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MULTIPHOTON JAYNES-CUMMINGS MODEL SOLVED VIA SUPERSYMMETRIC UNITARY TRANSFORMATION

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For multiphoton Jaynes-Cummings(JC) model Hamiltonian, we find its supersymmetric structure. Based on this, we introduce a supersymmetric unitary transformation to diagonalize the JC Hamiltonian, in so doing, its eigenvalues and eigenfunctions are obtained. The transition process is also calculated.

Keywords: jaynes-cummings model, supersymmetric unitary transformation, transition probability
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I. INTRODUCTION

The JC model[1] of a two-level atom interacting with a quantized single-model electromagnetic field plays an important role in quantum optics. It is perhaps the simplest solvable model that describes the essential physics of radiation-matter interaction. Many interesting quantum effects, like collapses and revivals,[2] squeezing,[3] etc. are shown in this model. Several generalized JC models have been proposed[4−6] since Jaynes and Cummings presented this model in 1963. These JC models have been also studied in detail. On the other hand, we find that in some JC models there exists a supersymmetry. Based on this, in supersymmetric quantum mechanics viewpoint, we have obtained the exact solution for super JC model by using a supersymmetric unitary transformation.[7] In this paper, we will further use the method to study the JC model for multiphoton process. The result will once again show that the method can play an important role in supersymmetric mechanics.

In the multiphoton process, its Hamiltonian[8] (rotating wave approximation) reads (let $\hbar$=1)

$$H = \omega a^+ a + \frac{\omega_0}{2}\sigma_z + g[(a^+)^k\sigma_- + a^k\sigma_+], \quad (1)$$

where $a^+(a)$ is the creation (annihilation) operator for the radiation field, and obey the commutation relation $[a, a^+]$=1, while the two-level atom is described by the Pauli operators $\sigma_\pm$ and σ_z, which satisfy the commutation relation $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$, g is the coupling constant and k is the photon number in each atom transition process; ω_0 and ω are respectively, the transition frequency and the model frequency. In the following discussion, we show how to solve the Hamiltonian (1) by a supersymmetric unitary transformation.

II. SUPERSYMMETRIC UNITARY TRANSFORMATION

To construct the supersymmetric unitary transformation operator we first define the supersymmetric transformation generators as follows:

$$N = a^+ a + \frac{k-1}{2}\sigma_z + \frac{1}{2} = \begin{pmatrix} a^+ a + \frac{k}{2} & 0 \\ 0 & aa^+ - \frac{k}{2} \end{pmatrix}, \quad (2)$$

$$\left.\begin{aligned} Q &= (a^+)^k\sigma_- = \begin{pmatrix} 0 & 0 \\ (a^+)^k & 0 \end{pmatrix} \\ Q^+ &= a^k\sigma_+ = \begin{pmatrix} 0 & a^k \\ 0 & 0 \end{pmatrix} \end{aligned}\right\}, \quad (3)$$

$$N' = \begin{pmatrix} a^k(a^+)^k & 0 \\ 0 & (a^+)^k a^k \end{pmatrix}. \quad (4)$$

It is easy to see that (N, N', Q, Q^+) form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.,

$$\left.\begin{aligned} &Q^2 = Q^{+^2} = 0 \quad [Q^+, Q] = N'\sigma_z \\ &[N, N'] = 0 \quad [N, Q] = Q \\ &[N, Q^+] = -Q^+ \quad \{Q^+, Q\} = N' \\ &\{Q, \sigma_z\} = \{Q^+, \sigma_z\} = 0 \\ &(Q^+ - Q)^2 = -N' \end{aligned}\right\}, \quad (5)$$

where $\{\ \}$ denotes the anticommutation bracket. With the help of Eqs(2)–(4), Eq.(1) can be rewritten as

$$H = \omega N + \frac{\omega - \delta}{2}\sigma_z + g(Q + Q^+) - \frac{\omega}{2}, \qquad (6)$$

where $\delta = k\omega - \omega_0$ is a detuning quantity.

Now, by the aid of supersymmetric transformation generators mentioned above, we construct the supersymmetric unitary transformation operator so that the Hamiltonian in Eq.(6) can be diagonalized. The supersymmitric unitary transformation operator is defined as the following

$$T = \exp\left\{ -\frac{\theta}{2} N'^{-\frac{1}{2}}(Q^+ - Q)\right\}, \qquad (7)$$

in which $N'^{-\frac{1}{2}}$ is

$$N'^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{a^k (a^+)^k}} & 0 \\ 0 & \frac{1}{\sqrt{(a^+)^k a^k}} \end{pmatrix}.$$

Using the formula $af(a^+a) = f(a^+a+1)a, a^+f(aa^+) = f(a^+a)a^+$, one can easily verify the following commutation relation

$$[N'^{-\frac{1}{2}}, Q] = [N'^{-\frac{1}{2}}, Q^+] = 0. \qquad (8)$$

Therefore, from Eqs(5) and (8), Eq.(7) can be expanded to the following form:

$$T = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\frac{1}{\sqrt{N'}}(Q^+ - Q). \qquad (9)$$

Based on quantum transformation,[9–11] we take the following supersymmetric unitary transformation

$$H' = T^{-1} H T.$$

From Eqs(5),(8) and (9), we obtain the following result

$$T^{-1} N T = N + \frac{1}{2}\sin\theta\frac{1}{\sqrt{N'}}(Q^+ + Q) + \sin^2\frac{\theta}{2}\sigma_z,$$

$$T^{-1}(Q^+ + Q)T = \cos\theta(Q^+ + Q) + \sin\theta\frac{1}{\sqrt{N'}}[Q^+ Q],$$

$$T^{-1}\sigma_z T = \cos\theta\sigma_z - \sin\theta\frac{1}{\sqrt{N'}}(Q^+ + Q).$$

Therefore, we have

$$H' = T^{-1}HT = \omega N - \frac{\omega}{2} + \left[\frac{\delta}{2}\frac{\sin\theta}{\sqrt{N'}} + g\cos\theta\right](Q^+ + Q) + \left[\frac{1}{\sqrt{N'}}(\frac{\omega}{2} - \frac{\delta}{2}\cos\theta) + g\sin\theta\right]\frac{1}{\sqrt{N'}}[Q^+, Q].$$

If we let

$$\frac{1}{\sqrt{N'}}\mathrm{tg}\,\theta = -\frac{2g}{\delta} \qquad (10)$$

(Note that Eq.(10) should be understood in the sense of eigenvalue and eigenvalue equation for the operator N'), we can obtain the diagonalized Hamiltonian as follows

$$H' = T^{-1}HT = \omega N - \frac{\omega}{2} + \left(\frac{\omega}{2} - \frac{\delta}{2\cos\theta}\right)\sigma_z. \qquad (11)$$

The corresponding eigenstates of H' read

$$\left.\begin{aligned} |\,\Psi'_{1,n}\rangle &= \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} \equiv |n, 1\rangle, \\ |\,\Psi'_{2,n}\rangle &= \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} \equiv |n+k, -1\rangle \end{aligned}\right\}. \qquad (12)$$

When $\frac{1}{\sqrt{(n+k)!/n!}}\mathrm{tg}\,\theta = -\frac{2g}{\delta}$, we get

$$\cos\theta = -\frac{\delta}{\left[4g^2\frac{(n+k)!}{k!} + \delta^2\right]^{1/2}}. \qquad (13)$$

Thus the eigenvalues and eigenstates in Eq.(1) are, respectively, given by

$$E_n = [\omega(n + \frac{k}{2}) \pm \frac{1}{2}\sqrt{\omega_1^2 + \delta^2}], \qquad (14)$$

where $\omega_1 = 2g[(n+k)!/n!]^{\frac{1}{2}}$ is Rabi frequency.

$$\left.\begin{aligned} |\,\Psi_{1,n}\rangle &= T|\,\Psi'_{1,n}\rangle = \cos\frac{\theta}{2}|n, 1\rangle + \sin\frac{\theta}{2}|n+k, -1\rangle, \\ |\,\Psi_{2,n}\rangle &= T|\,\Psi'_{2,n}\rangle = \cos\frac{\theta}{2}|n+k, -1\rangle - \sin\frac{\theta}{2}|n, 1\rangle \end{aligned}\right\}. \qquad (15)$$

It should be pointed out that state $|\,\Psi_{3,n}\rangle = |n, -1\rangle$ $(n \le k-1)$, which is not included in Eqs(15), is another eigenstate of H.

Ⅲ. THE ATOM-FIELD WAVE FUNCTION

Now, we can use the eigenstates of H, $|\Psi_{1,n}\rangle$, $|\Psi_{2,n}\rangle$ and $|\Psi_{3,n}\rangle$, to calculate the atom-field wave function $|\Psi(t)\rangle$.

Assuming that the system at $t=0$ is in $|\Psi(t)\rangle$, which is an arbitrary initial state, we write down the solution of the Schrödinger equation

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle. \tag{16}$$

Using the completeness condition of the whole space

$$\sum_{n=0}^{\infty}(|\Psi_{1,n}\rangle\langle\Psi_{1,n}| + |\Psi_{2,n}\rangle\langle\Psi_{2,n}|) + \sum_{n=0}^{k-1}|\Psi_{3,n}\rangle\langle\Psi_{3,n}| = 1, \tag{17}$$

and

$$\left.\begin{aligned} H|\Psi_{1,n}\rangle &= [(n+\frac{k}{2})\omega + \lambda_{n,k}]|\Psi_{1,n}\rangle \\ H|\Psi_{2,n}\rangle &= [(n+\frac{k}{2})\omega - \lambda_{n,k}]|\Psi_{2,n}\rangle \\ H|\Psi_{3,n}\rangle &= [(n-\frac{k}{2})\omega + \frac{1}{2}\delta]|\Psi_{3,n}\rangle \end{aligned}\right\}, \tag{18}$$

where $\lambda_{n,k} = \frac{1}{2}\sqrt{\omega_1^2+\delta^2}$, we have

$$\begin{aligned}|\Psi(t)\rangle = &\sum_{n=0}^{\infty}\exp^{-i[(n+\frac{1}{2}k)\omega]t}\Big\{\Big[\alpha_n\cos\frac{\theta}{2}e^{-i\lambda_{n,k}t} \\ &-\beta_n\sin\frac{\theta}{2}e^{i\lambda_{n,k}t}\Big]|n,1\rangle \\ &+\Big[\alpha_n\sin\frac{\theta}{2}e^{-i\lambda_{n,k}t}+\beta_n\cos\frac{\theta}{2}e^{i\lambda_{n,k}t}\Big]|n+k,-1\rangle\Big\} \\ &+\sum_{n=0}^{k-1}\gamma_n\exp^{-i[(n-\frac{k}{2})\omega+\frac{1}{2}\delta]t}|n,-1\rangle,\end{aligned} \tag{19}$$

where

$$\left.\begin{aligned} \alpha_n &= \langle\Psi_{1,n}|\Psi(0)\rangle \\ \beta_n &= \langle\Psi_{2,n}|\Psi(0)\rangle \\ \gamma_n &= \langle\Psi_{3,n}|\Psi(0)\rangle \end{aligned}\right\}, \tag{20}$$

When $|\Psi(0)\rangle = |z,-1\rangle$ (i.e. the field mode and the atom are initially in a coherent state $|z\rangle$ and a ground state $|-1\rangle$, respectively), the wave function is

$$\begin{aligned}|\Psi(t)\rangle = &\exp(-|z|^2/2)\sum_{n=0}^{\infty} \\ &\frac{z^{n+k}}{\sqrt{(n+k)!}}\exp\Big[-i(n+\frac{1}{2}k)\omega t\Big]\times \\ &\Big\{\frac{-ig[(n+k)!/n!]^{\frac{1}{2}}}{\lambda_{n,k}}\sin(\lambda_{n,k}t)|n,1\rangle \\ &+[\cos(\lambda_{n,k}t)-\frac{-i\delta}{2\lambda_{n,k}}\sin(\lambda_{n,k}t)]|n+k,-1\rangle\Big\} \\ &+\exp(-|z|^2/2)\sum_{n=0}^{k-1}\frac{z^n}{\sqrt{n!}}\exp\Big\{-i\Big[(n \\ &-\frac{1}{2}k)\omega+\frac{1}{2}\delta\Big]t\Big\}|n,-1\rangle.\end{aligned} \tag{21}$$

Similarly, if $\Psi(0)\rangle = |n,1\rangle$ (i.e. the field mode and the atom are initially in a Fock state $|n\rangle$ and an exited state $|1\rangle$, respectively), then

$$\begin{aligned}|\Psi(t)\rangle = &\sum_{n=0}^{\infty}\exp[-i(n+\frac{1}{2}k)\omega t]\Big\{\Big[\cos(\lambda_{n,k}t) \\ &+\frac{i\delta}{2\lambda_{n,k}}\sin(\lambda_{n,k}t)\Big]|n,1\rangle \\ &-[\frac{ig[(n+k)!/n!]^{\frac{1}{2}}}{\lambda_{n,k}}|n+k,-1\rangle\Big\}.\end{aligned} \tag{22}$$

Obviously, it is easy to calculate the probability at which the system can be in state $|n,1\rangle$ or state $|n+k,-1\rangle$,

$$|\langle n,1|\Psi(t)\rangle|^2 = \frac{4g^2(n+k)!/n!\cos^2\lambda_{n,k}t+\delta^2}{4\lambda_{n,k}^2}, \tag{23}$$

$$\begin{aligned}|\langle n+k,-1|\Psi(t)\rangle|^2 &= 1-|\langle n,1|\Psi(t)\rangle|^2 \\ &=\frac{g^2(n+k)!/n!\sin^2(\lambda_{n,k}t)}{\lambda_{n,k}^2}.\end{aligned} \tag{24}$$

In summary, we have obtained the exact solution for the multiphoton JC model by supersymmetric unitary transformation. As we have discussed above, we can see the universality and advantages of the method. It is worth pointing out that this method can also be applied to the study of parasupersymmetric quantum mechanics[12] and orthosupersymmetric quantum mechanic.[13]

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