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Berry phase in a generalized nonlinear two-level system*

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In this paper, we investigate the behaviour of the geometric phase of a more generalized nonlinear system composed of an effective two-level system interacting with a single-mode quantized cavity field. Both the field nonlinearity and the atom–field coupling nonlinearity are considered. We find that the geometric phase depends on whether the index k is an odd number or an even number in the resonant case. In addition, we also find that the geometric phase may be easily observed when the field nonlinearity is not considered. The fractional statistical phenomenon appears in this system if the strong nonlinear atom–field coupling is considered. We have also investigated the geometric phase of an effective two-level system interacting with a two-mode quantized cavity field.

Keywords: geometric phase, nonlinear system, atom–field coupling
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The concept of geometric phase was first discussed by Pancharatnam[1] in his study of interference of classical light in distinct states of polarization. Berry[2] found that the geometric phase could be generated along a closed curve in a parameter space of a quantum system under the adiabatic evolution. Simon[3] gave a simple geometric interpretation to the Berry phase in terms of differential geometry and fibre bundles. Since then, this important notion has become an interesting subject, a subject whose many different aspects one is interested in, resulting in many different generalizations and applications.[4−14] An important reason why one is interested in the concept of geometric phase is its relevance to geometric quantum computation.[15] Indeed, it is believed that the purely geometric characteristic of such phases potentially provides robustness against certain sources of noise.[16] In particular, the geometric phase of nonlinear systems has attracted a lot of attention due to its interesting property. For example, Wu *et al*[17] determined the geometric phase of non-eigenstates and found that the new geometric phase reduced to a statistical average of Berry's phase.

It is important that the geometric phases should be useful in quantum computing as a tool to achieve fault tolerance.[18] As quantum computing is always carried out in a real physical system, the nonlinearity of the system must be considered. This nonlinearity may play an important role in many experimental circumstances. In fact, it may even become crucial and unusual when geometrical phases are used to implement quantum protocols, such as communication and computational ones. In view of its importance, the authors[19] showed how to transmute the statistics of the system via the adiabatic geometric phase in a nonlinear Jaynes–Cummings model.[20−25] It is worthwhile to further study the property of geometric phase in the presence of nonlinearity, and we expect that it will give us more useful information from this study. In this paper, we investigate the geometric phase of a generalized nonlinear two-level system and develop more general standardized solution formulas via taking into account both the field nonlinearity and the intensity-dependent atom–field coupling nonlinearity. Then, we investigate the geometric phase of a generalized Jaynes–Cummings model which includes two forms of nonlinearities, i.e. the field nonlinearity and the atom–field coupling nonlinearity. In the rotating-wave approximation, this system is governed by the Hamiltonian ($\hbar = 1$)[26−28]

$$H_0 = \upsilon a^\dagger a + \frac{\omega}{2}\sigma_z + g[f(a^\dagger a)(a)^k\sigma_+ + (a^\dagger)^k f(a^\dagger a)\sigma_-] + R(a^\dagger a), \quad (1)$$

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where $a(a^\dagger)$ is the annihilation (creation) operator for the single-mode quantum field; ν is the frequency of the field; ω is the transition frequency between the eigenstates of the system; the σ_+, σ_-, etc. are the usual atomic transition operators satisfying $\sigma_{jk}\sigma_{mn} = \sigma_{jn}\delta_{mk}$, and $\sigma_z = \sigma_{++} - \sigma_{--}$, $\sigma_+ = \sigma_{+-}$, $\sigma_- = \sigma_{-+}$; k is any positive integer; $R(a^\dagger a)$ is the contribution to the nonlinearity of the field; $gf(a^\dagger a)$ denotes the intensity-dependent atom–field coupling; $f(a^\dagger a)$ and $R(a^\dagger a)$ each are a reasonable function of photon number operator n ($n = a^\dagger a$). Assuming $|n\rangle$ to denote the Fock state and $|\pm\rangle$ to represents atomic states which satisfy $\sigma_z|\pm\rangle = \pm|\pm\rangle$ and using the excellent method developed in Refs.[26,27] yield the eigenvalues of the system:

$$\begin{aligned} E_1 =& \frac{2\upsilon n + \upsilon k + R(n) + R(n+k)}{2} \\ &+ \sqrt{(\Delta_k)^2 + 4g^2[f(n)]^2\frac{(n+k)!}{n!}}, \\ E_2 =& \frac{2\upsilon n + \upsilon k + R(n) + R(n+k)}{2} \\ &- \sqrt{(\Delta_k)^2 + 4g^2[f(n)]^2\frac{(n+k)!}{n!}}, \end{aligned} \tag{2}$$

and their corresponding eigenstates:

$$\begin{aligned} |\psi_n^1\rangle &= \cos(\theta_n)|n,+\rangle + \sin(\theta_n)|n+k,-\rangle, \\ |\psi_n^2\rangle &= \cos(\theta_n)|n+k,-\rangle - \sin(\theta_n)|n,+\rangle, \end{aligned} \tag{3}$$

where the values of θ_n satisfy the following relations:

$$\cos(\theta_n) = \frac{\sqrt{\Theta^2 + \Delta_k\Theta}}{\sqrt{2}\Theta},$$

$$\sin(\theta_n) = \frac{\sqrt{\Theta^2 - \Delta_k\Theta}}{\sqrt{2}\Theta}, \tag{4}$$

with $\Theta = \sqrt{(\Delta_k)^2 + 4g^2[f(n)]^2(n+k)!/n!}$, $\Delta_k = \omega - \upsilon k + R(n) - R(n+k)$, and $f(n)$ and $R(n)$ each being a photon-number function whose form is dependent on physical models. For example, Hamiltonian (1) reduces to the well-known m-quantum Jaynes–Cummings model[29] when $R(a^\dagger a) = 0$ and $f(a^\dagger a) = 1$, and it can become the Kerr-type micromaser model[30] when $R(a^\dagger a) = n(n-1)$, $k = 1$ and $f(a^\dagger a) = 1$. To proceed further, now we first recount the method[31] for calculating the Berry phase in a fully quantized regime. The phase shift operator $U(\phi) = \mathrm{e}^{-\mathrm{i}\phi a^\dagger a}$, which is capable of changing the state of field, is needed in this nonlinear model to calculate the geometric phase. So the Hamiltonian described by Eq.(1) is slowly transformed into a parametrized one, $H(\theta) = U^\dagger(\phi)H_0U(\phi)$. This transformation is achieved by adiabatically adjusting the interaction. When ϕ cyclically varies from 0 to 2π, the eigenstates go back to the original ones with an extra geometric phase factor $\mathrm{e}^{\mathrm{i}\gamma}$. By changing ϕ slowly from 0 to 2π the eigenstates $\psi_{1,2}$ can acquire geometric phases which are calculated as follows:

$$\begin{aligned} \gamma_{1,2} =& \mathrm{i}\int_0^{2\pi} \mathrm{d}\phi\langle\psi_n^{1,2}|U^\dagger(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}U(\phi)|\psi_n^{1,2}\rangle \\ =& -\mathrm{i}^2\int_0^{2\pi}\mathrm{d}\phi\langle\psi_n^{1,2}|\hat{n}|\psi_n^{1,2}\rangle. \end{aligned} \tag{5}$$

Substituting expression (3) into expression (5) yields

$$\gamma_{1,2} = 2\pi\left[n + \frac{k}{2}\left(1 \mp \frac{\Delta_k}{\sqrt{(\Delta_k)^2 + 4g^2(f(n))^2(n+k)!/n!}}\right)\right]. \tag{6}$$

It is interesting that this nonlinear system acquires a phase, for the initial photon number is zero, which implies that the vacuum field may introduce a correction to the Berry phase. And the Berry phase is small when the total initial photon number is small, but it is considerable when the photon number is large. The phenomena are similar to those in the case in which the system is a linear quantum system.[31] It is noted that eigenstates $|\psi_n^1\rangle$ and $|\psi_n^2\rangle$ both acquire the same geometric phase ($\gamma_{1,2} = k\pi$) in a resonant regime. That is to say, geometric phases $\gamma_{1,2}$ depend on whether the index k is an odd number or even number when Δ_k is equal to zero. And the geometric phase is equal to 0 or 2π when k is an even number; the geometric phase is equal to π when k is an odd number. The physical origin of this phenomenon is that each exchange (absorption or emission) of one photon will contribute a phase of π to the total geometric phase.

Now we make a simple analysis of the influences of intensity-dependent process (the interaction coupling

coefficient between atom and photon is a function of photon number) and multi-photon degeneracy process on the geometric phase. To start with, we only consider a possible strong nonlinear atom–field coupling (the field nonlinearity is not considered). This kind of nonlinearity can naturally occur in strong-field scenarios where the nonlinear atom–field coupling is important. In this case, Hamiltonian (1) reduces to describing the Buck–Sukumar model[20] when $k = 1$, $R(n) = 0$ and $f(n) = \sqrt{n+1}$, and it has the following form:

$$\tilde{H} = \upsilon a^{\dagger}a + \frac{\omega}{2}\sigma_z + g[a\sqrt{n+1}\sigma_+ + a^{\dagger}\sqrt{n+1}\sigma_-], \quad (7)$$

and the geometric phases have explicit expressions as follows:

$$\Gamma_{1,2} = 2\pi\left[n + \frac{1}{2}\left(1 \mp \frac{\Delta_{k0}}{\sqrt{(\Delta_{k0})^2 + 4g^2(n+1)(n+1)}}\right)\right], \quad (8)$$

where $\Delta_{k0} = \omega - k\upsilon$. It is interesting to compare the geometric phase of a nonlinear system in the presence of nonlinear atom–field coupling with that of a linear quantum system.[31] The main change tendencies of the geometric phases are similar to each other. Figure 1 presents the dependence of geometric phase on the photon number n and the parameter Δ_{k0}/g. From Fig.1, we can clearly see that geometric phase Γ_1 tends to zero (or 2π) in the limit that $\Delta_{k0}/g \to \infty$ at different values of photon number n. We can also observe that the geometric phase changes more slowly as photon number n increases, indicating that photon number n has a considerable influence on geometric phase.

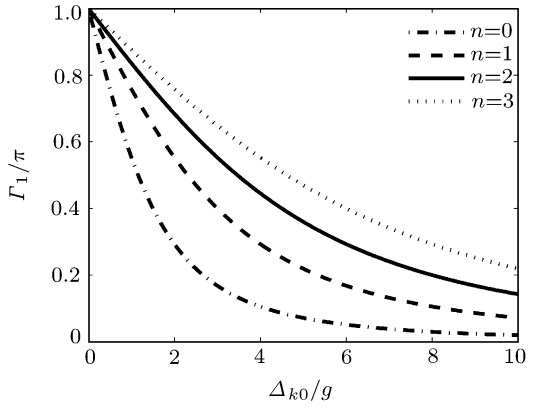


Fig.1. Geometric phase Γ_1 as a function of detuning Δ_{k0}/g at different values of photon number n with the other parameters being $R(n) = 0$, $\Delta_{k0} = \omega - k\upsilon$ and $k = 1$.

As another example, Hamiltonian (1) reduces to the well-known m-photon Jaynes–Cummings model[29] when $R(n) = 0$, $f(n) \equiv 1$ and k is any nonnegative integer. This problem may be described by an effective two-level atom coupling nonlinearly with a quantum field through k-photon absorptions and emissions, In the rotating wave approximation, this Hamiltonian describing such a model has the form

$$H' = \upsilon a^{\dagger}a + \frac{\omega}{2}\sigma_z + g[(a)^k\sigma_+ + (a^{\dagger})^k\sigma_-] \quad (9)$$

with the corresponding geometric phase being

$$\gamma'_{1,2} = 2\pi\left[n + \frac{k}{2}\left(1 \mp \frac{\Delta_{k0}}{\sqrt{(\Delta_{k0})^2 + 4g^2(n+k)!/n!}}\right)\right], \quad (10)$$

where $\Delta_{k0} = \omega - k\upsilon$. It is interesting to note that the geometric phases $\gamma'_{1,2}$ have a fractional statistical behaviour. In order to analyse the fractional statistics property, a similar method to that in Ref.[32] is adopted in our nonlinear quantum system. When the parameter of Hamiltonian $H(\phi)$ changes and cyclically returns to its original configuration, the wavefunction will return to the initial state with an extra phase factor $e^{i\beta 2\pi}$, where $\beta = \gamma_0/2\pi$ is called the statistical factor. $\beta = 0$, $\beta = 1$ and $\beta =$ fraction which are corresponding to Bose statistical factor, Fermi statistical factor, fractional statistical factor in the parametric space, respectively. The statistical factor has the following form:

$$\beta = \frac{k}{2}\left(1 - \frac{\Delta_{k0}}{\sqrt{(\Delta_k)^2 + 4g^2(n+k)!/n!}}\right). \quad (11)$$

In the resonant case in which $\Delta_{k0} = 0$, the statistical factor $\beta = \dfrac{k}{2}$. Thus we can obtain statistical factor $\beta = 1$ when $k = 2$, then the statistical factor displays the behaviour of the Fermi statistics. When the off-resonance case ($\Delta_{k0} \gg g$) is considered, the eigenstates of Hamiltonian (9) are completely separable, and obviously, the statistical factor $\beta \to 0$. Thus the Bose statistical property slowly emerges. We plot statistical factor as a function of Δ_{k0}/g at different values

of k in Fig.2. We can obviously observe that the statistical factors slowly change from one to zero as Δ_{k0}/g increases. This change indicates that the statistical property of the system changes from Fermi to Bose. Obviously, in the resonant case, if we change ϕ by a cyclic, we have a statistical factor $\beta = \dfrac{k}{2}$; if we change ϕ by $\dfrac{2}{3}$ cycle, then $\beta = \dfrac{k}{3}$, and so on. In this way, the statistical factor will be $\beta = k/2, k/3, k/4, \ldots$.

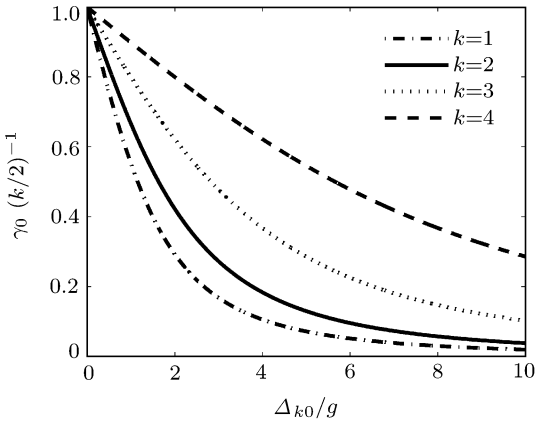


Fig.2. The fractional part of geometric phase $\{\gamma_{k0}(k/2)^{-1}\}$ as a function of Δ_{k0}/g at different values of k, with the other parameters being $R(n) = 0, f(n) = 1$, and $n = 0$.

Next, we introduce an effective two-level (fermionic) system coupled with two quantized bosonic fields. We consider a simple case, in which the two-level system interacts with one mode of the field with creation and annihilation operators $a^\dagger$ and a through a nonlinear Jaynes–Cummings model with an effective coupling efficient g, and the second mode of the field initially does not interact with the two-level system, nor does the first mode of the field. In the rotating-wave approximation, the initial Hamiltonian reads

$$\begin{aligned} H_0 =& \upsilon a^\dagger a + \upsilon b^\dagger b + \frac{\omega}{2}\sigma_z \\ &+ g[f(a^\dagger a)(a)^k\sigma_+ + (a^\dagger)^k f(a^\dagger a)\sigma_-] \\ &+ R(a^\dagger a), \end{aligned} \tag{12}$$

where $b^\dagger$ and b are the creation and annihilation operators of the second mode with the same frequency υ as that of the first mode. The eigenstates of Hamiltonian are

$$|\Psi^i_{n,n'}\rangle = |\psi^i_n\rangle \otimes |n'\rangle, \qquad i = 1, 2, \tag{13}$$

where $|\psi^i_n\rangle$ represents the eigenstate of Hamiltonian Eq.(1). The state vector is a product state of the Fock state $|n'\rangle$ of the second mode of the field and the nonlinear Jaynes–Cummings eigenstates of joint state of the first mode of field and the two-level system. In order to calculate the geometric phase of the system, we introduce a unitary transformation

$$U(\theta, \phi) = \exp(-\mathrm{i}\phi J_z)\exp(-\mathrm{i}\theta J_y), \tag{14}$$

where $J_z = \dfrac{1}{2}(a^\dagger a - b^\dagger b)$; $J_y = \dfrac{1}{2\mathrm{i}}(a^\dagger b - a b^\dagger)$; θ and ϕ are slowly varying parameters. The transformed Hamiltonian $H = U(\theta, \phi)H_0U^\dagger(\theta, \phi)$ describes the interaction between a two-level system and two modes of field through a nonlinear Jaynes–Cummings Hamiltonian where the coupling between the two-level system and each of modes is adiabatically changed through the parameters θ and ϕ. And the simultaneous eigenstates of H are $|\Psi^i_{(n,n')}(R)\rangle = \mathcal{U}(\theta, \phi)|\Psi^i_{(n,n')}\rangle$. According to the theory in Ref.[2], the eigenstates of this system acquire a purely geometric phase factor $\mathrm{e}^{\mathrm{i}\gamma_{n,n'}}$ in addition to the usual dynamical phase when parameters θ and ϕ change slowly and cyclically come back to their initial forms. The geometric phase $\gamma_{n,n'}$ can be directly determined in terms of an integral[14] $\gamma = \displaystyle\int_{R(0)}^{R(t)} A_i(R)\mathrm{d}R^i$, where a vector-valued function $A_i(R) = \mathrm{i}\langle\Psi^i_{(n,n')}(R)|\nabla|\Psi^i_{(n,n')}(R)\rangle$ is called Mead–Berry vector potential[33] and the term $A = A_i(R)\mathrm{d}R^i$ is called Mead–Berry connection, with R being a set of control parameters and $R = (\theta, \phi)$ in the system under consideration. Next we calculate the components of the Mead–Berry vector potential as follows:

$$\begin{aligned} A_\phi =& \mathrm{i}\langle\Psi^i_{(n,n')}|\mathcal{U}^\dagger(\theta, \phi)\frac{\partial}{\partial\phi}\mathcal{U}(\theta, \phi)|\Psi^i_{(n,n')}\rangle \\ =& -\mathrm{i}\langle\Psi^i_{(n,n')}|\sin\theta\hat{J}_x + \cos\theta\hat{J}_z|\Psi^i_{(n,n')}\rangle \\ =& -\mathrm{i}\cos\theta[n - n' + k\sin^2(\theta_n)], \end{aligned} \tag{15}$$

and

$$\begin{aligned} A_\theta =& \mathrm{i}\langle\Psi^i_{(n,n')}|\mathcal{U}^\dagger(\theta, \phi)\frac{\partial}{\partial\theta}\mathcal{U}(\theta, \phi)|\Psi^i_{(n,n')}\rangle \\ =& -\mathrm{i}\langle\Psi^i_{(n,n')}|\sin\theta\hat{J}_x + \cos\theta\hat{J}_y|\Psi^i_{(n,n')}\rangle \\ =& 0. \end{aligned} \tag{16}$$

If the value of θ is fixed and ϕ varies slowly from 0 to 2π, the geometric phase generated is calculated as follows:

$$\gamma^{1,2}_{n,n'} = \frac{\Omega}{2}\left[n - n' + \frac{k}{2}\left(1 \mp \frac{\Delta_k}{\Theta}\right)\right], \tag{17}$$

where $\Delta_k = \omega - \upsilon k + R(n) - R(n + k)$; $\Theta = \sqrt{(\Delta_k)^2 + 4g^2[f(n)]^2(n + k)!/n!}$; Ω is the solid angle enclosed in parameter space, which is parametrized by spherical polar angles θ and ϕ.

In this paper, with two forms of nonlinearities: the field nonlinearity, and the atom–field coupling

nonlinearity, taken into consideration, we have investigated the geometric phases of an effective two-level system interacting with a quantized field. We have analysed and discussed one mode and two modes in the adiabatic case, with the two forms of nonlinearities considered. In the resonant case, the proposed geometric phase displays an interesting property in which it is equal to zero when k is an even number and it is equal to π when k is an odd number. This phenomenon may be interpreted as follows: each exchange (absorption or emission) of one photon will contribute a phase factor π to the total geometric phase. As an example, we find that the geometric phase in the nonlinear case decreases more slowly when the detuning increases than in the linear case. Without the field nonlinearity taken into consideration, the geometric phase exhibits the fractional statistical property. In the appropriate parameters, the statistical factor can be continuously changed and the statistics of the system may be continuously transmuted from Bosons to Fermions.

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