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A new supersymmetric classical Boussinesq equation*

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In this paper, we obtain a supersymmetric generalization for the classical Boussinesq equation. We show that the supersymmetric equation system passes the Painlevé test and we also calculate its one- and two-soliton solutions.

Keywords: classical Boussinesq equation, supersymmetric integrable system, solitons, Painlevé analysis

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1. Introduction

Solitons and integrable systems have been studied extensively during the last 40 years and many differential or difference equations in physics and engineering have been found to be integrable.[1] It is interesting to note that the whole theory originated from the study of a single equation, namely the celebrated Korteweg–de Vries (KdV) equation.

The KdV equation has many interesting extensions. Among them, super-extensions have attracted much attention. The first example was constructed by Kupershmidt in 1984.[2] Then Manin and Radul, in their study of the supersymmetric Kadomtsev–Petviashvili (KP) hierarchy, presented a second candidate for the super KdV equation.[3] It was Mathieu who pointed out that while Kupershmidt's super KdV equation is a fermionic extension, Manin and Radul did construct a truly supersymmetric KdV (sKdV) equation.[4] Since then, many properties have been established for the sKdV equation.[5−10]

Apart from the KdV equation, the classical Boussinesq (CB) equation is another important equation. It is a dispersive generalization of the classical dispersiveless long wave equation and was derived from the theory of water waves by Broer and independently by Kaup,[11] so it is also called the dispersive water wave equation or Broer–Kaup equation. The above three names are not only ones coined for the classical Boussinesq equation. Indeed, the system appears in many different (but equivalent) forms and there are other names for it such as Jaulent–Modik system[12] or the two-boson system.[13] Now the reports on this equation are rich and many properties have been established for it. For instance, the CB system is known to have various solutions including standard solitons,[11] fusion and fission solutions,[14] explode-decay solitary wave solutions.[15] Also, it is the first equation having three local Hamiltonian structures[16] and it can be rewritten in so-called trilinear form[17] and Bäcklund (Darboux) transformations are constructed.[18,19] For $(2+1)$-dimensional and high-order Broer–Kaup equation, some properties have been presented too.[20,21] Interestingly, the CB hierarchy embodies many equations as particular examples. Kupershmidt called this system "the richest integrable system known to date".

The CB system may be formulated in the following form (among others):

$$u_t + uu_x + w_x = 0, \tag{1a}$$

$$w_t + u_{xxx} + (uw)_x = 0. \tag{1b}$$

It is interesting that this system enjoys two different bilinearizations (see Ref.[22] for example). That is, on the one hand, we take the following dependent variable transformations:

$$u = 2\left(\ln\left(\frac{\sigma}{\tau}\right)\right)_x, \quad w = 2(\ln(\sigma\tau))_{xx},$$

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then the CB system (1a) and (1b) is converted into

$$(D_t + D_x^2)\sigma\tau = 0, \tag{2a}$$
$$(D_x D_t + D_x^3)\sigma\tau = 0. \tag{2b}$$

On the other hand, we may use

$$u = 2\left(\ln\left(\frac{\sigma}{\tau}\right)\right)_x, \quad w = 1 + 2(\ln(\sigma\tau))_{xx},$$

then the CB system is converted into

$$(D_t + D_x^2)\sigma\tau = 0, \tag{3a}$$
$$(D_x D_t + D_x + D_x^3)\sigma\tau = 0. \tag{3b}$$

As pointed out in Ref.[23], while calculating the standard soliton based on Eqs.(3a, 3b), one may derive the fusion and fission solutions from Eqs.(2a, 2b). Therefore, two different forms take different responsibilities for the appearance of different solutions.

Recently, Brunelli and Das[24,25] proposed a supersymmetric generalization for the CB or two-boson system. It reads

$$\Phi_{0,t} = -(\mathcal{D}^4\Phi_0) + (\mathcal{D}(\mathcal{D}\Phi_0)^2) + 2(\mathcal{D}^2\Phi_1), \tag{4a}$$
$$\Phi_{1,t} = (\mathcal{D}^4\Phi_1) + 2(\mathcal{D}^2((\mathcal{D}\Phi_0)\Phi_1)), \tag{4b}$$

where $\Phi_0 = \Phi_0(x,t,\theta)$ and $\Phi_1 = \Phi_1(x,t,\theta)$ are Grassmann odd variables depending on super spatial variables x, θ and usual temporal variable t. $\mathcal{D}$ is the super derivative defined as $\mathcal{D} = \dfrac{\partial}{\partial\theta} + \theta\dfrac{\partial}{\partial x}$.

Like the CB system, its supersymmetric counterpart (4a, 4b) has many interesting properties.[25]

More recently, Liu and Yang[26] have proposed the following dependent variable transformation

$$\Phi_0 = \mathcal{D}\left(\ln\frac{\tau_2}{\tau_1}\right), \quad \Phi_1 = (\mathcal{D}\ln\tau_2)_x,$$

and shown that the system (4a, 4b) is bilinearized as

$$(D_t + D_x^2)\tau_1\tau_2 = 0, \tag{5a}$$
$$(SD_t + SD_x^2)\tau_1\tau_2 = 0, \tag{5b}$$

where the Hirota derivative is defined as

$$SD_t^m D_x^n f\cdot g = (\mathcal{D}_{\theta_1} - \mathcal{D}_{\theta_2})\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1,t_1,\theta_1)g(x_2,t_2,\theta_2)\Big|_{\substack{x_1=x_2\\ t_1=t_2\\ \theta_1=\theta_2}}.$$

Based on this bilinearization and the standard Hirota perturbation method, we have found for Eqs.(4a, 4b) a class of solutions which is analogous to the fission and fusion solutions of the CB system. Furthermore, a bilinear Bäcklund transformation is also constructed.

It is interesting to remark that our bilinearization for the supersymmetric CB system (4a, 4b) can be considered as a generalization of Eqs.(2a, 2b). In this paper, we will construct a new supersymmetric CB system by proposing a supersymmetric version for Eqs.(3a, 3b). Converting the bilinear form into nonlinear one, we will show that our system does reduce to the CB system.

This paper is organized as follows: in the next section, we will derive new equations from Eqs.(6a, 6b). In Section 3, we will perform a Painlevé analysis for the new equations and show that the equations pass the Painlevé test. In Section 4, using Hirota's bilinear method, we will calculate one-soliton, two-soliton solutions to the new equations. The last section summarizes the results briefly.

2. New equation

We start with the following equations in bilinear form:

$$(D_t + D_x^2)\tau_1\cdot\tau_2 = 0, \tag{6a}$$
$$(SD_t + S + SD_x^2)\tau_1\cdot\tau_2 = 0, \tag{6b}$$

which can be considered as a supersymmetric extension for the Hirota bilinear equations (3a, 3b). We now consider the bilinear equations (6a, 6b) as a new system and convert them into nonlinear evolution version. To this end, we define new variables m, n as

$$m = \ln\left(\frac{\tau_2}{\tau_1}\right), \quad n = \ln\tau_2.$$

Then our bilinear equations (6a), (6b) can be reformulated as

$$m_t + m_{xx} - m_x^2 - 2n_{xx} = 0, \tag{7a}$$

$$\begin{aligned}&(\mathcal{D}m)m_t - (\mathcal{D}m_t) + 2(\mathcal{D}n_t) - (\mathcal{D}m_{xx})\\ &+ (\mathcal{D}m)(-2n_{xx} + m_{xx} - m_x^2 - 1)\\ &- 4(\mathcal{D}n_x)m_x + 2m_x(\mathcal{D}m_x) = 0.\end{aligned} \tag{7b}$$

Substituting Eq.(7a) into Eq.(7b), we eliminate m_t and obtain

$$(\mathcal{D}n_t) - (\mathcal{D}n_{xx}) - \frac{1}{2}(\mathcal{D}m) - 2m_x(\mathcal{D}n_x) = 0. \tag{8}$$

Then differentiating Eqs.(7a) and (8), we obtain

$$m_{xt} = 2m_x m_{xx} + 2n_{xxx} - m_{xxx}, \tag{9a}$$

$$\begin{aligned}(\mathcal{D}n_{xt}) = (\mathcal{D}n_{xxx}) &+ \frac{1}{2}(\mathcal{D}m_x)\\ &+ 2m_{xx}(\mathcal{D}n_x) + 2m_x(\mathcal{D}n_{xx}).\end{aligned} \tag{9b}$$

Now we introduce new variables $M = m_x$, $\Lambda = (\mathcal{D}n_x)$. Then Eqs.(9a) and (9b) yield

$$M_t = 2MM_x + 2(\mathcal{D}\Lambda_x) - M_{xx}, \tag{10a}$$

$$\Lambda_t = \Lambda_{xx} + \frac{1}{2}(\mathcal{D}M) + 2(M\Lambda)_x. \tag{10b}$$

The above system is our supersymmetric candidate for the CB system. To compare it with the version of Brunelli and Das, we rewrite the system (4a, 4b) as

$$M_{0,t} = 2M_0M_{0,x} + 2(\mathcal{D}\Phi_{1,x}) - M_{0,xx}, \tag{11a}$$

$$\Phi_{1,t} = \Phi_{1,xx} + 2(M_0\Phi_1)_x, \tag{11b}$$

where $M_0 = \mathcal{D}\Phi_0$. We see that while the first equation of the above system is the same as Eq.(10a), the second one is different from Eq.(10b). Indeed, this second equation is an exact derivative with respect to x but Eq.(10b) is not.

To show that our system is indeed a supersymmetric extension of the CB system, we take $M = u + \theta\xi$, $\Lambda = \zeta + \theta w$. Then Eqs.(10a) and (10b) in components read

$$u_t = (2w + u^2 - u_x)_x, \tag{12a}$$

$$w_t = \left(w_x + \frac{1}{2}u + 2uw + 2\xi\zeta\right)_x, \tag{12b}$$

$$\xi_t = (-\xi_x + 2u\xi + 2\zeta_x)_x, \tag{12c}$$

$$\zeta_t = \zeta_{xx} + \frac{1}{2}\xi + 2(u\zeta)_x. \tag{12d}$$

It is easy to see that when the fermionic fields vanish, Eqs.(12c)–(12d) hold identically and the other two equations reduce to

$$u_t = (2w + u^2 - u_x)_x, \tag{13a}$$

$$w_t = \left(2uw + w_x + \frac{1}{2}u\right)_x. \tag{13b}$$

which are nothing but the CB system up to a simple invertible change of dependent variables.

3. Painlevé analysis

A natural question about our sCB system proposed in the last section is wether it is integrable or not. To this end, we appeal to Painlevé analysis, which has been a very useful method to deal with the classical soliton equations.[1,27–30] Indeed, such an analysis often leads to many interesting results such as Bäcklund transformations, Hirota's bilinear form or even Lax representations for a given system. Painlevé analysis has been extended into super integrable systems by Mathieu[5] and applied to some super systems by others.[31] In this section, we will perform such an analysis for our sCB system (12a–12d).

We start with the following series expansions for the field variables:

$$\begin{aligned}u &= \phi^\alpha \sum_{j=0}^{\infty} u_j(t)\phi^j, \quad w = \phi^\beta \sum_{j=0}^{\infty} w_j(t)\phi^j,\\ \xi &= \phi^\gamma \sum_{j=0}^{\infty} \xi_j(t)\phi^j, \quad \zeta = \phi^\lambda \sum_{j=0}^{\infty} \zeta_j(t)\phi^j,\end{aligned} \tag{14}$$

where we assume the singularity surface of the solutions to have the form

$$\phi = x + g(t), \tag{15}$$

that is, we are using Kruskal's ansatz to simplify the analysis. We further assume that the zeroth order coefficients in the expansions do not vanish, namely,

$$u_0 \neq 0, \quad w_0 \neq 0, \quad \xi_0 \neq 0, \quad \zeta_0 \neq 0.$$

In passing, we point out that while the coefficients $u_j(t)$ and $w_j(t)$ are bosonic, $\xi_j(t)$ and $\zeta_j(t)$ are fermionic.

Substituting

$$u \sim u_0\phi^\alpha, \quad w \sim w_0\phi^\beta, \quad \xi \sim \xi_0\phi^\gamma, \quad \zeta \sim \zeta_0\phi^\lambda$$

into Eqs.(12a)–(12d) and using the leading term analysis give the following possibilities for $\alpha, \beta, \gamma, \lambda$:

Case (i): $\alpha = -1, \quad \beta = 0, \quad \gamma = 1, \quad \lambda = 2,$

Case (ii): $\alpha = -1, \quad \beta = 0, \quad \gamma = 1, \quad \lambda = 3,$

Case (iii): $\alpha=-1,\quad \beta=0,\quad \gamma=2,\quad \lambda=2,$

Case (iv): $\alpha=-1,\quad \beta=0,\quad \gamma=1,\quad \lambda=1,$

Case (v): $\alpha=-1,\quad \beta=-2,\quad \gamma=1,\quad \lambda=3,$

Case (vi): $\alpha=-1,\quad \beta=-2,\quad \gamma=2,\quad \lambda=4,$

Case (vii): $\alpha=-1,\quad \beta=-2,\quad \gamma=1,\quad \lambda=1.$

To show that our sCB system passes the Painlevé test, we should work case by case. Since calculations involved here are tedious, we will give the details only for the Case (i) in the Appendix. For all the seven cases, we next list their resonances, and show the compatibility for the corresponding α,β,γ and λ.

For Case (i), the resonances arise at

$$j=-3,-1,-1,0,0,1,2,2.$$

It is known that $j=-1$ corresponds to the arbitrariness in the location of the singularity surface. The other five non-negative resonance values correspond exactly five arbitrary functions

$$\xi_0,\zeta_0,w_1,u_2,w_2;$$

and the other lower order coefficients are given by

$$\begin{aligned}&u_0=-1,\quad w_0=-\frac{1}{4},\quad u_1=\frac{1}{2}\phi_t,\quad \xi_1=\zeta_0,\\&\zeta_1=-\frac{1}{4}\xi_0,\quad \xi_2=-\frac{3}{10}\xi_0+\frac{2}{5}u_2\xi_0-\frac{1}{10}\xi_{0,t},\\&\zeta_2=-\frac{1}{12}\zeta_0-u_2\zeta_0+\frac{1}{6}\zeta_{0,t}.\end{aligned}$$

For Case (ii), the resonances arise at

$$j=-1,-1,-2,-3,0,1,2,2;$$

and there are four arbitrary functions

$$\zeta_0,w_1,u_2,w_2.$$

The other lower order coefficients read

$$\begin{aligned}&u_0=-1,\quad w_0=-\frac{1}{4},\quad \xi_0=-4\zeta_0,\quad u_1=\frac{1}{2}\phi_t,\\&\xi_1=\zeta_1=0,\quad \xi_2=\frac{6}{5}\zeta_0-\frac{8}{5}u_2\zeta_0+\frac{2}{5}\zeta_{0,t},\\&\zeta_2=-\frac{1}{20}\zeta_0-\frac{3}{5}u_2\zeta_0+\frac{1}{15}\zeta_{0,t}.\end{aligned}$$

For Case (iii), we find that the resonances occur at

$$j=-1,-1,-1,-4,0,1,2,2;$$

and the arbitrary functions are

$$\zeta_0,w_1,u_2,w_2.$$

The lower order coefficients read

$$\begin{aligned}&u_0=-1,\quad w_0=-\frac{1}{4},\quad \xi_0=\zeta_0,\\&u_1=\frac{1}{2}\phi_t,\quad \xi_1=\zeta_1=0,\\&\xi_2=-\frac{1}{9}\zeta_0-u_2\zeta_0+\frac{1}{6}\zeta_{0,t},\\&\zeta_2=-\frac{1}{12}\zeta_0-u_2\zeta_0+\frac{1}{6}\zeta_{0,t}.\end{aligned}$$

For Case (iv) we have

$$j=-1,-3,0,0,1,1,2,2;$$

and the arbitrary functions are

$$\xi_0,\zeta_0,w_1,\xi_1,u_2,w_2.$$

Then the low order coefficients read

$$\begin{aligned}&u_0=-1,\quad w_0=-\frac{1}{4},\quad u_1=\frac{1}{2}\phi_t,\quad \zeta_1=\xi_1,\\&\xi_2=-\frac{3}{10}\xi_0+\frac{2}{5}u_2\xi_0-\frac{12}{5}u_2\zeta_0-\frac{1}{10}\xi_{0,t}+\frac{3}{5}\zeta_{0,t},\\&\zeta_2=-\frac{1}{4}\xi_0-2u_2\zeta_0+\frac{1}{2}\zeta_{0,t}.\end{aligned}$$

For Case (v), we obtain

$$j=-1,-2,-5,0,1,2,3,4;$$

and the arbitrary functions are

$$\xi_0,\xi_1,u_2,u_3,u_4.$$

Then the low order coefficients read

$$\begin{aligned}&u_0=1,\quad w_0=-1,\quad \zeta_0=-\frac{1}{20}\xi_0,\quad u_1=\frac{1}{2}\phi_t,\\&w_1=0,\quad \zeta_1=-\frac{1}{36}\xi_1,\quad w_2=u_2-\frac{1}{4},\\&\xi_2=-\frac{3}{10}\xi_0+2u_2\xi_0+\phi_t\xi_1-\frac{1}{2}\xi_{0,t},\\&\zeta_2=\frac{3}{560}\xi_0-\frac{3}{140}u_2\xi_0-\frac{1}{56}\phi_t\xi_1+\frac{1}{140}\xi_{0,t},\\&w_3=\frac{1}{4}\phi_{tt},\quad \xi_3=-\frac{1}{9}\xi_1+u_2\xi_1+u_3\xi_0-\frac{1}{6}\xi_{1,t},\\&\zeta_3=\frac{1}{720}\xi_1-\frac{1}{180}u_2\xi_1+\frac{1}{720}\xi_{1,t},\\&w_4=-\frac{1}{2}u_2^2+\frac{1}{2}u_4+\frac{1}{4}u_{2,t},\end{aligned}$$

$$
\begin{aligned}
\xi_4 =& -\frac{1}{12}\phi_{tt}\xi_1 - \frac{1}{12}\phi_t\xi_{1,t} - \frac{1}{6}u_{2,t}\xi_0 - \frac{1}{2}u_2\xi_{0,t} \\
&+\frac{41}{840}\xi_{0,t} + \frac{1}{24}\xi_{0,tt} - \frac{19}{70}u_2\xi_0 - \frac{5}{84}\phi_t\xi_1 \\
&+\frac{1}{56}\xi_0 + \frac{2}{3}u_2\phi_t\xi_1 + \frac{4}{3}u_2^2\xi_2 + \frac{2}{3}u_3\xi_1 + \frac{2}{3}u_4\xi_0, \\
\zeta_4 =& -\frac{1}{6048}\xi_0 - \frac{1}{2835}\xi_{0,t} + \frac{1}{756}u_2\xi_0 + \frac{5}{9072}\phi_t\xi_1 \\
&+\frac{1}{2268}\phi_{tt}\xi_1 + \frac{1}{2268}\phi_t\xi_{1,t} + \frac{13}{11340}u_{2,t}\xi_0 \\
&+\frac{1}{378}u_2\xi_{0,t} - \frac{23}{90720}\xi_{0,tt} - \frac{5}{2268}u_2\phi_t\xi_1 \\
&-\frac{43}{5670}u_2^2\xi_0 + \frac{2}{405}u_4\xi_0.
\end{aligned}
$$

For Case (vi), we obtain

$$j = -1, -1, -3, -6, 0, 2, 3, 4;$$

and the arbitrary functions are

$$\xi_0, u_2, u_3, u_4.$$

The low order coefficients read

$$
\begin{aligned}
&u_0 = 1, \quad w_0 = -1, \quad \zeta_0 = -\frac{1}{36}\xi_0, \quad u_1 = \frac{1}{2}\phi_t, \\
&w_1 = \xi_1 = \zeta_1 = 0, \quad w_2 = u_2 - \frac{1}{4}, \\
&\xi_2 = -\frac{1}{9}\xi_0 + u_2\xi_0 - \frac{1}{6}\xi_{0,t}, \\
&\zeta_2 = \frac{1}{720}\xi_0 - \frac{1}{180}u_2\xi_0 + \frac{1}{720}\xi_{0,t}, \quad w_3 = \frac{1}{4}\phi_{tt}, \\
&\xi_3 = \frac{2}{3}u_3\xi_0, \quad \zeta_3 = 0, \quad w_4 = -\frac{1}{2}u_2^2 + \frac{1}{2}u_4 + \frac{1}{4}u_{2,t}, \\
\xi_4 =& \frac{1}{120}\xi_{0,tt} - \frac{1}{20}u_{2,t}\xi_0 - \frac{2}{15}u_2\xi_{0,t} + \frac{7}{720}\xi_{0,t} \\
&-\frac{13}{180}u_2\xi_0 + \frac{1}{240}\xi_0 + \frac{1}{2}u_4\xi_0 + \frac{1}{2}u_2^2\xi_0, \\
\zeta_4 =& \frac{1}{504}u_4\xi_0 + \frac{1}{1680}u_2\xi_{0,t} - \frac{31}{12600}u_2^2\xi_0 \\
&+\frac{1}{4200}u_2\xi_0 - \frac{1}{25200}\xi_{0,tt} + \frac{1}{3600}u_{2,t}\xi_0 \\
&-\frac{1}{20160}\xi_{0,t} - \frac{1}{33600}\xi_0.
\end{aligned}
$$

For the last case, Case (vii), we have

$$j = -1, -3, 0, 0, 1, 2, 3, 4;$$

and the arbitrary functions are

$$\xi_0, \zeta_0, \xi_1, u_2, u_3, u_4.$$

The low order coefficients read

$$
\begin{aligned}
&u_0 = 1, \quad w_0 = -1, \quad u_1 = \frac{1}{2}\phi_t, \\
&w_1 = \zeta_1 = 0, \quad w_2 = u_2 - \frac{1}{4}, \\
&\xi_2 = -\frac{3}{10}\xi_0 + 2u_2\xi_0 - \frac{12}{5}u_2\zeta_0 - \frac{1}{2}\xi_{0,t} + \frac{3}{5}\zeta_{0,t}, \\
&\zeta_2 = -\frac{1}{20}\xi_0 - \frac{2}{5}u_2\zeta_0 + \frac{1}{10}\zeta_{0,t}, \quad w_3 = \frac{1}{4}\phi_{tt}, \\
&\xi_3 = -\frac{1}{9}\xi_1 + u_3\xi_0 - \frac{4}{3}u_3\zeta_0 + u_2\xi_1 - \frac{1}{6}\xi_{1,t}, \\
&\zeta_3 = -\frac{1}{36}\xi_1 - \frac{1}{3}u_3\zeta_0, \quad w_4 = -\frac{1}{2}u_2^2 + \frac{1}{2}u_4 + \frac{1}{4}u_{2,t}, \\
\xi_4 =& -\frac{1}{6}u_{2,t}\xi_0 - \frac{1}{2}u_2\xi_{0,t} + \frac{1}{24}\xi_{0,tt} + \frac{16}{105}u_{2,t}\zeta_0 \\
&+\frac{16}{35}u_2\zeta_{0,t} + \frac{41}{840}\xi_{0,t} - \frac{4}{105}\zeta_{0,tt} - \frac{19}{70}u_2\xi_0 \\
&+\frac{1}{7}u_2\zeta_0 + \frac{1}{56}\xi_0 - \frac{1}{28}\zeta_{0,t} - \frac{128}{105}u_2^2\zeta_0 \\
&-\frac{20}{21}u_4\zeta_0 + \frac{4}{3}u_2^2\xi_0 + \frac{2}{3}u_3\xi_1 + \frac{2}{3}u_4\xi_0, \\
\zeta_4 =& -\frac{1}{70}u_{2,t}\zeta_0 - \frac{3}{70}u_2\zeta_{0,t} + \frac{1}{140}\xi_{0,t} + \frac{1}{280}\zeta_{0,tt} \\
&-\frac{3}{140}u_2\xi_0 + \frac{3}{70}u_2\zeta_0 + \frac{3}{560}\xi_0 - \frac{3}{280}\zeta_{0,t} \\
&+\frac{4}{35}u_2^2\zeta_0 - \frac{2}{7}u_4\zeta_0.
\end{aligned}
$$

From the above result, we know that Eqs.(12a–12d) pass the Painlevé test. Therefore, the sCB system is a candidate for the integrable supersymmetric equation.

4. Soliton solutions

Since our sCB system (10a, 10b) has Eqs.(6a, 6b) as its Hirota's bilinear form, we may adopt the standard perturbation method to find possible soliton-like solutions. By tedious but straightforward calculation, we find one-soliton and two-soliton solutions, which are listed in the following.

- **One-soliton**:

$$
\begin{aligned}
\tau_1 &= 1 + \mathrm{e}^{\eta + \theta\xi}, \\
\tau_2 &= 1 + \mathrm{e}^{\eta + \theta\xi + m},
\end{aligned}
$$

where $\eta = kx - \omega t + c$, $\omega^2 = k^2 + k^4$, and $\mathrm{e}^m = \dfrac{\omega - k^2}{\omega + k^2}$.

- **Two-soliton**:

$$\begin{aligned}\tau_1 &= 1+\mathrm{e}^{\eta_1+\theta\xi_1}+\mathrm{e}^{\eta_2+\theta\xi_2}\\ &\quad+\mathrm{e}^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)+A_{12}},\\ \tau_2 &= 1+\mathrm{e}^{\eta_1+\theta\xi_1+m_1}+\mathrm{e}^{\eta_2+\theta\xi_2+m_2}\\ &\quad+\mathrm{e}^{\eta_1+\eta_2+\theta(\xi_1+\xi_2)+m_1+m_2+A_{12}},\end{aligned}$$

where $\eta_i = k_i x - \omega_i t + c_i, \omega_i^2 = k_i^2 + k_i^4$, and $\mathrm{e}^{m_i} = \dfrac{\omega_i - k_i^2}{\omega_i + k_i^2}, i = 1, 2$. The coupling constant A_{12} is given by

$$\mathrm{e}^{A_{12}} = \frac{k_1-k_2}{k_1+k_2}\frac{k_1^2\omega_2 - k_2^2\omega_1}{k_1^2\omega_2 + k_2^2\omega_1}, \quad \text{or}$$

$$\mathrm{e}^{A_{12}} = \frac{(k_1\omega_2 - k_2\omega_1)^2 - k_1^2k_2^2(k_1-k_2)^2}{(k_1\omega_2 - k_2\omega_1)^2 - k_1^2k_2^2(k_1+k_2)^2}.$$

These two expressions will coincide with the help of the relations between ω_i and k_i.

5. Discussions

In this paper, we have constructed a supersymmetric extension for the CB system. We have shown that our system has passed the Painlevé test. Furthermore, we have also calculated the one-soliton and two-soliton solutions. This evidence suggests that our system is integrable. It would be interesting to find other integrability properties of this system, such as Bäcklund transformation and Lax representation.

Appendix The Painlevé analysis of the system (10a, 10b) for Case (i)

A leading order singularity analysis yields

$$\alpha = -1, \quad \beta = 0, \quad \gamma = 1, \quad \lambda = 2.$$

Then, from Eq.(14), we have

$$\begin{aligned}u &= \sum_{j=0}^{\infty} u_j(t)\phi^{j-1}, \quad w = \sum_{j=0}^{\infty} w_j(t)\phi^j,\\ \xi &= \sum_{j=0}^{\infty} \xi_j(t)\phi^{j+1}, \quad \zeta = \sum_{j=0}^{\infty} \zeta_j(t)\phi^{j+2}. \quad \text{(A1)}\end{aligned}$$

Substituting Eqs.(15) and (A1) into Eqs.(12a–12d). We obtain the recursion relations

$$(j-1)(j-2)u_j - 2\sum_{m+n=j}(n-1)u_mu_n + u_{j-2,t} + (j-2)(\phi_t u_{j-1} - 2w_{j-2}) = 0, \quad \text{(A2)}$$

$$\frac{1}{2}(j-1)u_j + j(j-1)w_j + 2\sum_{m+n=j}(j-1)u_mw_n - w_{j-2,t} + 2\sum_{m+n=j-4}(j-1)\xi_m\zeta_n - (j-1)\phi_t w_{j-1} = 0, \quad \text{(A3)}$$

$$-2\sum_{m+n=j} ju_m\xi_n - 2j(j+1)\zeta_{j-1} + j(j+1)\xi_j + \xi_{j-2,t} + j\phi_t\xi_{j-1} = 0, \quad \text{(A4)}$$

$$(j+1)(j+2)\zeta_j + \frac{1}{2}\xi_{j-1} - \zeta_{j-2,t} + 2\sum_{m+n=j}(j+1)u_m\zeta_n - (j+1)\phi_t\zeta_{j-1} = 0. \quad \text{(A5)}$$

When $j = 0$, from the relations (A2–A5), we obtain

$$\begin{aligned}&u_0 = -1, \quad w_0 = -\frac{1}{4},\\ &\xi_0 = \text{arbitrary}, \quad \zeta_0 = \text{arbitrary}. \quad \text{(A6)}\end{aligned}$$

For the purpose of resonance analysis, we substitute Eq.(A6) into Eqs.(A2–A5) and rewrite Eqs.(A2–A5) in a more convenient form

$$(j+1)(j-2)u_j = F_1, \quad \text{(A7)}$$

$$(j-1)(j-2)w_j = F_2, \quad \text{(A8)}$$

$$2j\xi_0u_j - j(j+3)\xi_j = F_3, \quad \text{(A9)}$$

$$j(j+1)\zeta_j + 2(j+1)\zeta_0u_j = F_4, \quad \text{(A10)}$$

where F_1, F_2, F_3 and F_4 are functions independent of u_j, w_j, ξ_j and ζ_j. From the above Eqs.(A7–A10), the resonances are found to arise at

$$j = -3, -1, -1, 0, 0, 1, 2, 2.$$

The compatibility conditions at resonances have been verified in Section 3.

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