*[5] proposed OGY method, Ditto *et al*[6] first realized the stable control for fixed points in magnet-elastic system by means of OGY method in the same year. Since then, the investigation in chaos control and synchronization has been developed rapidly,[7−11] and many schemes were proposed, such as adaptive control approach，observer method，nonlinear feedback method and active backstepping method. However, the designs of the controllers mostly need state feedback. Since the state vectors are not all measurable in practical application, so the use of the controllers is strictly constrained.

The aim of this paper is to design an output feedback controller to synchronize the unified chaotic system. Compared with existing results, the controllers designed in this paper have such advantages as small feedback gain, simple structure and less conservation.

This paper is organized as follows. In Section 2, we describe the unified chaotic system and its mathematical models. In Section 3 single state output feedback controller is used to implement chaos synchronization for the unified chaotic system. Numerical simulations are provided in Section 4 to illustrate and verify the method. Finally, concluding remarks are given in Section 5.

2. Description of system and mathematical models

Consider the following unified chaotic system:[12]

$$\begin{aligned}\dot{x}_1 &= (25\alpha+10)(x_2-x_1),\\ \dot{x}_2 &= (28-35\alpha)x_1+(29\alpha-1)x_2-x_1x_3,\\ \dot{x}_3 &= x_1x_2-(8+\alpha)x_3/3, \qquad (1)\\ y &= x_2, \qquad (2)\end{aligned}$$

and response system

$$\begin{aligned}\dot{z}_1 &= (25\alpha+10)(z_1-z_2)+u_1,\\ \dot{z}_2 &= (28-35\alpha)z_1+(29\alpha-1)z_2-z_1z_3+u_2,\\ \dot{z}_3 &= z_1z_2-(8+\alpha)z_3/3, \qquad (3)\end{aligned}$$

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$$\bar{y} = z_2. \tag{4}$$

Where $x_1,\ x_2,\ x_3$ are state variables, y is the output variable. $\alpha \in [0,1]$ is the system parameter. When $\alpha \in (0,\ 0.8)$ it is Lorenz system,[13] when $\alpha \in (0.8,\ 1]$ it is a Chen system[14] and when $\alpha = 0.8$ it is a Lü system.[15] Equation (1) is the drive system and the equation (3) is the response system.

From Eqs.(1) and (4), we can get the following error dynamical system:

$$\begin{aligned}
\dot{e}_1 &= (25\alpha+10)(e_2-e_1)+u_1,\\
\dot{e}_2 &= (28-35\alpha)e_1+(29\alpha-1)e_2-e_1e_3-x_3e_1\\
&\quad -x_1e_3+u_2,\\
\dot{e}_3 &= e_1e_2+x_2e_1+x_1e_2-(8+\alpha)e_3/3, \qquad (5)\\
e_y &= \bar{y}-y=e_2, \qquad (6)
\end{aligned}$$

where $e_i = z_i - x_i (i = 1, 2, 3)$.

Our goal is to design an output feedback controller with only one state such that the trajectory of the response system can asymptotically exponentially approach the drive system, i.e.

$$\lim_{t\to\infty} ||e(t)|| = \lim_{t\to\infty} ||z(t)-x(t)|| = 0.$$

Since the trajectory of system (1) is bounded, so there exist positive constants M_i such that $|x_i| < M_i (i = 1,\ 2,\ 3)$.

Lemma 1 For any real scalar $\varepsilon > 0$ and any vectors with appropriate dimensions a and b, the following inequality holds:

$$2|a\,b| \le \varepsilon a^2 + \varepsilon^{-1}b^2.$$

Lemma 2[7] (Schur complements) For a given symmetrical matrix S, the following conditions are equivalent:

i) $S < 0$;

ii) $S_{11} < 0,\ S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$;

iii) $S_{22} < 0,\ S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

In the next sections we will study chaos synchronization.

3. Design of synchronization controller

Theorem 1 We choose the output feedback controllers with only one state as follows:

$$u_1 = k_1e_y,\ \ k_1 = -(10+25\alpha),$$

$$u_2 = k_2e_y,\ \ k_2 = 1-29\alpha-\mu,\ \mu>0, \tag{7}$$

then the two chaotic systems will be globally exponentially synchronized.

Proof Substituting Eq.(7) into Eq.(5), we have

$$\begin{aligned}
\dot{e}_1 &= -(25\alpha+10)e_1,\\
\dot{e}_2 &= (28-35\alpha-x_3)e_1-\mu e_2-e_1e_3-x_1e_3,\\
\dot{e}_3 &= e_1e_2+x_2e_1+x_1e_2-(8+\alpha)e_3/3. \qquad (8)
\end{aligned}$$

From the first equation of Eq.(8), we have

$$e_1(t) = e_1(0)\mathrm{e}^{-(25\alpha+10)t}. \tag{9}$$

Construct a Lyapunov function as

$$V = e_2^2 + e_3^2,$$

then we have

$$\begin{aligned}
\dot{V} &= 2(e_2\dot{e}_2+e_3\dot{e}_3)\\
&= 2[-\mu e_2^2+(28-35\alpha-x_3)e_1e_2+x_2e_1e_3\\
&\quad -(8+\alpha)e_3^2/3]\\
&\le -2\,[\mu e_2^2+(8+\alpha)e_3^2/3]+2M_2|e_1||e_3|\\
&\quad +2(|28-35\alpha|+M_3)|e_1||e_2|. \qquad (10)
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&2M_2|e_1||e_3|+2(|28-35\alpha|+M_3)|e_1||e_2|\\
&\le M_2(\varepsilon|e_3|^2+\varepsilon^{-1}|e_1|^2)\\
&\quad +(|28-35\alpha|+M_3)(\varepsilon|e_2|^2+\varepsilon^{-1}|e_1|^2). \qquad (11)
\end{aligned}$$

Substituting Eq.(11) into Eq.(10) we have

$$\begin{aligned}
\dot{V} &\le -[2\mu-(|28-35\alpha|+M_3)\varepsilon]e_2^2\\
&\quad -[2(8+\alpha)/3-M_2\varepsilon]e_3^2+(|28-35\alpha|\\
&\quad +M_3+M_2)\varepsilon^{-1}|e_1|^2. \qquad (12)
\end{aligned}$$

Choose an appropriate ε which satisfies the following condition:

$$\varepsilon < \min\left(\frac{2\mu}{|28-35\alpha|+M_3},\ \frac{2(8+\alpha)}{3M_2}\right),$$

and define

$$\lambda = \min\begin{pmatrix}2\mu-(|28-35\alpha|+M_3)\varepsilon\\ 2(8+\alpha)/3-M_2\varepsilon\end{pmatrix} > 0,$$

$$\gamma = (\,|28-35\alpha|+M_3+M_2)\varepsilon^{-1},$$

then

$$\dot{V} \le -\lambda V + \gamma|e_1|^2.$$

Substituting Eq.(9) into Eq.(12), we have

$$\begin{aligned}\dot{V} &\leq -\lambda V\gamma|e_1(0)|^2\mathrm{e}^{-2\bar{\alpha}t},\\ \bar{\alpha} &= 25\alpha+10.\end{aligned}\tag{13}$$

From equation (13), we have

$$\begin{aligned}V &\leq \mathrm{e}^{-\lambda t}V_0+\gamma\int_0^t \mathrm{e}^{-\lambda(t-\tau)}|e_1(0)|^2\mathrm{e}^{-2\bar{\alpha}\tau}\mathrm{d}\tau\\ &= \mathrm{e}^{-\lambda t}V_0+\frac{\gamma}{\lambda-2\bar{\alpha}}|e_1(0)|^2\mathrm{e}^{-\lambda t}(\mathrm{e}^{(\lambda-2\bar{\alpha})t}-1)\\ &= \mathrm{e}^{-\lambda t}V_0+\frac{\gamma}{\lambda-2\bar{\alpha}}|e_1(0)|^2(\mathrm{e}^{-2\bar{\alpha}t}-\mathrm{e}^{-\lambda t})\to 0.\end{aligned}$$

So $\lim\limits_{t\to\infty} e_2(t)=\lim\limits_{t\to\infty} e_3(t)=0$, since Eq.(9), we know that

$$\lim_{t\to\infty} e_1(t)=\lim_{t\to\infty} e_2(t)=\lim_{t\to\infty} e_3(t)=0.$$

This implies that response system (3) and the drive system (1) are globally exponentially synchronized.

Theorem 2 When $\alpha=0$, select the feedback controllers as follows:

$$u_1=-10e_y\ ,\ u_2=u_3=0,\tag{14}$$

then the response system (3) and the drive system (1) are globally exponentially synchronized.

Proof When $\alpha=0$, error system (5) becomes

$$\begin{aligned}\dot{e}_1 &= 10(e_2-e_1)+u_1,\\ \dot{e}_2 &= 28e_1-e_2-e_1e_3-x_3e_1-x_1e_3,\\ \dot{e}_3 &= e_1e_2+x_2e_1+x_1e_2-8e_3/3.\end{aligned}$$

Since $u_1=-10e_y=-10e_2$, so we have

$$\begin{aligned}\dot{e}_1 &= -10e_1,\\ \dot{e}_2 &= 28e_1-e_2-e_1e_3-x_3e_1-x_1e_3,\\ \dot{e}_3 &= e_1e_2+x_2e_1+x_1e_2-8e_3/3.\end{aligned}\tag{15}$$

From the first equation of Eq.(15) , we known that e_1 is exponentially stable

$$e_1(t)=e_1(0)\mathrm{e}^{-10t}.\tag{16}$$

Choose a Lyapunov function as

$$V=e_2^2+e_3^2,$$

then we have

$$\begin{aligned}\dot{V} &= 2(e_2\dot{e_2}+e_3\dot{e_3})\\ &= 2[28e_1e_2-e_2^2-x_3e_1e_2+x_2e_1e_3-8e_3^2/3]\\ &\leq -2[e_2^2+8e_3^2/3]+56|e_1||e_2|\\ &\quad +2M_3|e_1||e_2|+2M_2|e_1||e_3|.\end{aligned}\tag{17}$$

By Lemma 1, we have

$$\begin{aligned}&56|e_1||e_2|+2M_3|e_1||e_2|+2M_2|e_1||e_3|\\ &\leq 28(\varepsilon^{-1}e_1^2+\varepsilon e_2^2)+M_3(\varepsilon^{-1}e_1^2+\varepsilon e_2^2)\\ &\quad +M_2(\varepsilon^{-1}e_1^2+\varepsilon e_3^2).\end{aligned}\tag{18}$$

Substituting Eq.(18) into Eq.(17), we have

$$\begin{aligned}\dot{V} &\leq -(2-28\varepsilon-M_3\varepsilon)e_2^2-(16/3-M_2\varepsilon)e_3^2\\ &\quad +(28+M_3+M_2)\varepsilon^{-1}e_1^2.\end{aligned}\tag{19}$$

Choose an appropriate ε which satisfies the following condition:

$$\varepsilon<\min\left(\frac{2}{28+M_3}\ ,\ \frac{16}{3M_2}\right),$$

and define

$$\begin{aligned}\lambda &= \min\ (2-28\varepsilon-M_3\varepsilon\ ,\ 16/3-M_2\varepsilon)\ >0\\ \gamma &= (28+\ M_3+M_2)\varepsilon^{-1},\end{aligned}$$

then

$$\dot{V}\leq -\lambda V+\gamma e_1^2.\tag{20}$$

By the same approach as theorem 1, we also can get

$$\lim_{t\to\infty} e_1(t)=\lim_{t\to\infty} e_2(t)=\lim_{t\to\infty} e_3(t)=0.$$

So the response system (3) and the unified chaotic system (1) are exponentially synchronized.

Theorem 3 Select the feedback controllers as follows:

$$u_1=0\ ,\ u_2=-k_2e_2,\tag{21}$$

where

$$k_2>\frac{25\alpha+10+\rho(28-35\alpha+M_3)}{2\varepsilon\,\rho}+29\alpha-1,\tag{22}$$

$$\rho<\min\left(\left(\frac{16+2\alpha}{3M_2}\right)^2,\ \left(\frac{20+50\alpha}{M_2}\right)^2\right),\tag{23}$$

$$\varepsilon<\frac{50\alpha+20-M_2\rho^{1/2}}{25\alpha+10+\rho(28-35\alpha+M_3)},\tag{24}$$

then the response system (3) and the unified chaotic system (1) are exponentially synchronized.

Proof Substituting $u_1=0\ ,\ u_2=-k_2e_2$ into Eq.(5), we have

$$\begin{aligned}\dot{e}_1 &= (25\alpha+10)(e_2-e_1),\\ \dot{e}_2 &= (28-35\alpha-x_3)e_1+(29\alpha-1-k_2)e_2\\ &\quad -e_1e_3-x_1e_3,\\ \dot{e}_3 &= e_1e_2+x_2e_1+x_1e_2-(8+\alpha)e_3/3.\end{aligned}\tag{25}$$

Choose a Lyapunov function as

$$\begin{aligned} V &= e_1^2 + \rho(e_2^2 + e_3^2), \\ \dot{V} &= -2(10+25)e_1^2 + 2(10+25)e_1e_2 \\ &\quad +2\rho(28-35\alpha)e_1e_2 + 2\rho x_2 e_1 e_3 - 2\rho x_3 e_1 e_2 \\ &\quad +\rho(58\alpha - 2 - 2k_2)e_2^2 - \rho(16+2\alpha)e_3^2/3 \\ &= -2(10+25\alpha)e_1^2 - \rho(2-58\alpha+2k_2)e_2^2 \\ &\quad -\rho(16+2\alpha)e_3^2/3 + 2\rho x_2 e_1 e_3 \\ &\quad +2[10+25\alpha+\rho(28-35\alpha-x_3)]e_1e_2. \end{aligned}$$

From lemma 1

$$\begin{aligned} &2[10+25\alpha+\rho(28-35\alpha-x_3)]|e_1||e_2| \\ &\le [10+25\alpha+\rho(28-35\alpha-x_3)] \\ &\quad \times(\varepsilon e_1^2 + \varepsilon^{-1}e_2^2), \end{aligned} \tag{26}$$

$$\begin{aligned} 2\rho|x_2||e_1e_3| &= 2|x_2|\rho^{1/4}|e_1|\rho^{3/4}|e_3| \\ &\le M_2(\sqrt{\rho}e_1^2 + \rho^{3/2}e_3^2). \end{aligned} \tag{27}$$

From Eqs.(26) and (27), we have

$$\begin{aligned} \dot{V} \le & -[(20+50\alpha)(1-\varepsilon) - M_2\sqrt{\rho} \\ & -\rho(28-35\alpha-x_3)\varepsilon]e_1^2 - \rho[2-58\alpha+2k_2 \\ & -(10+25\alpha+\rho(28-35\alpha+M_3))\varepsilon^{-1}]e_2^2 \\ & -\rho[(16+2\alpha)/3 - M_2\sqrt{\rho}]e_3^2. \end{aligned} \tag{28}$$

From Eq.(24), we have

$$\begin{aligned} &(10+25\alpha)(2-\varepsilon) - M_2\sqrt{\rho} - \rho(28-35\alpha-x_3)\varepsilon) \\ &= \lambda_1 > 0. \end{aligned} \tag{29}$$

From Eq.(22), we have

$$\begin{aligned} &\rho[2-58\alpha+2k_2 - (10+25\alpha \\ &+\rho(28-35\alpha+M_3))\varepsilon^{-1}] \\ &= \lambda_2 > 0. \end{aligned} \tag{30}$$

From Eq.(23), we have

$$(16+2\alpha)/3 - M_2\sqrt{\rho} = \lambda_3 > 0. \tag{31}$$

Substituting Eqs.(30), (31) and (32) into Eq.(29), then we have

$$\begin{aligned} &\dot{V} \le -\lambda V, \quad \lambda = \min\{\lambda_1,\ \lambda_2,\ \lambda_3\}, \\ &V \le V(0)\mathrm{e}^{-\lambda t}. \end{aligned}$$

So the response system (3) and the unified chaotic system (1) are globally exponentially synchronized.

4. Numerical simulation

Consider the system (1) when $\alpha = 0$,

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1), \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - (8/3)x_3. \end{aligned} \tag{32}$$

System (32) is Lorenz system,[13] and the error system is

$$\begin{aligned} \dot{e}_1 &= 10(e_2 - e_1) + u_1, \\ \dot{e}_2 &= 28e_1 - e_2 - e_1e_3 - x_3e_1 - x_1e_3 + u_2, \\ \dot{e}_3 &= e_1e_2 + x_2e_1 + x_1e_2 - 8e_3/3 + u_3. \end{aligned} \tag{33}$$

The output feedback controllers are selected below, respectively

i) u_1 , u_2 , u_3 are selected as Eq.(7) in theorem 1, namely

$$u_1 = -10e_y\ ,\ u_2 = (1-\mu)e_y\ ,\ \mu > 0\ ,\ u_3 = 0,$$

simulation results are shown in Fig.1 and Fig.2.

ii) u_1 , u_2 , u_3 are selected as Eq.(14) in theorem 2, namely

$$u_1 = -10e_y\ ,\ u_2 = u_3 = 0,$$

simulation results are shown in Fig.3 and Fig.4.

The simulations demonstrate that the designed controllers can implement global and exponential asymptotical synchronization of chaos for the unified chaotic system.

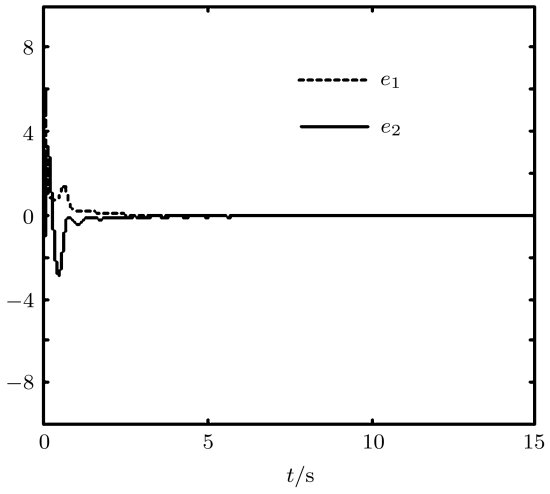


Fig.1. Simulation results of e_1, e_2.

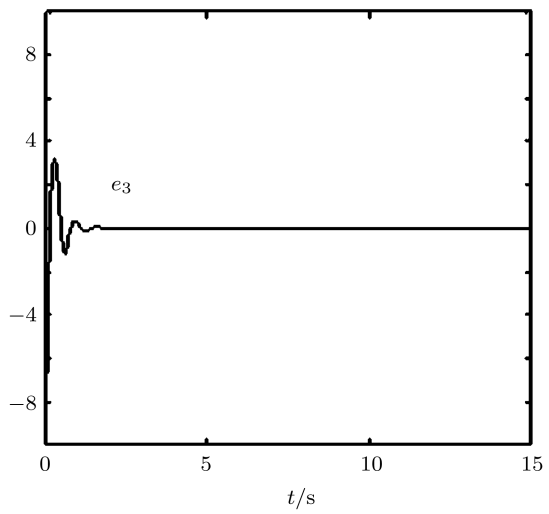


Fig.2. Simulation result of e_3.

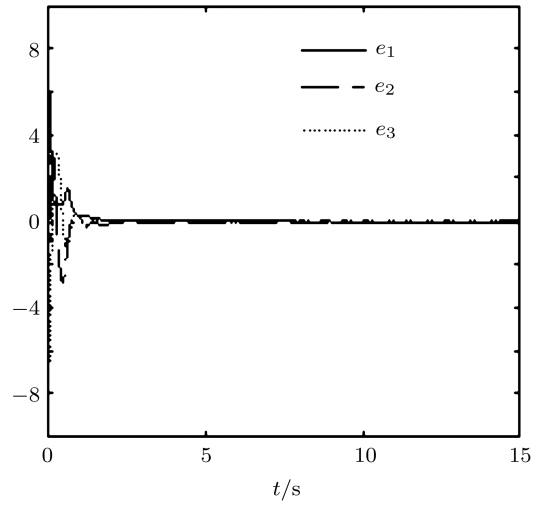


Fig.4. Simulation results of errors e_1, e_2, e_3.

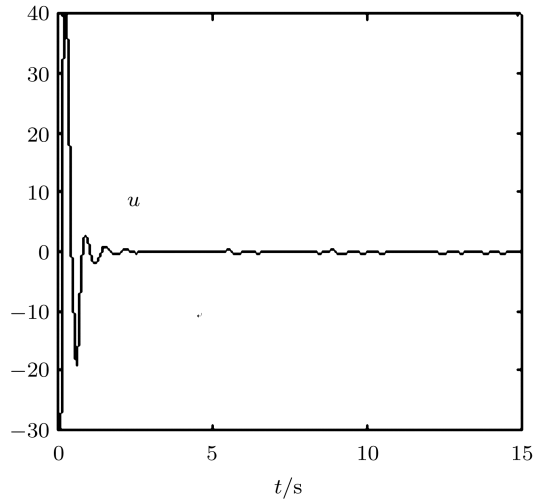


Fig.3. Simulation results of u.

5. Conclusion

In this paper, output feedback controllers with only one state are presented to synchronize a unified chaotic system when the states are not all measurable. Compared with existing results, the designed controllers in this paper have some advantages such as small feedback gain, simple structure and less conservation.

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