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q-NONLINEAR CAVITY FIELD STATES GENERATED BY THE EXCITATIONS ON A q-COHERENT STATE *

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It is shown that photon-excitation state defined by $|\alpha,k\rangle_q = \hat{a}_q^{\dagger k}|\alpha\rangle_q (k = 1,2,3...)$ up to a normalization constant can be produced in nonlinear processes in q-nonlinear cavities. The mathematical and quantum statistical properties of this state are studied in detail. We show that this state, along with the number states absented in it, forms a complete set. We also show by numerical method that this state exhibits quantum-squeezing for some values of $|\alpha|$ and always reveals quantum-antibunching effect.

Keywords: q-nonlinear cavity, photon-excitation state, completeness relation, nonclassical property.
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It is well known that the quantum harmonic oscillators (QHO) is one of the most fundamental objects in quantum physics. In quantum optics the QHO describes, in particular, the signal mode of the quantized cavity field. Owing to the fact that experimentally prepared cavities generally have some degrees of anharmonicity, a q-deformation quantum harmonic oscillator (q-QHO) model[1] described by Hamiltonian $\widehat{H}_q = (\hat{a}_q\hat{a}_q^\dagger + \hat{a}_q^\dagger\hat{a}_q)/2$ is introduced in the hope of providing a better description of physical situation involved, with the anharmonicity in the cavity potential being parameterized by the deformation parameter q [2], where the q-analogue annihilation and creation operators $\hat{a}_q$ and $\hat{a}_q^\dagger$, along with the q-number operator $\hat{N}_q$, form the q-deformed Heisenberg-Weyl algebra: $\hat{a}_q\hat{a}_q^\dagger - q\hat{a}_q^\dagger\hat{a}_q = q^{\hat{N}_q}$, $[\hat{N}_q,\hat{a}_q] = -\hat{a}_q, [\hat{N}_q,\hat{a}_q^\dagger] = \hat{a}_q^\dagger$. In the contraction limit $q \to 1$ this algebra reduces to the usual Heisenberg-Weyl algebra. As a consequence, the interaction Hamiltonian of atom-field system has the form[3]

$$\hat{H} = g(\hat{\sigma}_+\hat{a}_q + \hat{\sigma}_-\hat{a}_q^\dagger), \tag{1}$$

where g is the coupling constant and $\sigma_\pm$ are two-level atom operators. Let us consider the passage of the excited atom through a q-nonlinear cavity in initial q-analogue coherent state (q-CS)[4]:

$$|\alpha\rangle_q = A(z)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle_q, \quad A(\alpha) = (\sum_{n=0}^{\infty}\frac{|\alpha|^2}{[n]!})^{-\frac{1}{2}}, \tag{2}$$

where $\alpha = |\alpha|\exp(i\theta)$ is a complex number and $\{|n\rangle_q, n = 0,1,2,...\}$ are the basis of q-Fock space. The function $[x]$ is defined as $[x] = (q^x - q^{-x})/(q - q^{-1})$, the q-factorial $[n]!$ is given by the equation $[n]! = [n]\cdot[n-1]\cdots[2]\cdot[1]$. Note that $[x]$ is invariant under $q \leftrightarrow 1/q$. Since the coupling constant g is generally small, the state of atom-field system at time $t > 0$ can be approximated by

$$\begin{aligned}|\psi(t)\rangle_q &\cong |\alpha\rangle_q|e\rangle - i\hat{H}t|\alpha\rangle_q|e\rangle \\ &= |\alpha\rangle_q|e\rangle - igt\hat{a}_q^\dagger|\alpha\rangle_q|g\rangle,\end{aligned} \tag{3}$$

which is valid for interaction times such as $gt \ll 1$. Therefore, if we send an excited atom through the q-nonlinear cavity, the state of the q-nonlinear cavity field is reduced to the state $\hat{a}_q^\dagger|\alpha\rangle_q$ by detecting the atom to be in the ground state $|g\rangle$. As a consequence, in principle, the $k-$photon excitation state of q-CS:

$$\begin{aligned}|\alpha,+k\rangle_q &= C_k\hat{a}_q^{\dagger k}|\alpha\rangle_q \\ &= C_kA(\alpha)\sum_{n=0}^{\infty}\frac{\alpha^n\sqrt{[n+k]!}}{[n]!}|n+k\rangle_q,\end{aligned} \tag{4}$$

can be produced in the similar passage by sending the excited atom through the cavity with the initial field state $\hat{a}_q^{\dagger k-1}|\alpha\rangle_q$, where C_k with $C_0 = 1$ is the normalization constant. One can easily prove that the state (4) is nothing but the eigenstate of the operator

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$\hat{F}_k = \hat{a}_q^{\dagger k}\hat{a}_q\hat{a}_q^{\dagger -k}$ with eigenvalue α, i.e.,

$$\hat{a}_q^{\dagger k}\hat{a}_q\hat{a}_q^{\dagger -k}|\alpha,+k\rangle_q = \alpha|\alpha,+k\rangle_q. \tag{5}$$

Hence, the set $\{|\alpha,+k\rangle_q, k=1,2,3...\}$ can be treated as another family of the q-analogue photon-added coherent states besides the states $\{\hat{a}_q^{-k}|\alpha\rangle_q, k=1,2,3...\}$ studied in Ref. [5], where $\hat{a}_q^{-1}$ and $\hat{a}^{\dagger -1}$, defined by the equations

$$\begin{aligned}\hat{a}_q^{-1}|n\rangle_q &= \frac{1}{\sqrt{[n+1]}}|n+1\rangle_q,\\ \hat{a}_q^{\dagger -1}|n\rangle_q &= (1-\delta_{n,0})\frac{1}{\sqrt{[n]}}|n-1\rangle_q,\end{aligned} \tag{6}$$

are the generalized inverse of q-boson operators.[5,6]

Like the various quantum states of the q-QHO such as the q-CS,[3] the q-even(odd) coherent states,[7] q-generalized coherent state [8] and the orthogonalized eigenstate[9] of $\hat{a}_q^k (k \geq 3)$ etc.[13], the state (4) will be interesting because of its mathematical and nonclassical features, though it has not been realized experimentally until now. At first, unlike the q-CS and q-even(odd)CS, the state (4) cannot form a complete set as the number states $\{|j\rangle_q, j=0,1,2,...k-1\}$ are absent in it. However, it, along with the number states $\{|j\rangle_q, j=0,1,2,...k-1\}$, does form a complete set. By using Eq.(5), the completeness relation of q-CS[10] and the following relations

$$\begin{aligned}\hat{a}_q^k\hat{a}_q^{-k} &= \hat{a}_q^{\dagger -k}\hat{a}_q^{\dagger k} = \hat{I},\\ \hat{a}_q^{-k}\hat{a}_q^k &= \hat{a}_q^{\dagger k}\hat{a}_q^{\dagger -k} = \hat{I} - \sum_{i=0}^{k-1}|j\rangle_{qq}\langle j|,\end{aligned} \tag{7}$$

the corresponding completeness relation of the state (4) can be expressed as

$$\int C_k^{-2}|\alpha,+k\rangle_{qq}\langle\alpha,+k|(\hat{a}_q^{\dagger k}\hat{a}_q^k)^{-1}\mathrm{d}\mu(\alpha) + \sum_{j=0}^{k-1}|j\rangle_{qq}\langle j| = \hat{I}, \tag{8}$$

with $d_q\mu(\alpha) = \frac{1}{2\pi}\exp_q(|\alpha|^2)\exp_q(-|\alpha|^2)d_q|\alpha|^2\mathrm{d}\theta$, where $\hat{I}$ is unit operator, the integral over $\mathrm{d}\theta$ is a normal integration but the integration over $|\alpha|^2$ is a q-integration.[10] Therefore, the state (4), along with the number states $\{|j\rangle_q, j=0,1,2..,k-1\}$ mathematically form a complete Hilbert space. Secondly, some information about the physical meaning of the q-nonlinearity may be gained by comparing the quantum statistical properties of the state (4) with those of the photon-added coherent state for the usual cavity field studied in Ref. [11]. In addition, this state can reveal the q-dependent quantum-squeezing and thus possess the latent application in optical communication and in precision and sensitive measurement.[12] According to Ref.[9], the state is said to be quantum-squeezed if

$$S_j = (\Delta\hat{X}_j)^2 - \frac{1}{4}\langle[\hat{a}_q,\hat{a}_q^\dagger]\rangle < 0, (j=1 \text{ or } 2), \tag{9}$$

where operators $\hat{X}_1 = (\hat{a}_q + \hat{a}_q^\dagger)/2$ and $\hat{X}_2 = i(\hat{a}_q^\dagger - \hat{a}_q)/2$ with commutation relation $[\hat{X}_1,\hat{X}_2] = \frac{i}{2}[\hat{a}_q,\hat{a}_q^\dagger]$ are the q-analogue quadrature operators. For the q-CS, i.e., $|\alpha,+k\rangle_q$ with $k=0$, it is easily seen that it does not exhibit the quantum-squeezing effect as $S_1 = S_2 \equiv 0$. By calculating the expectation values of operators $\hat{a}_q$, $\hat{a}_q^\dagger$, $\hat{a}_q^2$, $\hat{a}_q^{\dagger 2}$, $\hat{a}_q^\dagger\hat{a}_q$, and $\hat{a}_q^{\dagger 2}\hat{a}_q^2$ in the state $|\alpha,+k\rangle_q$, we find that for the state $|\alpha,+k\rangle_q$, quantum-squeezing in the quadrature $\hat{X}_1$ exhibits whenever

$$\begin{aligned}&\cos(2\theta) < \zeta_k(q,|\alpha|),\\ &\zeta_k(q,|\alpha|) = \frac{V_k^2 - B_kW_k/|\alpha|^2}{-V_k^2 + B_kU_k},\end{aligned} \tag{10}$$

with

$$\begin{aligned}V_k &= \sum_{n=0}^{\infty}\frac{[n+k+1]!}{[n]![n+1]!}|\alpha|^{2n},\\ W_k &= \sum_{n=0}^{\infty}\frac{[n+k][n+k]!}{([n]!)^2}|\alpha|^{2n},\\ U_k &= \sum_{n=0}^{\infty}\frac{[n+k+2]!}{[n]![n+2]!}|\alpha|^{2n},\\ B_k &= \sum_{n=0}^{\infty}\frac{[n+k]!}{([n]!)^2}|\alpha|^{2n}.\end{aligned}$$

While quantum-squeezing in the quadrature $\hat{X}_2$ exhibits whenever

$$\cos(2\theta) > -\zeta_k(q,|\alpha|). \tag{11}$$

Therefore, the state $|\alpha,+k\rangle_q$ shows quantum-squeezing for some values of α and q in which $|\zeta_k(q,|\alpha|)| < 1$ is satisfied. We see easily from Eqs. (10) and (11) that the state $|\alpha,+k\rangle_q$ shows quantum-squeezing in $\hat{X}_1$ if

$$\frac{1}{2}\arccos\{\xi_k(q,|\alpha|)\} < \theta < \pi - \frac{1}{2}\arccos\{\xi_k(q,|\alpha|)\}; \tag{12}$$

and in $\hat{X}_2$ if

$$-\frac{1}{2}\arccos\{-\xi_k(q,|\alpha|)\} < \theta < \frac{1}{2}\arccos\{-\xi_k(q,|\alpha|)\}, \tag{13}$$

where the angle θ defined by $\alpha = |\alpha|\exp(i\theta)$ is limited to the interval from 0 to π or from $-\pi/2$ to $\pi/2$. We have plotted the regions of squeezing in $\hat{X}_1$ and $\hat{X}_2$ in Fig. 1 for the state $|\alpha, +k\rangle_q$ with $k = 1, 2, 3$ and 4. This figure clearly shows that for larger $|\alpha|$ (about ≥ 2.0) the quantum fluctuation of $\hat{X}_1$ is always squeezed in the region $45°<\theta<135°$, while that of $\hat{X}_2$ is always squeezed in the region $-45° < \theta < 45°$. The regions of squeezing for smaller $|\alpha|$ are clearly influenced by the q- nonlinearity. It is also seen from Fig. 1 that the $|\alpha|$ regions corresponding to quantum-squeezing in $\hat{X}_1$ and in $\hat{X}_2$ may exist, are broaden with the decrease of parameter q and the increase of k.

Similarly, we can determine the quantum-photon number distribution in a q-nonlinear cavity field state by calculating the second-order quantum-correlation function[9]

$$g_q^{(2)}(0) \equiv \frac{\langle \hat{a}_q^{\dagger 2} \hat{a}_q^2 \rangle}{\langle \hat{a}_q^{\dagger} \hat{a}_q \rangle^2}. \tag{14}$$

The state is said to be of quantum-antibunching (QAB) if $g_q^{(2)}(0) < 1$. For the q-CS,i.e., state (4) with $k = 0$, it is easily seen that it does not show QAB as $g_q^{(2)}(0) \equiv 1$. However, for the state (4) with $k \neq 0$ the second-order quantum-correlation function reads

$$g_q^{(2)}(0) = \frac{B_k M_k}{W_k^2},$$
$$M_k = \sum_{n=0}^{\infty} \frac{[n+k][n+k-1][n+k]!}{([n]!)^2} |\alpha|^{2n}. \tag{15}$$

In Fig. 2 we have plotted the variation of q-dependent second-order correlation function $g_q^{(2)}(0)$ versus $|\alpha|$ for the state (4) with $k = 1, 2, 3, 4$ and $q = 1.0, 0.8$ respectively. We conclude from these numerical results that the states $\{|\alpha, +k\rangle_q\}$ always show QAB effect. This effect is weakened by the q-nonlinearity for larger $|\alpha|$ values (about $|\alpha| > 1.0$). For smaller $|\alpha|$ it is seen that the feature of QAB for the state $|\alpha, +k\rangle_q$ is strongly dependent on the excited number k. Especially, when $|\alpha| \to 0$ the QAB effect always exist except for $k = 1$. This effect is not always strengthened with increasing k e.g. the QAB for the state $|\alpha, +2\rangle_q$ is stronger than that for the states $|\alpha, +3\rangle_q$. The reason is that the probabilistic amplitude $a_n(k)$ of each Fock state $|n\rangle_q$ depends on the excited number k, i.e., $a_n = 0$ for $n < k$ and $a_n = (\alpha^{(n-k)}\sqrt{[n]!})/([n]!\sum_{n=k}^{\infty}|\alpha|^{2(n-k)}[n]!/[n-k]!^2)$ for $n \geq k$. As a consequence, the QAB effect deduced from the collective coherence of all number states present in $|\alpha, +k\rangle_q$ reveals complicated $k-$dependent feature. This must be shown by numerical method.

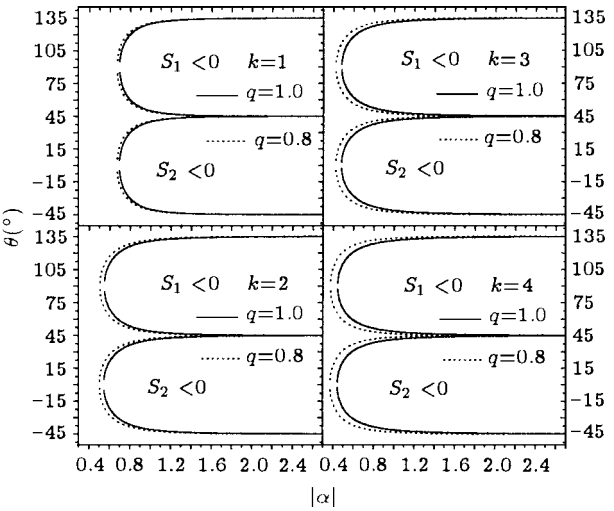


Fig.1. Regions showing quantum-squeezing in $\hat{X}_1$ ($S_1 < 0$) and $\hat{X}_2$ ($S_2 < 0$) in ($|\alpha|, \theta$) plane for the states $|\alpha, +k\rangle_q$ with $k = 1, 2, 3$, and 4.

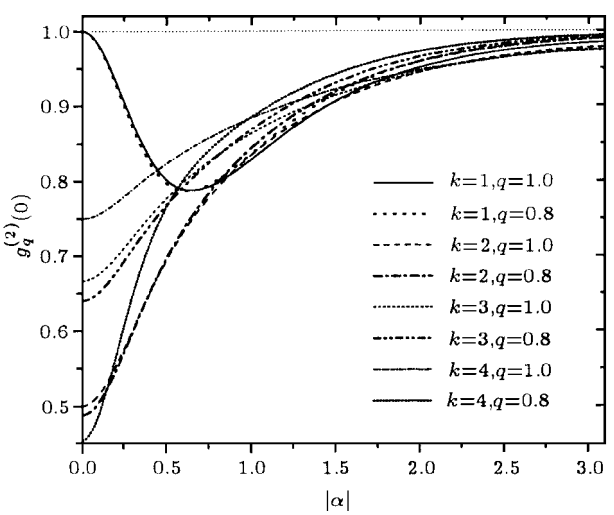


Fig.2. Plots of the second-order quantum-correlation function $g_q^{(2)}(0)$ versus $|\alpha|$ for the states $|\alpha, +k\rangle_q$ with $k = 1, 2, 3, 4$ and $q = 1.0, 0.8$.

In summary, we have studied the k-photon excitation states of the q-CS by the repeated application of q-boson creation operator $\hat{a}^{\dagger}$ to the q-CS. These states, another family of q-analogue photon-added coherent states besides the states $\{\hat{a}_q^{-k}|\alpha\rangle_q, k = 1, 2, ...\}$ studied in Ref. [4], can be produced in nonlinear process in q-nonlinear cavities. It is shown that each of the states $\{|\alpha, +k\rangle_q, k = 1, 2, ...\}$, along with the number states $\{|j\rangle_q, j = 0, 1, 2, ...k-1\}$, does form a complete set. We have shown by the numerical method that these states reveal quantum-antibunching effects for any α, q and exhibit q-dependent quantum-squeezing effects

in $\hat{X}_1$ or in $\hat{X}_2$ for some values of α. The $|\alpha|$ region corresponding to quantum-squeezing may exist, is broadened with decreasing q and increasing k. While the q-nonlinearity weakens the quantum-antibunching effect for larger $|\alpha|$ values.

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