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The density wave in a new anisotropic continuum model*

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In this paper the new continuum traffic flow model proposed by Jiang *et al* is developed based on an improved car-following model, in which the speed gradient term replaces the density gradient term in the equation of motion. It overcomes the wrong-way travel which exists in many high-order continuum models. Based on the continuum version of car-following model, the condition for stable traffic flow is derived. Nonlinear analysis shows that the density fluctuation in traffic flow induces a variety of density waves. Near the onset of instability, a small disturbance could lead to solitons determined by the Korteweg–de-Vries (KdV) equation, and the soliton solution is derived.

Keywords: high-order continuum model, density waves, KdV equation
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1. Introduction

Traffic jams are an important issue from the viewpoint of transportation efficiency and reduction in pollution, which have thus attracted much attention recently. Lots of studies have been conducted with different traffic models, such as the cellular automaton models, car-following models, gas kinetic models and hydrodynamic models (see Refs.[1–16]). The first significant development in continuum models was developed independently by Lighthill and Whitham,[7,8] and Richards[9] (the LWR model, for short). In the LWR model, the reason for the waves described as "kinematic" not "dynamic" lies in that only the kinematic properties of traffic flow are concerned with and traffic waves possess only one wave velocity, while "dynamic" waves possess at least two. The idea of the LWR model was pursued further until 1971, and it was Payne[10] who introduced a high-order continuum traffic flow model including a dynamic equation, which is derived from the car-following theory:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{\mu}{\rho T}\frac{\partial \rho}{\partial x}+\frac{V(\rho)-v}{T}, \tag{1}$$

where T is the relaxation time, and $\mu=-0.5\partial V/\partial\rho$ is the anticipation coefficient. The first term on the right-hand side of Eq.(1) is called the anticipation term, which reflects a driver's reaction to the car in front of him/her. The second term on the right-hand side represents a relaxation to the equilibrium state. The Payne model improved the LWR model by incorporating the equation of motion and taking into account the acceleration and inertial effects. The model could be used to describe the amplification of small disturbances in heavy traffic and allow fluctuations of speed around the equilibrium state. However, as was pointed out by Daganzo[11], one of characteristic speeds in the Payne model is greater than the macroscopic traffic flow velocity, that is to say,the vehicle behind can influence vehicle ahead, even in single-lane traffic. In general, a fast vehicle cannot force a slow one to speed up. The drivers' reaction mainly to front vehicles is referred as the anisotropy of traffic flow. Later on, a few other high-order continuum models have been developed.[12−15]

An anisotropic macroscopic continuum model was put forward by Jiang *et al*,[16] which comprised two partial differential equations as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{V(\rho)-v}{T}+c_0\frac{\partial v}{\partial x}, \tag{3}$$

where c_0 represents the propagation speed of the dis-

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turbance. It differs from the others in the equation of motion Eq.(3), in which the speed gradient term replaces the density gradient term as the anticipation term. The anisotropy of the model is shown by the method of characteristics. We rewrite Eqs.(2) and (3) as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = E, \tag{4}$$

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} v & \rho \\ 0 & v - c_0 \end{pmatrix},$$
$$E = \begin{pmatrix} 0 \\ (V(\rho) - v)/T \end{pmatrix}.$$

The eigenvalues of the matrix A are derived from

$$\det|A - \lambda I| = 0, \tag{5}$$

where I is the identity matrix. So we have $\lambda_1 = v, \quad \lambda_2 = v - c_0$. Since $c_0 \geq 0$, it follows that the characteristic speeds $\mathrm{d}x/\mathrm{d}t$ are always less than or equal to the macroscopic flow velocity v, which demonstrates the fundamental principle that vehicle flows are anisotropic and response only to frontal stimuli.

The vehicle density could fluctuate for various reasons to form a density wave in traffic flow, which leads to the traffic jams. Different nonlinear wave equations have been derived to describe the corresponding density waves, among which the Burgers, Korteweg–de-Vries (KdV) and mKdV equations depict the density waves appearing in the distinct regions respectively. A lot of research work has been done in this aspect for the car-following model,[17−20] while for the continuum model little work has been reported. Kurtze and Hong[21] derived the KdV equation from the hydrodynamic model and showed that the traffic soliton appears near the neutral stability line. In 2000, Berg *et al*[22] developed a systematic method for relating the car-following model with the continuum model of road traffic. In the light of this relation, the density waves in car following models were studied from another viewpoint.[23,24]

In this paper, as an extended work to the anisotropic continuum model proposed by Jiang *et al*,[16] the nonlinear analysis methods are applied to analyse the model. We derive the KdV equation by using the relation between the car following model and the continuum model. The KdV soliton solution is firstly given to describe the traffic jam.

2. Model and stability analysis

The optimal velocity model was put forward by Bando *et al*,[25] in which the acceleration of every car is determined by its velocity v_n and the optimal speed $V(\Delta x_n)$ as follows:

$$\dot{v}_n = a[V(\Delta x_n) - v_n], \tag{6}$$

where Δx_n is the headway between the car considered and the car ahead, $a = 1/T$ is the driver's sensitivity, and the over-dot represents the derivative with respect to time. The optimal velocity function is selected as

$$V(\Delta x_n) = \tanh(\Delta x_n - h_\mathrm{c}) + \tanh(h_\mathrm{c}). \tag{7}$$

It is obvious that the headway Δx is related to the traffic flow density ρ. The approximate relation is[22]

$$\Delta x \approx \frac{1}{\rho} - \frac{\rho_x}{2\rho^3} - \frac{\rho_{xx}}{6\rho^4}. \tag{8}$$

The first term denotes the classical transformation for relating them; the second term is similar to a pressure term in gas kinetics and acts to destabilize the traffic flow; the third term smoothes variations in traffic density and has a stabilizing effect on traffic flow, which counteracts the pressure term. We rewrite Eq.(3) as

$$\begin{aligned} &\frac{\partial v}{\partial t} + (v - c_0)\frac{\partial v}{\partial x} \\ &= a[\bar{V}(\rho) - v] + a\bar{V}'(\rho)\left(\frac{\rho_x}{2\rho} + \frac{\rho_{xx}}{6\rho^2}\right), \end{aligned} \tag{9}$$

where

$$\bar{V}(\rho) = V(1/\rho),$$
$$\frac{\partial V(\Delta x)}{\Delta x}|_{\Delta x = 1/\rho} = -\rho^2\bar{V}'(\rho).$$

Suppose that traffic flow is initially in a state differing infinitesimally from the homogeneous flow,

$$\begin{pmatrix} \rho(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \rho_\mathrm{h} \\ v_\mathrm{h} \end{pmatrix} + \sum_k \begin{pmatrix} \hat{\rho}_\mathrm{k} \\ \hat{v}_\mathrm{k} \end{pmatrix} \exp(\mathrm{i}kx + \sigma_k t). \tag{10}$$

Substituting Eq.(10) into Eqs.(2) and (9), we have

$$(\sigma_k + \mathrm{i}kv_\mathrm{h})\hat{\rho}_k + \mathrm{i}k\rho_\mathrm{h}\hat{v}_k = 0, \tag{11}$$

$$\begin{aligned} &\sigma_k\hat{v}_k + \mathrm{i}k(v_\mathrm{h} - c_0)\hat{v}_k \\ &= a[\bar{V}'(\rho_\mathrm{h})\hat{\rho}_k - \hat{v}_k] \\ &\quad + a\bar{V}'(\rho_\mathrm{h})\left(\frac{\mathrm{i}k\hat{\rho}_k}{2\rho_\mathrm{h}} - \frac{k^2\hat{\rho}_k}{6\rho_\mathrm{h}^2}\right). \end{aligned} \tag{12}$$

Linearizing in $\hat{\rho}_k$ and $\hat{v}_k$, we find that σ_k satisfies the quadratic equation

$$(\sigma_k + \mathrm{i}kv_\mathrm{h})^2 + (a - \mathrm{i}kc_0)(\sigma_k + \mathrm{i}kv_\mathrm{h}) + \mathrm{i}ka\rho_\mathrm{h}\bar{V}'(\rho_\mathrm{h})\left(1 + \frac{\mathrm{i}k}{2\rho_\mathrm{h}} - \frac{k^2}{6\rho_\mathrm{h}^2}\right) = 0. \quad (13)$$

The traffic flow will remain stable as long as both roots of σ_k have negative real parts, and thus we have

$$a > -2\rho_\mathrm{h}^2\bar{V}'(\rho_\mathrm{h})\left(1 - \frac{k^2}{6\rho_\mathrm{h}^2}\right)^2 - 2c_0\rho_\mathrm{h}\left(1 - \frac{k^2}{6\rho_\mathrm{h}^2}\right), \quad (14)$$

$$\mathrm{Im}(\sigma_k) \approx -k[v_\mathrm{h} + \rho_\mathrm{h}\bar{V}'(\rho_\mathrm{h})] + O(k^3). \quad (15)$$

The neutral stability condition of the traffic flow is

$$a_\mathrm{s} = -2\rho_\mathrm{h}^2\bar{V}'(\rho_\mathrm{h}) - 2c_0\rho_\mathrm{h}. \quad (16)$$

From Eq.(15), we can see that the critical disturbance travels with a speed

$$c(\rho_\mathrm{h}) = v_h + \rho_\mathrm{h}\bar{V}'(\rho_\mathrm{h}), \quad (17)$$

which is in agreement with the kinematic wave speed mentioned in Ref.[16].

3. Nonlinear analysis

We focus on the system behaviour near the neutral stability condition determined by Eq.(16), and transform the reference system of the wave by introducing the coordinate[23]

$$z = x - ct. \quad (18)$$

Then Eqs.(2) and (9) change respectively into

$$-c\rho_z + q_z = 0, \quad (19)$$

$$-cv_z + (v - v_0)v_z = a[\bar{V}(\rho) - v] + a\bar{V}'(\rho)\left(\frac{\rho_z}{2\rho} + \frac{\rho_{zz}}{6\rho^2}\right), \quad (20)$$

where q is the product of the density and speed, and

$$v_z = \frac{c\rho_z}{\rho} - \frac{q\rho_z}{\rho^2}. \quad (21)$$

Equation (20) can be written as

$$-\frac{\rho_z}{\rho^3}q^2 + \left[\frac{(c_0 + 2c)\rho_z}{\rho^2} + \frac{a}{\rho}\right]q = \frac{(c^2 + cc_0)\rho_z}{\rho} + a\bar{V}(\rho) + a\bar{V}'(\rho)\left(\frac{\rho_z}{2\rho} + \frac{\rho_{zz}}{6\rho^2}\right). \quad (22)$$

We expand the flow q as

$$q = \rho\bar{V} + b_1\rho_z + b_2\rho_{zz}. \quad (23)$$

The leading term represents the homogeneous stable flow q. The parameters b_1 and b_2 can be determined through the balance of Eq.(22) in ρ_z and ρ_{zz}, which turns out

$$b_1 = \frac{\bar{V}'}{2} + \frac{(\bar{V} - c)^2 + c_0(c - \bar{V})}{a}, \quad (24)$$

$$b_2 = \frac{\bar{V}'}{6\rho}. \quad (25)$$

Near the neutral stability condition, taking Eqs.(16) and (17) into account, we have $b_1 = 0$, so Eq.(19) turns into

$$-c\rho_z + (\rho\bar{V})_z + \frac{\bar{V}'}{6\rho_h}\rho_{zzz} = 0. \quad (26)$$

Let $\rho = \rho_\mathrm{h} + \hat{\rho}(x,t)$, so we have

$$\rho\bar{V} \approx \rho_\mathrm{h}\bar{V}(\rho_\mathrm{h}) + (\rho\bar{V})_\rho|_{\rho=\rho_\mathrm{h}}\hat{\rho} + \frac{1}{2}(\rho\bar{V})_{\rho\rho}|_{\rho=\rho_\mathrm{h}}\hat{\rho}^2. \quad (27)$$

Substituting Eq.(27) into Eq.(26) and dropping the "hat" we get

$$-c\rho_z + [(\rho\bar{V})_\rho + (\rho\bar{V})_{\rho\rho}\rho]\rho_z + \frac{\bar{V}'}{6\rho_\mathrm{h}}\rho_{zzz} = 0, \quad (28)$$

where c and $\bar{V}'/6\rho_\mathrm{h}$ are negative. In order to derive the standard KdV equation from Eq.(28), we make the following transformations:

$$U = -[(\rho\bar{V})_\rho + (\rho\bar{V})_{\rho\rho}\rho],$$
$$X = \sqrt{-\frac{6\rho_\mathrm{h}}{\bar{V}'}}x, \quad T = -\sqrt{-\frac{6\rho_h}{\bar{V}'}}t.$$

Inserting these into the above equation, we obtain the regularized equation

$$U_\mathrm{T} + UU_\mathrm{X} + U_\mathrm{XXX} = 0. \quad (29)$$

The solution of this KdV equation is

$$U = -3c\mathrm{sech}^2\left[\frac{\sqrt{-c}}{2}(X + cT)\right]. \quad (30)$$

Finally, we get the soliton solution

$$\rho = \frac{3c}{(\rho\bar{V})_{\rho\rho}}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{6c\rho_h}{\bar{V}'}}(x - ct)\right] - \frac{(\rho\bar{V})_\rho}{(\rho\bar{V})_{\rho\rho}}. \quad (31)$$

4. Summary

The new anisotropic continuum model overcomes the characteristic speed problem, and the wrong-way travel issue does not exist in this model. As an extended work to it, the linear stability analysis is given. The neutral stability condition is obtained, which indicates that the model is unstable with small disturbance as the sensitivity a is less than or equal to some fixed value. So the density wave results in the traffic congestion. Through transforming the headway into density by using an asymptotic approximation, the KdV equation has been derived to describe the traffic behaviour near the neutral stability condition. The result is consistent with that in previous work.[23] Comparing the two results, we find that the neutral stability condition is different, but the density wave solution is the same.

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