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THE GROWTH RATE AND STATISTICAL FLUCTUATION OF BOSE-EINSTEIN CONDENSATE FORMATION*

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Using the generating function method to solve the master equation of Bose-Einstein condensate and to evaluate the growth rate, statistical fluctuation of condensate atoms, we find out that there is a plateau in the growth rate curve and a super-Poisson distribution observed.

Keywords: Bbose-Einstein condensate, atom laser, growth rate
PACC: 0530, 3450, 4250

I. INTRODUCTION

At sufficiently low temperature, the process of Bose-Einstein condensate (BEC) formation may be conceived of the dilute Bose gas atoms continuous transtition to condensate. Assuming there are n atoms in condensate ground state $|n\rangle$, the transition probability that a gas atom drops into condensate $|n\rangle$ and makes it transit into $|n+1\rangle$ can be enhanced by the number of atoms n in the state $|n\rangle$. This is analogous to the growth mechanism of laser, the photons in the mode will stimulate more photons lasing in the same mode, namely stimulated emission. The stimulation of Bose atoms results in amplification of matter wave, i.e. atom laser gain. This is the experiment conducted at MIT (1997–1998)[1] and a series of theoretical studies on condensate growth by Gardiner *et al.*[2−4], who derived the master equation satisfied by matrix density operator ρ of the condensate and the rate equation of atom laser. Based on the rate equation, it is possible to obtain the evolution of condensate n versus time t, but impossible to evaluate the condensate variation. In this paper, after giving a physical explanation of density master equation, we obtain its analytical solution by the generating function method.[5,6] Furthermore, we find out the evolutions of $< n >$ and $\frac{\mathrm{d}\ln < n >}{\mathrm{d}t}$ with time t, and statistical fluctuation $< (\Delta n)^2 >$ is also obtained. Our main results are that the growth rate γ curve has a plateau, the statistical fluctuation shows super-Poisson distribution, and γ decreases progressively with the increase of condensate n_{eq}.

II. MASTER EQUATION SATISFIED BY MATRIX DENSITY OPERATOR ρ OF THE CONDENSATE

According to Ref.[4], the master equation satisfied by matrix density operator ρ of the condensate is written as

$$\begin{aligned}\dot{\rho}(n) =& 2nW^+(n-1)\rho(n-1) \\ &+ 2(n+1)W^-(n+1)\rho(n+1) \\ &- [2(n+1)W^+(n) + 2nW^-(n)]\rho(n). \end{aligned} \quad (1)$$

In order to obtain an analytical solution we adopt the approxmation

$$\begin{aligned} W^+(n-1) &\approx W^+(n) \approx W^+(< n >), \\ W^-(n-1) &\approx W^-(n) \approx W^-(< n >), \end{aligned} \quad (2)$$

and set $A = 2W^+(< n >), B = 2W^-(< n >)$, then Eq.(1) becomes

$$\begin{aligned}\dot{\rho}(n) =& A[n\rho(n-1) - (n+1)\rho(n)] \\ &+ B[(n+1)\rho(n+1) - n\rho(n)]. \end{aligned} \quad (3)$$

In terms of annihilation and creation operators b, b^+ of the condensate, Eq.(3) can be expressed as

$$\begin{aligned}\dot{\rho} =& A[2b^+\rho b - \rho b b^+ - b b^+ \rho] \\ &+ B[2b\rho b^+ - \rho b^+ b - b^+ b\rho]. \end{aligned} \quad (4)$$

This is a familiar form of interaction equation of laser and heat bath, A and B are determined by the mechanism of exchanging photon between them. For current atom laser, the first term in Eq.(4) represents that thermal atoms 1 and 2 are scattered by collision into thermal atom 3 and condensate atom 4, the transition probability $W^+ \propto \phi^+_{1n}\phi_{1n}\phi^+_{2n}\phi_{2n}\phi_{3n}\phi^+_{3n}\phi_{4c}\phi^+_{4c}$. It has

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been derived[4]

$$W^+ = \frac{u^2}{(2\pi)^5\hbar^2}\int \mathrm{d}^3k_1\int \mathrm{d}^3k_2\int \mathrm{d}^3k_3\int \mathrm{d}^3k_4\delta \cdot (k_1+k_2-k_3-k_4) \cdot \delta(\Delta\omega_{123}-\mu(n)/\hbar)F(k_1,0)F(k_2,0) \cdot [1+F(k_3,0)]|\overline{\xi}_n(k)|^2, \tag{5}$$

where $\overline{\xi}_n = \frac{1}{2\pi^{3/2}}\int \mathrm{d}^3x\mathrm{e}^{\mathrm{i}k\cdot x}\xi_n(x)$ is the condensate wave function of momentum space. Similarly, the second term in Eq.(4) represents the inverse physical process that thermal atom 3 and condensate atom 4 translate by collision into thermal atoms 1 and 2. The transition probability is $W^- \propto \phi_{1n}\phi^+_{1n}\phi_{2n}\phi^+_{2n}\phi^+_{3n}\phi_{3n}\phi^+_{4c}\phi_{4c}$, this is

$$W^- = \frac{u^2}{(2\pi)^5\hbar^2}\int \mathrm{d}^3k_1\int \mathrm{d}^3k_2\int \mathrm{d}^3k_3\int \mathrm{d}^3k_4\delta \cdot (k_1+k_2-k_3-k_4) \cdot \delta(\Delta\omega_{123}-\mu(n)/\hbar)(1+F(k_1,0)) \cdot (1+F(k_2,0))F(k_3,0)|\overline{\xi}_n(k)|^2. \tag{6}$$

Ⅲ. SOLUTION OF THE MASTER EQUATION

Using the method of solution of maser eqution in Refs.[5] and [6], we introduce the generating function

$$G(z)=\sum_{n=0}^{\infty} z^n\rho(n). \tag{7}$$

It is easy to prove the identities

$$\begin{aligned}&\sum_{n=0}^{\infty} nz^n\rho(n-1) = z(z\frac{\partial}{\partial z}+1)G(z),\\ &\sum_{n=0}^{\infty}(n+1)z^n\rho(n) = (z\frac{\partial}{\partial z}+1)G(z),\\ &\sum_{n=0}^{\infty}(n+1)z^n\rho(n+1) = \frac{\partial}{\partial z}G(z).\end{aligned} \tag{8}$$

Multiplying Eq.(3) by z^n and taking the summation over n, as a result of Eqs. (7) and (8), we obtain the equation satisfied by $G(z)$

$$\begin{aligned}&(\frac{\partial}{\partial t}+(A(1-z)z+B(z-1))\frac{\partial}{\partial z})G(z)\\ =&A(z-1)G(z).\end{aligned} \tag{9}$$

Setting

$$z_0 = \frac{(z-1)B\mathrm{e}^{-ct}+B-Az}{B-Az+(z-1)A\mathrm{e}^{-ct}},\ c=B-A, \tag{10}$$

the following relations can be deduced easily

i. $[\frac{\partial}{\partial t}+(z-1)(B-Az)\frac{\partial}{\partial z}]z_0 = 0.$

ii. $z_0|_{t\to 0} = z.$

iii. $\frac{\partial z_0}{\partial t} = \frac{(B-A)^2\mathrm{e}^{-ct}}{(B-Az+(z-1)A\mathrm{e}^{-ct})^2},$

$$\left.\frac{\partial z_0}{\partial z}\right|_{t\to 0} = 1,\quad \left.\frac{\partial z_0}{\partial z}\right|_{z=1} = \mathrm{e}^{-ct}.$$

iv. $\frac{\partial^2 z_0}{\partial z^2} = \frac{2A(B-A)^2\mathrm{e}^{-ct}(1-\mathrm{e}^{-ct})}{(B-Az+(z-1)A\mathrm{e}^{-ct})^3},$

$$\begin{aligned}\frac{\partial^2 z_0}{\partial z^2} &= \Big|_{t\to 0} = 0,\\ \frac{\partial^2 z_0}{\partial z^2} &= \Big|_{z=1} = \frac{2A}{B-A}\mathrm{e}^{-ct}(1-\mathrm{e}^{-ct}).\end{aligned} \tag{11}$$

A general solution of Eq.(9) is

$$G(z) = G_0(z_0)\exp[\int_{z_0}^{z}\frac{A}{B-Az}\mathrm{d}z], G_0(1)=1. \tag{12}$$

Inserting Eq.(12) in Eq. (9) we obtain

$$\begin{aligned}&[\frac{\partial}{\partial t}+(A(1-z)z+B(z-1))\frac{\partial}{\partial z}]G\\ =&(A(1-z)z+B(z-1))\frac{A}{B-Az}G = A(z-1)G.\end{aligned} \tag{13}$$

From Eqs.(12)and (11)we obtain

$$\begin{aligned}<n> &= \left.\frac{\partial G}{\partial z}\right|_{z=1} = [\frac{\partial G_0}{\partial z_0}\frac{\partial z_0}{\partial z}\\ &\quad +G_0\frac{A}{B-Az}(1-\frac{\partial z_0}{\partial z})]_{z=1}\\ &=<n>_0\ \mathrm{e}^{-ct}+\frac{A}{B-A}(1-\mathrm{e}^{-ct}),\end{aligned} \tag{14}$$

$$\begin{aligned}<(\Delta n)^2> &=<n^2>-<n>^2\\ &=<n(n-1)>+<n>-<n>^2\\ &=\left.\frac{\partial^2 G}{\partial z^2}\right|_{z=1}+<n>-<n>^2,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\left.\frac{\partial^2 G}{\partial z^2}\right|_{z=1} &= [\frac{\partial^2 G_0}{\partial z^2}+\frac{\partial^2 H}{\partial z^2}G_0+\frac{\partial H}{\partial z}\frac{\partial G_0}{\partial z}\\ &\quad +(\frac{\partial G_0}{\partial z}+\frac{\partial H}{\partial z}G_0)\frac{\partial H}{\partial z}]\mathrm{e}^H\Big|_{z=1},\end{aligned}$$

$$H=\int_{z_0}^{z}\frac{A}{B-Az}\mathrm{d}z,$$

$$\begin{aligned}\left.\frac{\partial^2 G_0}{\partial z^2}\right|_{z=1} &= \left.[\frac{\partial^2 G}{\partial z^2}(\frac{\partial z_0}{\partial z})^2+\frac{\partial G_0}{\partial z_0}\frac{\partial^2 z_0}{\partial z^2}]\right|_{z=1}\\ &=<n(n-1)>_0\ \mathrm{e}^{-2ct}\\ &\quad +<n>_0\ \mathrm{e}^{-ct}\frac{2A}{B-A}(1-\mathrm{e}^{-ct}),\end{aligned}$$

$$\left.\frac{\partial H}{\partial z}\frac{\partial G_0}{\partial z}\right|_{z=1} =< n >_0 \mathrm{e}^{-ct}\frac{A}{B-A}(1-\mathrm{e}^{-ct}),$$

$$\left.(\frac{\partial G_0}{\partial z}+\frac{\partial H}{\partial z}G_0)\frac{\partial H}{\partial z}\right|_{z=1} =< n >_0 \mathrm{e}^{-ct} +\frac{A}{B-A}(1-\mathrm{e}^{-ct})\frac{A}{B-A}(1-\mathrm{e}^{-ct}),$$

$$\left.\frac{\partial^2 H}{\partial z^2}\right|_{z=1} = \left.\frac{\partial}{\partial z}(\frac{A}{B-Az}-\frac{A}{B-Az_0}\frac{\partial z_0}{\partial z})\right|_{z=1} = \frac{A^2}{(B-A)^2}(1-\mathrm{e}^{-ct})^2. \quad (16)$$

Based on Eqs. (14)–(16), we have

$$\begin{aligned}< (\Delta n)^2 > =& < (\Delta n)^2 >_0 \mathrm{e}^{-2ct} \\ &+ \frac{A+B}{B-A}\mathrm{e}^{-ct}(1-\mathrm{e}^{-ct}) < n > \\ &+ \frac{A}{B-A}(1-\mathrm{e}^{-ct}) + \frac{A^2}{(B-A)^2}(1-\mathrm{e}^{-ct})^2.\end{aligned} \quad (17)$$

Since A and B are functions of $< n >$, Eqs.(14) and (17) provide actually iterative formulas to evaluate $< n >$ and $< (\Delta n)^2 >$. Dividing time space $(0, t)$ into some small intervals of Δt, according to Eqs.(14) and (17), in each interval the change of $< n >$ and $< (\Delta n)^2 >$ can be written as

$$\begin{aligned}< n >_{k+1} =& < n >_k \mathrm{e}^{-c_k\Delta t} \\ &+ \frac{A_k}{B_k-A_k}(1-\mathrm{e}^{-c_k\Delta t}),\end{aligned}$$

$$\begin{aligned}< (\Delta n)^2 >_{k+1} =& < (\Delta n)^2 >_k \mathrm{e}^{-2c_k\Delta t} \\ &+ \frac{A_k+B_k}{B_k-A_k}\mathrm{e}^{-c_k\Delta t}(1-\mathrm{e}^{-c_k\Delta t}) < n > \\ &+ \frac{A_k}{B_k-A_k}(1-\mathrm{e}^{-c_k\Delta t}) + \frac{A_k^2}{(B_k-A_k)^2}(1-\mathrm{e}^{-c_k\Delta t})^2, \\ A_{k+1} =& A(< n >_{k+1}), \\ B_{k+1} =& B(< n >_{k+1}).\end{aligned} \quad (18)$$

As $c_k = B_k - A_k < 0$, Eq.(18) indicates that in the interval Δt initial values $< n >_k$ and $< (\Delta n)^2 >_k$ are enlarged by multipliers $\exp(-c_k\Delta t)$ and $\exp(-2c_k\Delta t)$ respectively. During the study of statistical distribution properties of atoms, the following definition of reduced $< \tilde{n} >_k$ and $< (\Delta\tilde{n})^2 >_k$ are more appropriate

$$< \tilde{n} >_k = \frac{< n >_k}{\exp(-\sum_{j=1}^{k} c_j\Delta t)},$$

$$< (\Delta\tilde{n})^2 >_k = \frac{< (\Delta n)^2 >_k}{\exp(-2\sum_{j=1}^{k} c_j\Delta t)}. \quad (19)$$

And we introduce the factor Q_k characterizing the statistical property of atom distribution, defined by

$$Q_k = \frac{< (\Delta\tilde{n})^2 >_k}{< \tilde{n} >_k}. \quad (20)$$

IV. THE GROWTH RATE AND STATISTICAL FLUCTUATION OF CONDENSATE

From Refs. [2] and [4], we have

$$A = 2W^+ = \frac{8m(\overline{a}kT)^2}{\pi\hbar^3}\mathrm{e}^{2\mu(n_{\mathrm{eq}})/kT}\left\{\frac{\mu(n)}{kT}K_1(\frac{\mu(n)}{kT})\right\}, \quad (21)$$

$$B/A = W^-/W^+ = \exp\{(\mu(n)-\mu(n_{\mathrm{eq}}))/kT\},$$

where n is the number of atoms in condensate, n_{eq} is the final equilibrium value of n. First of all the factor in Eq.(21) can be written as $\frac{8m(\overline{a}kT)^2}{\pi\hbar^3} = \frac{4\omega}{\pi}\left(\frac{\overline{a}}{\alpha}\right)^2\sigma^{-2}$, $\alpha = \sqrt{\frac{\hbar}{2m\omega}}$, $\sigma = \frac{\hbar\omega}{kT}$. For soldium atom,[2] $m = 3.84\times10^{-26}$kg, $\omega = 2\pi\times415s^{-1}$, $\overline{a} = 4.9$nm, we obtain $\alpha = 1.817\times10^{-6}$m. With these parameters and our numerical solution,[7] the relation between the calculated chemical potential μ and the number of atoms n in condensate can be fitted by $\mu(n)$=$0.124(n+511)^{2/5}$. Substituting $\mu(n)$ in Eq.(21), for temperature $T = 2\mu$K, $n_{\mathrm{eq}} = 5\times10^5$, we can evaluate the evolutions of the number of atoms $< n >$ and its growth rate $\gamma = \frac{\mathrm{d}\ln < n >}{\mathrm{d}t}$ by solving Eqs.(18)–(20). The results are plotted in Figs.1 and 2, respectively. From Fig.2 we can see that, at the beginning of condensate formation, the growth rate is very large, in the middle of the process it shows a plateau and approaches to zero at the end. The growth rate at the plateau can represent the growth rate of condensate. Figure 3 shows the evolution of fluctuation Q with time. Initially it assumes the Poisson distribution Q=1, then increases rapidly to super-Poisson plateau Q=1.33. When n_{eq} is changed, the results are similar to the curves in Figs.1–3.

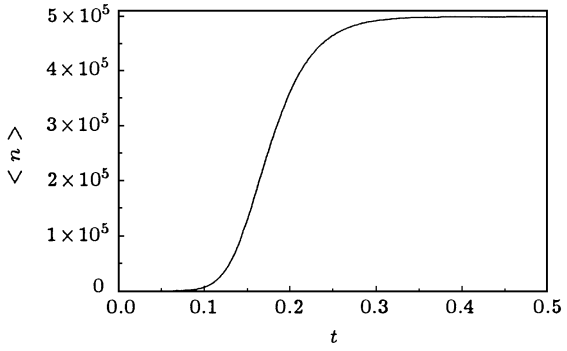


Fig.1. Condensate growth for sodium.

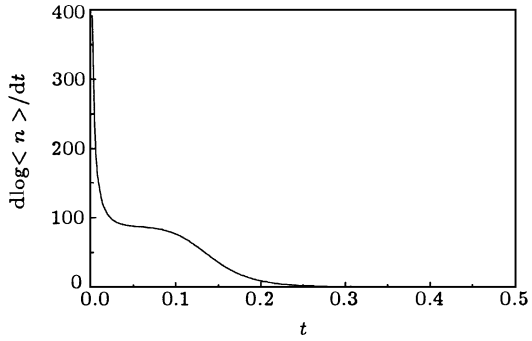


Fig.2. Evolution of condensate growth rate γ with time t.

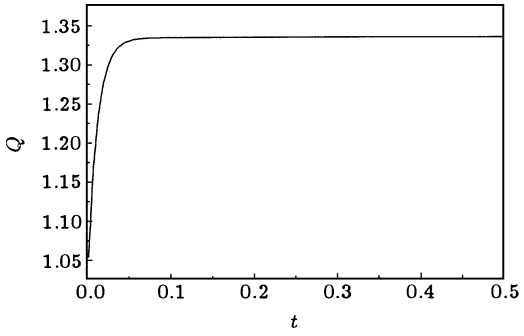


Fig.3. Fluctuation Q versus time t.

Now it is worthy to mention that, during the derivation of Eq.(21), the effect atom mean free path is neglected. Actually, although there are n_{eq} atoms in the condensate, the contributive atoms in the hard-sphere model to GP equation are $\tilde{n}_{\rm eq}$, the atoms located in the neighborhood of the mean free path l, instead of $n_{\rm eq}$. And $\tilde{n}_{\rm eq}$ is related to $n_{\rm eq}$ by the formula

$$\begin{aligned}\tilde{n}_{\rm eq} &= n_{\rm eq}\frac{\int_0^R r^2 \mathrm{e}^{-r/l}\mathrm{d}r}{\int_0^R r^2\mathrm{d}r} \\ &= n_\lambda\left(1-(1+R/l+\frac{1}{2}(R/l)^2)\mathrm{e}^{-R/l}\right), \quad (22)\\ n_\lambda &= 6l^3\frac{n_{\rm eq}}{R^3}, \quad R/l=(6n_{\rm eq}/n_\lambda)^{1/3}.\end{aligned}$$

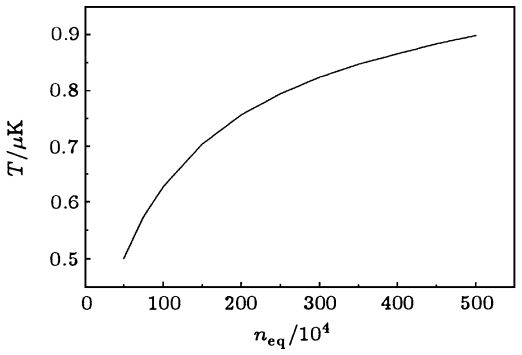


Fig.4. The final temperature T versus the number of condensate atoms in equilibrium $n_{\rm eq}$.

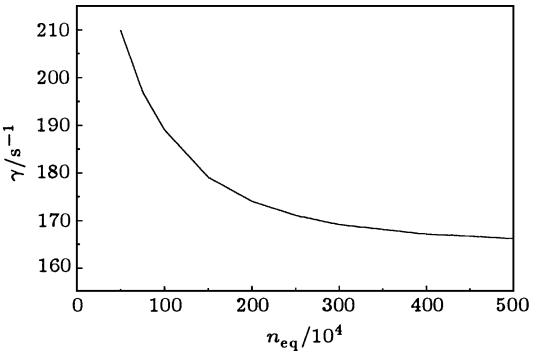


Fig.5. Condensate growth rate γ versus the number of condensate atoms in equilibrium $n_{\rm eq}$.

Given n_λ and $n_{\rm eq}$ the effective $\tilde{n}_{\rm eq}$ can be calculated through Eq.(22). Thus taking into account the mean free path effect, $n_{\rm eq}$ in Eq.(21) should be replaced by $\tilde{n}_{\rm eq}$, the formulas for A and B/A being modified also. According to Ref.[8] the mean free path $l = 1/\sqrt{2}\pi n d^2$, we assume the atomic density $n = \dfrac{n_{\rm eq}}{(4/3)\pi R^3} = 10^{15}$ and atomic diameter $d = 4.9\,\mathrm{nm}$, $n_\lambda = 8\pi l^3 n \approx 3\times10^5$. Referring to the experiment,[1] we may assume the temperature T corresponding different $n_{\rm eq}$ as shown in Fig.4, the final growth rates calculated for different $n_{\rm eq}$ are depicted by the curve in Fig.5. The relation between γ and $n_{\rm eq}$ shown in Fig.5 agrees with the experimental result on the whole.[1]

V. CONCLUSION

In conclusion, we have solved the BEC master equation by the generating function method and obtained the evolutions of growth rate, statistical fluctuation of condensate atoms. The calculated results show a plateau in growth rate configuration, super-Poisson distribution fluctuation; and γ decreases progressively with the increase of $n_{\rm eq}$.

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