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TIME-DEPENDENT LANDAU SYSTEM AND NON-ADIABATIC BERRY PHASE IN TWO DIMENSIONS

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By applying the time-independent unitary transformation, the time-dependent Landau system is transformed into a product of a time-independent Landau system's Hamiltonian and a factor only depending on time, which can be solved exactly. Both the invariant operator and the eigenstate are obtained. In the periodical time-dependent case, the non-adiabatic Berry's phase is also presented.

Keywords: unitary transformation, Landau system, non-adiabatic Berry's phase
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I. INTRODUCTION

In recent years the quantum mechanical motion of a charged particle in a uniform magnetic field, i.e., the Landau system has attracted much attention of researchers, and many interesting properties of this system were revealed.[1−3] As we know, if the charged particle is trapped in a static magnetic field which does not vary with space and time, the system is analytically solvable.[4] An interesting problem is that the magnetic field acting on the charged particle varies with time. It will be seen this system is also exactly solvable and the non-abiabatic Berry's phase can be got if the change of magnetic field is periodic in time.

II. GENERAL METHOD FOR SOLVING TIME-DEPENDENT QUANTUM SYSTEM

The general method of solving time-dependent system is given in Ref[5]. Consider the time-dependent Schrödinger equation ($\hbar$=1)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{1}$$

and introduce the following unitary transformation[6]

$$|\psi(t)\rangle' = U^{-1}(t)|\psi(t)\rangle. \tag{2}$$

It is obvious that if the Hamiltonian goes through the following transformation

$$U^{-1}(t)H(t)U(t) - \mathrm{i}U^{-1}(t)\frac{\partial U(t)}{\partial(t)} \equiv H(t)', \tag{3}$$

the Schrödinger equation will maintain its original form, i.e.,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle' = H(t)'|\psi(t)\rangle', \tag{4}$$

Our aim is to choose several appropriate unitary transformations to make the time-dependent Hamiltonian become the general Hill-Floquet-Bloch (HFB) form, i.e. the TD Hamiltonian H' is transformed into the form

$$H(t)' = \eta(t)H'_{\mathrm{o}}, \tag{5}$$

where H'_{o} is a time-independent Hamiltonian, but $\eta(t)$ is a time-dependent function. In this case the solution for $|\psi\rangle'$ takes the following form

$$|\psi(t)\rangle' = \exp[-\mathrm{i}H'_{\mathrm{o}}\int_{t_0}^{t}\eta(s)\mathrm{d}s]|\phi\rangle, \tag{6}$$

where $|\phi\rangle$ can be expressed by the eigenstates of the time-independent Hamiltonian H'_{o}, i.e. $H_{\mathrm{o}}|\phi_n\rangle = E_n|\phi_n\rangle$ and $|\phi\rangle = \sum a_n|\phi_n\rangle$, in which a_n are some constants. Combining Eq.(2) and Eq.(6), the solution for the Shrödinger equation(1) is

$$|\psi(t)\rangle = U(t)\sum a_n\exp[-\mathrm{i}E_n\int_{t_0}^{t}\eta(s)\mathrm{d}s]|\phi_n\rangle. \tag{7}$$

With the help of Eq.(3), we find that $I(t) = U(t)H_{\mathrm{o}}U^{-1}(t)$ is the invariant of the system, i.e. $I(t)$ satisfies the equation $\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0$. It is easy to prove that the eigenfunction of $I(t)$ is

$$|\lambda_n(t)\rangle = U(t)|\phi_n\rangle, \tag{8}$$

and its eigenvalues are the same as those of H_{o}. In order to give non-adiabatic Berry's phase, the Hamiltonian should be periodical, i.e. $H(t+T) = H(t)$. If the transformation $U(t)$ has the same period as the Hamiltonian $H(t)$ and the initial state of the system is chosen as the eigenstate of H'_{o}, it can be seen from

Eq.(7) that the state should return to itself after a period T up to a Hill's phase $\alpha_n(T)$,

$$\alpha_n(T) = E_n \int_0^T \eta(s)\mathrm{d}s. \tag{9}$$

The dynamic phase is the time integral of the instantaneous expectation value of the Hamiltonian $\delta_n = \int_0^T \langle\psi_n(t)|H(t)|\psi(t)\rangle \mathrm{d}t$. Following Ref.[5] the non-adiabatic Berry's phase is acquired by using Eqs.(3), (7) and (9),

$$\beta_n = \alpha_n - \delta_n = -\mathrm{i}\int_0^T \langle\phi_n|U^{-1}\frac{\partial U}{\partial t}|\phi_n\rangle \mathrm{d}t, \tag{10}$$

which shows that the non-adiabatic Berry's phase is independent of the eigenvalue of the Hamiltonian H_o, but depends on its eigenstate and corresponding unitary transformation U.

Ⅲ. TIME-DEPENDENT LANDAU SYSTEM AND ITS SOLUTION

Consider a charged particle moving in a magnetic field B along the z-axis. For the present purpose, we can neglect the free motion of the particle in the z direction. Thus, if the vector potential of magnetic field varies with time, working in the symmetric gauge, $A(t) = \frac{B(t)}{2}(-y, x, 0)$, the Hamiltonian of this two-dimensional (2D) system can be written as[7]

$$H = H_\mathrm{space}(t) + H_\mathrm{spin}(t), \tag{11}$$

$$H_\mathrm{space}(t) = \frac{\boldsymbol{P}^2}{2m} + \frac{1}{2}m\omega^2(t)\boldsymbol{r}^2 + \frac{\omega_c}{2}l_z, \tag{12}$$

$$H_\mathrm{spin}(t) = -2\mu_B B(t) s_z, \tag{13}$$

where $\omega^2(t) = \omega_c^2/4$ and $\omega_c = \dfrac{eB(t)}{mc}$ is the cyclotron's frequency. The momentum and position of the particle are given by the 2D vectors $\boldsymbol{p}$ and $\boldsymbol{r}$, respectively; l_z is the z component of the angular momentum.

H_spin is the Zeeman energy, and the eigenstates of $H_\mathrm{spin}(t)$ are just the spinors of the charged particle. Due to $[H_\mathrm{spin}(t), H(t)]$=0, for simplicity, we also neglect the influence of $H_\mathrm{spin}(t)$ and select the mass of the particle to be a unit hereafter. In order to get the spacial state, the following consecutive transformations are introduced, which make the spacial state $|\psi(t)\rangle$ become $|\psi_1(t)\rangle, |\psi_2(t)\rangle$ and $|\psi_3(t)\rangle$ respectively.

$$|\psi(t)\rangle = U_1|\psi_1(t)\rangle = \exp[-\mathrm{i}\lambda(t)l_z]|\psi_1(t)\rangle, \tag{14}$$

$$|\psi_1(t)\rangle = U_2|\psi_2(t)\rangle = \exp[\mathrm{i}C(t)\boldsymbol{r}^{\to 2}]|\psi_2(t)\rangle, \tag{15}$$

and

$$\begin{aligned}|\psi_2(t)\rangle &= U_3|\psi_3(t)\rangle \\ &= \exp[\mathrm{i}D(t)(\boldsymbol{r}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{r})]|\psi_3(t)\rangle.\end{aligned} \tag{16}$$

where $\lambda(t), C(t)$ and $D(t)$ are real functions of time. These functions are chosen in such a way that the Hamiltonian $H_\mathrm{space}(t)$, after these transformations, becomes the HFB's form according to the method in section (II). In addition, it is important to note that the transforms in Eqs.(15) and (16) also comprise the squeezing transforms due to the change of the magnetic field, as pointed out by the authors of Refs.[8] and [9].

It's easy to be shown that under these transformations the coordinate and momentum operators will change in the following way:

$$U_1^{-1}xU_1 = x\cos\lambda + \mathrm{i}y\,\sin\lambda;$$

$$U_1^{-1}yU_1 = y\cos\lambda - \mathrm{i}x\,\sin\lambda;$$

$$U_1^{-1}p_xU_1 = p_x\cos\lambda + \mathrm{i}p_y\sin\lambda;$$

$$U_1^{-1}p_yU_1 = p_y\cos\lambda - \mathrm{i}p_x\sin\lambda;$$

$$U_1^{-1}\boldsymbol{r}^2U_1 = \boldsymbol{r}^2;\ \ U_1^{-1}\boldsymbol{p}^2U_1 = \boldsymbol{p}^2; \tag{17}$$

$$U_1^{-1}\boldsymbol{r}U_2 = \boldsymbol{r}^2;\ \ U_1^{-1}\boldsymbol{p}U_2 = \boldsymbol{p} + 2C(t)\boldsymbol{r}; \tag{18}$$

$$U_3^{-1}\boldsymbol{r}U_3 = \boldsymbol{r}^2\exp[-2D(t)];$$

$$U_3^{-1}\boldsymbol{p}U_3 = \boldsymbol{p}\,\exp[2D(t)]. \tag{19}$$

Using Eqs.(3) and (17), we get the Hamiltonian:

$$\begin{aligned}H_1(t) &= U_1^{-1}H_\mathrm{space}(t)U_1 - \mathrm{i}U_1^{-1}\frac{\partial U_1}{\partial t} \\ &= \frac{\boldsymbol{P}^2}{2} + \frac{1}{2}\omega^2(t)\boldsymbol{r}^2 + (\frac{\omega_c}{2} - \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t})l_z.\end{aligned} \tag{20}$$

Obviously, if $\lambda(t)$ is chosen to satisfy

$$(\frac{\omega_c}{2} - \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}) = 0, \tag{21}$$

the term of l_z disappears in Eq.(20). That is, the magnetic field is removed by taking an appropriate rotating frame. Note that the rotational angular function $\lambda(t)$ has been determined by Eq.(21). During the process of getting Eq.(20), the Hamiltonian $H_1(t)$ becomes $H_2(t)$, $H_3(t)$ respectively after using U_2 and U_3 transformations whose generators are the monopole compression and dilatation operators:

$$H_2(t) = U_2^{-1}H_1(t)U_2 - \mathrm{i}U_2^{-1}\frac{\partial U_2}{\partial t}$$

$$= \frac{\boldsymbol{P}^2}{2} + [\frac{\partial C(t)}{\partial t} + 2C^2(t)]\boldsymbol{r}^2 + C(t)(\boldsymbol{r}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{r}), \quad (22)$$

and

$$H_3(t) = U_3^{-1}H_2(t)U_3 - \mathrm{i}U_3^{-1}\frac{\partial U_3}{\partial t}$$

$$= \exp[4D(t)][\frac{\boldsymbol{P}^2}{2} + \exp(-4D(t))[\frac{\mathrm{d}C(t)}{\mathrm{d}t} + 2C^2(t)]$$
$$+ [C(t) + \frac{\mathrm{d}D(t)}{\mathrm{d}t}](\boldsymbol{r}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{r}), \quad (23)$$

In order to get $H_3(t)$ in a HFB's form, $C(t)$, $D(t)$ are chosen to satisfy

$$C(t) + \frac{\mathrm{d}D(t)}{\mathrm{d}t} = 0, \quad (24)$$

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} + 2C^2(t) = \frac{\Omega^2}{2}\exp[8D(t)], \quad (25)$$

where Ω is a constant. Setting $\rho(t)=\exp[-2D(t)]$, Eq.(25) can be rewritten as:

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}^2t} + \frac{1}{2}\omega^2(t) = \frac{\Omega^2}{\rho^3}. \quad (26)$$

After these choices, H_3 is shown in the HFB's form

$$H_3(t) = \frac{1}{\rho^2(t)}[\frac{\boldsymbol{P}^2}{2} + \frac{1}{2}\Omega^2\boldsymbol{r}^2]\frac{1}{\rho^2(t)}H_0, \quad (27)$$

where H_0 is the Hamiltonian of the Landau system in 2D space and $\frac{1}{\rho^2(t)}$ is determined by the auxiliary equation (26). Thus, the time-dependent Hamiltonian $H_{\mathrm{space}}(t)$ with a term $\frac{1}{2}\omega_c(t)l_z$ can be transformed into a product of a time-independent Landau system and a time-dependent factor.

According to Eq.(6), the solution of the Shrödinger equation of $H_3(t)$ is given by

$$|\psi_3(t)\rangle = \exp[-\mathrm{i}H_0\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\mathrm{d}\tau]|\psi_3(t_0)\rangle. \quad (28)$$

To get an explicit solution, let us assume the initial state as: $|\psi_3(t_0)\rangle = \sum_n a_n|\phi_n\rangle$, where $|\phi_n\rangle$ is the eigenstate of the time-independent Landau system's Hamiltonian H_0. We have

$$|\psi_3(t)\rangle = \sum_n a_n\exp[-\mathrm{i}E_n\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\mathrm{d}\tau]|\phi_n\rangle. \quad (29)$$

The exact solution of the Hamiltonian $H_{\mathrm{space}}(t)$ can now be found by combining the above results. We finally obtain

$$|\psi(t)\rangle = U(t)|\psi_3(t)\rangle$$

$$= \sum_n a_n\exp[-\mathrm{i}E_n\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\mathrm{d}\tau]U(t)|\phi_n\rangle$$
$$= \sum_n a_n|\,\Psi_n(t)\rangle, \quad (30)$$

where $U(t) = U_1U_2U_3$ and $|\,\Psi_n(t)\rangle = \exp[-\mathrm{i}E_n\int_{t_0}^{t}\cdot\frac{1}{\rho^2(\tau)}\mathrm{d}\tau]U(t)|\phi\rangle$.

We can see from Eq.(30) that if the eigenstates and eigenvalues of H_0 are known, then the solutions of the time-dependent Hamiltonian $H_{\mathrm{space}}(t)$ can be obtained. As an example, we consider the ground state and its eigenvalue of the Landau system, which are given as[7]

$$|\phi_0\rangle = \exp[-\omega_L\boldsymbol{r}^2/2]\boldsymbol{r}^{|m|},$$

$$E_0 = (m + |m| + 1)\omega_L, \quad (31)$$

where ω_L is the Lamor frequency. If $|\phi_0\rangle$ is chosen as the initial state $|\psi_3(t_0)\rangle$, the evolution of the state can be written, by using Eq.(7), as

$$|\psi_0(t)\rangle = \exp[-\mathrm{i}\int_{t_0}^{t}\frac{1}{2}\omega_c(\tau)\mathrm{d}\tau l_z]\exp[-\mathrm{i}\frac{\rho(t)}{2\rho(t)}\boldsymbol{r}^2]$$
$$\cdot\exp[-\mathrm{i}\frac{1}{2}\ln\rho(t)(\boldsymbol{r}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{r})]$$
$$\cdot\exp[-\mathrm{i}E_0\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\mathrm{d}\tau]\exp[-\omega_L\boldsymbol{r}^2/2]\boldsymbol{r}^{|m|}. \quad (32)$$

It can be easily proved that the invariant operator of the Hamiltonian $H_{space}(t)$ is

$$I(t) = U(t)H_0U^{-1}(t) = \frac{1}{2m}(\rho\boldsymbol{p} - \dot{\rho}\boldsymbol{r})^2 + \frac{1}{2\rho^2}\Omega_0^2\boldsymbol{r}^2, \quad (33)$$

and $|\,\Psi_n(t)\rangle$ is the eigenstate of $I(t)$ with eigen value $E_n(\Omega_0 = \omega_L)$.

In order to give non-adiabatic Berry's phase, a periodical Hamiltonian $H_{\mathrm{space}}(t+T) = H_{\mathrm{space}}(t)$ is

considered. According to Eq.(10) the non-adiabatic Berry's phase of the system is

$$\beta(T) = -\int_0^T \langle\phi_0(\tau)|\boldsymbol{r}^2|\phi_0(\tau)\rangle \oint \rho \mathrm{d}\rho. \qquad (34)$$

Equation (34) shows that due to the time-dependent magnetic field acting on the Landau system, non-adiabatic Berry's phase will appear.

REFERENCES

[1] M. S. Abdalla, *Phys. Rev.*, **A44**(1991),2024.

[2] V. M. Tkachuk, S. I. Vakachuk, *Phys. Lett.*, **A228** (1997), 141.

[3] H. Y. Fan, H. Zou, Y. Fan, *Chin. Phys. Lett.*, **16**(1999), 706.

[4] L. Quiroga, D. R. Ardila, *Solid State Commun.*, **86** (1998),775.

[5] M. Maamache, *J. Phys.* A, **39**(1998),161.

[6] F. Calogero, *J. Math. Phys.*, **12**(1971),419.

[7] H. Jing, Q. Y. Shi, *Acta Physics Sinica*, **48**(1999),1581 (in Chinese).

[8] H. Y. Fan, Y. Zhang, *Mod. Phys. Lett.*, **B11**(1997),239.

[9] H. Y. Fan, Y. Fan, *Commun. Theor. Phys.*, **30**(1998),301.