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EFFECTS OF NONHOLONOMIC CONSTRAINT ON LIE SYMMETRIES AND CONSERVED QUANTITIES OF LAGRANGIAN SYSTEMS*

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After a Lagrangian system is constrained by nonholonomic constraints, the determining equations , the structure equation and the form of conserved quantities corresponding to the Lie symmetries will change. Some symmetries vanish and under certain conditions some Lie symmetries still remain.

Keywords: lagrangian system, nonholonomic constraint, lie symmetry, conserved quantity.
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I. INTRODUCTION

Studies on symmetry and conserved quantities not only have mathematical significance, but also are more important in the physical field. Noether symmetry and Lie symmetry are two of those important symmetries. There have been significant progresses about the research on Noether symmetry.[1,2] Monographs[3−5] about Lie symmetry from the mathematical point of view have also come forth. Since Lutzky and others introduced Lie's invariance theory of the differential equations under the infinitesimal transformations into the studies on mechanical systems in 1970's, researches on Lie symmetries of mechanical system have made great progress.[6−8] However, after nonholonomic constraints are imposed on a Lagrangian system, how Lie symmetries and conserved quantities of the system change is still not clear. So in this paper, we will study the effects of nonholonomic constraints on Lie symmetries and conserved quantities of Lagrangian systems.

II. EQUATIONS OF MOTION OF THE SYSTEM

The equations of motion of a Lagrangian system can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=0,\ \ (s=1,\cdots,n) \tag{1}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian of the system, q_s are generalized coordinates. Suppose that it is subjected to the g ideal nonholonomic constraints of Chetaev's type

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\ \ (\beta=1,\cdots,g), \tag{2}$$

then the equations of motion can be expressed in Routh form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s},\ \ (s=1,\cdots,n). \tag{3}$$

Suppose that the system is not singular, i.e.

$$\Delta=\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0, \tag{4}$$

then we can obtain all generalized accelerations from Eqs. (1)

$$\ddot{q}_s=\frac{\Delta_{ls}}{\Delta}\Bigg\{-[k,m;l]\dot{q}_k\dot{q}_m+\left(\frac{\partial B_k}{\partial q_l}-\frac{\partial B_l}{\partial q_k}\right)\dot{q}_k-\frac{\partial B_l}{\partial t}+\frac{\partial T_0}{\partial q_l}-\frac{\partial A_{kl}}{\partial t}\dot{q}_k\Bigg\}, \tag{5}$$

where Δ_{ls} is the algebraic complement of the $(l,\,s)$ element of Δ, $[k,\,m;\,l]$ is the Christoffel symbol of first kind as follows:

$$[k,m;l]=\frac{1}{2}\left(\frac{\partial A_{kl}}{\partial q_m}+\frac{\partial A_{ml}}{\partial q_k}-\frac{\partial A_{km}}{\partial q_l}\right), \tag{6}$$

where A_{ks} are the coefficients of quadratic forms of velocity in the kinetic energy, B_s are the coefficients of linear forms of velocity in the kinetic energy, T_0 are the terms without $\dot{q}_s$ in the kinetic energy. Similarly, Eq.(2) can also be expressed explicitly

$$\ddot{q}_s=\frac{\Delta_{ls}}{\Delta}\Bigg\{-[k,m;l]\dot{q}_k\dot{q}_m+\left(\frac{\partial B_k}{\partial q_l}-\frac{\partial B_l}{\partial q_k}\right)\dot{q}_k-\frac{\partial B_l}{\partial t}+\frac{\partial T_0}{\partial q_l}-\frac{\partial A_{kl}}{\partial t}\dot{q}_k+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_l}\Bigg\}. \tag{7}$$

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Differentiating the constraint equations (2) with respect to t and substituting Eqs. (7) into it, we obtain algebraic equations of multipliers λ_β[9]

$$\begin{aligned}&\frac{\Delta_{sl}}{\Delta}\frac{\partial f_\gamma}{\partial \dot q_l}\frac{\partial f_\beta}{\partial \dot q_s}+\frac{\partial f_\gamma}{\partial q_l}\dot q_l\\&+\frac{\partial f_\gamma}{\partial t}+\frac{\partial f_\gamma}{\partial \dot q_l}\frac{\Delta_{sl}}{\Delta}\Big\{-[k,m;s]\dot q_k\dot q_m\\&+\left(\frac{\partial B_k}{\partial q_s}-\frac{\partial B_s}{\partial q_k}\right)\dot q_k-\frac{\partial B_s}{\partial t}+\frac{\partial T_0}{\partial q_s}-\frac{\partial A_{ks}}{\partial t}\dot q_k\Big\}=0,\\&(\beta=1,\cdots,g).\end{aligned}\tag{8}$$

From Eqs. (8) we can solve λ_β as functions of $t,\boldsymbol{q},\dot{\boldsymbol{q}}$

$$\lambda_\beta=\lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\tag{9}$$

then we have generalized nonholonomic constraint forces

$$\Lambda_s=\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}.\tag{10}$$

Equations(5) and (7) can be written simply as

$$\ddot q_s=g_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\qquad(s=1,\cdots,n);\tag{11}$$

$$\ddot q_s=h_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\qquad(s=1,\cdots,n).\tag{12}$$

Equations. (12) are called equations of motion of the corresponding holonomic system of nonholonomic system (2), (3). Apparently,

$$h_s=g_s+\frac{\Delta_{ls}}{\Delta}\Lambda_l.\tag{13}$$

Ⅲ. INFINITESIMAL TRANSFORMATIONS AND DETERMINING EQUATIONS

Choose the infinitesimal transformations of time and coordinates as

$$\begin{aligned}&t^*=t+\Delta t,\\&q_s^*(t^*)=q_s(t)+\Delta q_s,\qquad(s=1,\cdots,n),\end{aligned}\tag{14}$$

or their expanded forms

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\&q_s^*(t^*)=q_s+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad(s=1,\cdots,n),\end{aligned}\tag{15}$$

where ε is the infinitesimal parameter and ξ_0,ξ_s are the generators of infinitesimal transformations. Then we introduce the generator vector of infinitesimal transformations

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}\tag{16}$$

and its first extended form

$$X^{(1)}=X^{(0)}+(\dot\xi_s-\dot q_s\dot\xi_0)\frac{\partial}{\partial \dot q_s}.\tag{17}$$

According to Lie's invariance theory of ordinary differential equations under infinitesimal transformations, the invariances of Eqs.(11) and (12) under the infinitesimal transformations (15) lead up to the satisfaction of the following determining equations respectively

$$\ddot\xi_s-\dot q_s\ddot\xi_0-2\dot\xi_0 g_s=X^{(1)}(g_s),\ (s=1,\cdots,n),\tag{18}$$

$$\ddot\xi_s-\dot q_s\ddot\xi_0-2\dot\xi_0 h_s=X^{(1)}(h_s),\ (s=1,\cdots,n).\tag{19}$$

The invariance of constraint equations (2) under the infinitesimal transformations (15) can be expressed as following restriction equations:

$$X^{(1)}(f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}))=0,\quad(\beta=1,\cdots,g).\tag{20}$$

We hereinafter give the definition of Lie symmetry.

Definition 1 If generators ξ_0,ξ_s satisfy the determining equations (18), then the corresponding symmetries are called Lie symmetries of Lagrangian system (1).

Definition 2 If generators ξ_0,ξ_s satisfy the determining equations (19), then the corresponding symmetries are called Lie symmetries of the corresponding holonomic system of nonholonomic system (2), (3).

Definition 3 If generators ξ_0,ξ_s satisfy the determining equations (19) and restriction equations (20), then the corresponding symmetries are called Lie symmetries of the nonholonomic system (2), (3).

From the definitions 2 and 3, we have

Proposition 1 Lie symmetries of the nonholonomic system (2), (3) must be Lie symmetries of its corresponding holonomic system (12), but the inverse does not always hold.

When nonholonomic constraints are exerted on the Lagrangian systems, some Lie symmetries still remain. We have

Proposition 2 If infinitesimal generators ξ_0,ξ_s and generalized constraint forces Λ_l satisfy

$$-2\dot\xi_0\frac{\Delta_{ls}}{\Delta}\Lambda_l=X^{(1)}\left(\frac{\Delta_{ls}}{\Delta}\Lambda_l\right),\ (s=1,\cdots,n),\tag{21}$$

then Lie symmetries of the Lagrangian system (11) are also Lie symmetries of system (12).

Proof. Substituting Eqs.(13) into the determining equations Eqs.(18) and taking note the relation (21), we obtain Eq.(19).

Proposition 3 If infinitesimal generators ξ_0,ξ_s are corresponding to Lie symmetries of the Lagrangian

system (11) and satisfy Eqs. (20), (21), then these symmetries are also Lie symmetries of nonholonomic system (2), (3).

IV. STRUCTURE EQUATION AND CONSERVED QUANTITIES

Lie symmetries do not always generate conserved quantities. The subsequent propositions give the condition under which Lie symmetries generate conserved quantities and the form of conserved quantities.

Proposition 4[7] For those infinitesimal generators ξ_0, ξ_s satisfying Eqs.(18), if there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation

$$L\dot{\xi}_0 + X^{(1)}(L) + \dot{G} = 0, \tag{22}$$

then the Lagrangian system (11) has conserved quantities as follows

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G = \text{const.} \tag{23}$$

Proposition 5[8] For those infinitesimal generators ξ_0, ξ_s satisfying Eqs.(19), if there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation:

$$L\dot{\xi}_0 + X^{(1)}(L) + \Lambda_s(\xi_s - \dot{q}_s\xi_0) + \dot{G} = 0, \tag{24}$$

then the corresponding holonomic system (12) of the nonholonomic system (2), (3) has conserved quantities as Eq.(23).

Proposition 6[8] For those infinitesimal generators ξ_0, ξ_s satisfying both Eqs.(19) and (20), if there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation Eq.(24), then the nonholonomic system (2), (3) has the same conserved quantities as Eq.(23).

Proposition 7 If the Lagrangian system (11) and the corresponding holonomic system (12) of nonholonomic system (2), (3) have the same Lie symmetries, and the condition

$$\Lambda_s(\xi_s - \dot{q}_s\xi_0) = 0 \tag{25}$$

is satisfied, then these two systems have the same conserved quantities.

Proposition 8 If the lagrangian system (11) and the nonholonomic system (2) (3) have the same Lie symmetries and the condition (25) is satisfied, then these two systems have the same conserved quantities.

V. ILLUSTRATIVE EXAMPLE

Suppose that a Lagrangian system is given by

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3, \tag{26}$$

and a nonholonomic constraint is exerted on system (26)

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0. \tag{27}$$

We make a study on Lie symmetries and conserved quantities of these two systems.

The equations of motion of the Lagrangian system are

$$\begin{aligned} m\ddot{q}_1 &= 0, \\ m\ddot{q}_2 &= 0, \\ m\ddot{q}_3 &= -mg, \end{aligned} \tag{28}$$

and the equations of motion of the nonholonomic system are

$$\begin{aligned} m\ddot{q}_1 &= 2\lambda\dot{q}_1, \\ m\ddot{q}_2 &= 2\lambda\dot{q}_2, \\ m\ddot{q}_3 + mg &= -2\lambda\dot{q}_3. \end{aligned} \tag{29}$$

From Eqs. (27) and (29), we obtain

$$\begin{aligned} \Lambda_1 &= 2\lambda\dot{q}_1 = -\frac{mg\dot{q}_1}{2\dot{q}_3}, \\ \Lambda_2 &= 2\lambda\dot{q}_2 = -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ \Lambda_1 &= -2\lambda\dot{q}_1 = \frac{1}{2}mg \end{aligned} \tag{30}$$

The conditions (21) give

$$\begin{aligned} -2\dot{\xi}_0\left(-\frac{g\dot{q}_1}{2\dot{q}_3}\right) &= X^{(1)}\left(-\frac{g\dot{q}_1}{\dot{q}_3}\right), \\ -2\dot{\xi}_0\left(-\frac{g\dot{q}_2}{2\dot{q}_3}\right) &= X^{(1)}\left(-\frac{g\dot{q}_2}{\dot{q}_3}\right), \\ -2\dot{\xi}_0\left(-\frac{1}{2}g\right) &= 0. \end{aligned} \tag{31}$$

The determining equations (18) give

$$\begin{aligned} \ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 &= 0, \\ \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 &= 0, \\ \ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 - 2\dot{\xi}_0(-g) &= 0. \end{aligned} \tag{32}$$

The determining equations (19) give

$$\begin{aligned} \ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\frac{g\dot{q}_1}{2\dot{q}_3}\right) &= X^{(1)}\left(-\frac{g\dot{q}_1}{2\dot{q}_3}\right), \\ \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\frac{g\dot{q}_2}{2\dot{q}_3}\right) &= X^{(1)}\left(-\frac{g\dot{q}_2}{2\dot{q}_3}\right), \\ \ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 - 2\dot{\xi}_0(-\frac{1}{2}g) &= 0. \end{aligned} \tag{33}$$

Equations (32) have following solutions

$$\xi_0 = 1, \xi_1 = \xi_2 = \xi_3 = 0, \tag{34}$$

$$\xi_1 = 1, \xi_0 = \xi_2 = \xi_3 = 0, \tag{35}$$

$$\xi_2 = 1, \xi_0 = \xi_1 = \xi_3 = 0, \tag{36}$$

$$\xi_3 = 1, \xi_0 = \xi_1 = \xi_2 = 0, \tag{37}$$

$$\xi_1 = t, \xi_0 = \xi_2 = \xi_3 = 0, \tag{38}$$

$$\xi_2 = t, \xi_0 = \xi_1 = \xi_3 = 0, \tag{39}$$

$$\xi_3 = t, \xi_0 = \xi_1 = \xi_2 = 0. \tag{40}$$

Equations (34)-(40) are corresponding to Lie symmetries of the Lagrangian system (26). Equations(34)-(37) also satisfy conditions (31). By proposition 2, they are also Lie symmetries of the corresponding holonomic system of the nonholonomic system.

The restriction equations (20) give

$$\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) - \dot{q}_3(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0) = 0. \tag{41}$$

Equations (34)-(37) satisfy the condition (41), so they are also corresponding to Lie symmetries of the nonholonomic system.

The condition (25) gives

$$-\frac{mg}{2\dot{q}_3}[\dot{q}_1(\xi_1 - \dot{q}_1\xi_0) + \dot{q}_2(\xi_2 - \dot{q}_2\xi_0) - \dot{q}_3(\xi_3 - \dot{q}_3\xi_0)] = 0. \tag{42}$$

The solution (34) satisfies the condition (42), so when the solution (34) is chosen as generators of infinitesimal transformations, by proposition 8 these two systems have the same conserved quantity

$$I = L - \frac{\partial L}{\partial \dot{q}_s} = -\frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3 = \text{const.} \tag{43}$$

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