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ELECTROSTATIC POTENTIAL OF STRONGLY NONLINEAR COMPOSITES: HOMOTOPY CONTINUATION APPROACH*

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The homotopy continuation method is used to solve the electrostatic boundary-value problems of strongly nonlinear composite media, which obey a current-field relation of $J = \sigma E + \chi|E|^2 E$. With the mode expansion, the approximate analytical solutions of electric potential in host and inclusion regions are obtained by solving a set of nonlinear ordinary different equations, which are derived from the original equations with homotopy method. As an example in dimension two, we apply the method to deal with a nonlinear cylindrical inclusion embedded in a host. Comparing the approximate analytical solution of the potential obtained by homotopy method with that of numerical method, we can obverse that the homotopy method is valid for solving boundary-value problems of weakly and strongly nonlinear media.

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I. INTRODUCTION

The study of nonlinear composite media has received much attention recently because of its potential applications in engineering and physics.[1−5] In particular, the electrostatic potentials of nonlinear composite systems consisting of two or more mateials are estimated by means of different methods,[6−10] which are suitable for different nonlinear composite problems over the past few years. For example, the weakly nonlinear composites, in which the linear part is dominant in the constitutive relation, can be coped with the perturbation method.[11] The strongly nonlinear composites, which obey a local current-field relation of the form $J = \chi|E|^2 E$, can be treated by the parameter variational method.[12] In this work, we study the electrostatic potential of nonlinear composite media in which the inclusion and host regions are subject to the current-field relation of the form $J = \sigma E + \chi|E|^2 E$. As we know, the boundary-value problem of strongly nonlinear (i.e., $(\chi|E|^2/\sigma) > 1$) composite media is extremely difficult to solve. As the nonlinearity appears as the leading form of the behavior rather than as a small correction to a predominant liner response, the conventional perturbation method fails. However, Gu and Yu[13] have introduced the homotopy continuation method, which has been proposed to deal with various problems in physics and mechanics,[14−16] to handle electrostatic boundary-value problems of nonlinear media for two cases: one is nonlinear inclusion in linear host, the other is linear inclusion in nonlinear host. In the first case, the local potential has been obtained exactly by the perturbation method. The aim of our paper is to obtain the approximate analytical electrostatic potential of strongly nonlinear and weakly nonlinear composite media by means of homotopy method.

Following Ref.[13], for convenience of discussion, we assume that in some regions of the nonlinear composite media at zero frequency, the current density J is related to the local electric field E by a nonlinear equation

$$J = \sigma E + \chi|E|^2 E, \tag{1}$$

where σ and χ are the first and third conductivity coefficients of the medium. We denote σ_m, χ_m as the coefficients in host and σ_i, χ_i as those in inclusion. For electrostatic problem, the current density J and field E satisfy

$$\nabla \cdot J = 0, \tag{2}$$

$$\nabla \times E = 0. \tag{3}$$

Therefore, from Eq.(3), there exists an electric potential φ such that

$$E = -\nabla\varphi. \tag{4}$$

The boundary conditions for the continuity of potential and current density are applied on the surfaces of inclusion

$$\varphi^m = \varphi^i \qquad \text{on } \partial\Omega_i, \tag{5}$$

$$n \cdot J^m = n \cdot J^i \qquad \text{on } \partial\Omega_i, \tag{6}$$

where the superscripts m and i denote the quantities in the host and inclusion region, respectively; $\partial\Omega_i$ denotes the surface of the inclusion.

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From the above nonlinear equations, the analytical solution of the electrostatic potential in strongly nonlinear region is difficult to obtain because of the boundary-value problem of strongly nonlinear media. In this paper, we introduce a mode function expansion to overcome the difficulty of matching the boundary-value, i.e., the nonlinear partial differential equations (PDEs) are transformed into a set of ordinary differential equations (ODEs). With the homotopy method, in the strongly and weakly nonlinear composite media, the approximate analytical solutions of the electrostatic potential are derived from the ODE, and the validity of the formalism is tested by comparing the results obtained by homotopy method with that of the numerical method for various values of parameter $\chi|E|^2/\sigma$.

Ⅱ. THE MODE FUNCTION EXPANSION

In Ref.[13], the mode expansion method was used to deal with nonlinear electrostatic problems. The idea of the mode method is to obtain a set of ODEs by constructing a complete and orthogonal set of basis functions according to the given boundary conditions and substituting them into original PDEs. In order to perform the mode expansion, without loss of generality, let us consider a boundary-value problem in tow-dimension of a cylindrical inclusion of radius r_1 suspended in a host medium with the application of an external uniform field E_0. Based on the basis functions (in this case, the basis functions are the trigonometric functions), the electric potentials in the host region and the inclusion region are expanded as

$$\varphi^{\alpha}(r,\theta) = \varphi_1^{\alpha}(r)\cos\theta + \varphi_3^{\alpha}(r)\cos 3\theta + \ldots, \quad \alpha = i, m, \tag{7}$$

where r and θ are the radial and the angular variables, respectively; $\varphi_n^{\alpha}(r)$ is called the mode function.

To solve the governing equations (1)–(4), we substitute the mode expansion Eq.(7) into Eqs.(1)–(4) as well as into the boundary conditions (5) and (6), and give a set of differential equations containing odd powers of the trigonometric functions of angular variable θ. Due to the symmetry of the problem, all powers of the trigonometric functions can be expressed by $\cos[(2n-1)\theta], n = 1, 2, \ldots$. Therefore, by comparing the coefficients of trigonometric functions, the governing equations and the boundary conditions are transformed into a set of nonlinear ODEs,

$$h_n^i(r) = 0 \qquad \text{in } \Omega_i, \tag{8}$$

$$h_n^m(r) = 0 \qquad \text{in } \Omega_m, \tag{9}$$

$$\chi_i g_n^i(r) = \chi_m g_n^m(r)|_{r=1}, \tag{10}$$

$$\varphi_n^i(r) = \varphi_n^m(r)|_{r=1}, \tag{11}$$

where $r = r_1 = 1$; h_n^{α} and g_n^{α} are functions of $\varphi_n^{\alpha}(r)$ and their derivatives. Equations (8) and (9) are called mode functional equations. For the convenience of discussion, truncating the equations by a finite number of terms, we get an approximate analytical solution of the original problem. For instance, we keep the lowest approximation, i.e., the first mode. The resulting equations in the inclusion and host regions are as follows:

$$\sigma_{\alpha} F_1\left[\varphi_1^{\alpha}\right] + \chi_{\alpha} F_3\left[\varphi_1^{\alpha}\right] = 0 \quad \alpha = i, m \quad \text{in } \Omega_{\alpha}, \tag{12}$$

$$\varphi_1^i(r) = \varphi_1^m(r)|_{r=1}, \tag{13}$$

$$\sigma_i J_1\left[\varphi_1^i\right] + \chi_i J_3\left[\varphi_1^i\right] = \sigma_m J_1\left[\varphi_1^m\right] + \chi_m J_3\left[\varphi_1^m\right]|_{r=1}, \tag{14}$$

where $F_1[\varphi_1] = -(4r^2\varphi_1 - 4r^3\varphi_{1r} - 4r^4\varphi_{1rr})/4r^4$,

$F_3[\varphi_1] = (-3\varphi_1^3 - r\varphi_1^2\varphi_{1r} + r^2\varphi_1\varphi_{1r}^2 + 3r^3\varphi_{1r}^3 + r^2\varphi_1^2\varphi_{1rr} + 9r^4\varphi_{1r}^2\varphi_{1rr})/4r^4$,

$J_1[\varphi_1] = \varphi_{1r}$ and $J_3[\varphi_1] = (1/4r^2)\varphi_1^2\varphi_{1r} + 3\varphi_{1r}^3/4$.

Ⅲ. HOMOTOPY METHOD AND TESTEMENT BY NUMERICAL SOLUTION

To solve the boundary-value problem with homotopy method, we introduce two homotopy functions, $\Phi_m(r,p)$ and $\Phi_i(r,p)$, to the mode functional equations (12)–(14), and obtain the homotopy equations in the inclusion and host regions as follows, respectively:

$$\sigma_i F_1[\Phi_i(r,p)] + \chi_i F_3[\Phi_i(r,p)] = (1-p)[\sigma_i F_1[\varphi_0^i] + \chi_i F_3[\varphi_0^i]] \quad \text{for} \quad r < 1 \tag{15}$$

and

$$\sigma_m F_1[\Phi_m(r,p)] + \chi_m F_3[\Phi_m(r,p)] = (1-p)[\sigma_m F_1[\varphi_0^m] + \chi_m F_3[\varphi_0^m]] \quad \text{for} \quad r \geq 1, \tag{16}$$

where the initial guess potential, φ_0^α, is arbitrarily chosen. The parameter p varies from zero to unity. Matching the boundary conditions on the surface of inclusion, the continuity of potential and flow are, respectively:

$$\Phi_i(r,p) - \Phi_m(r,p) = (1-p)[\varphi_0^i(r) - \varphi_0^m(r)]|_{r=1} \tag{17}$$

and

$$\begin{aligned} &\sigma_m J_1[\Phi_m(r,p)] + \chi_m J_3[\Phi_m(r,p)] \\ &-\sigma_i J_1[\Phi_i(r,p)] - \chi_i J_3[\Phi_i(r,p)] \\ =\ &(1-p)\{\sigma_m J_1[\varphi_0^m(r)] + \chi_m J_3[\varphi_0^m(r)] \\ &-\sigma_i J_1[\varphi_0^i(r)] - \chi_i J_3[\varphi_0^i(r)]\}|_{r=1}. \end{aligned} \tag{18}$$

It is obvious from the homotopy equations and boundary conditions that the homotopy functions change from the initial function φ_0^α to the $\Phi_\alpha(r,1)$ as the parameter p varies from zero to unity. We can see that the solution of the original equations are recovered from the homotopy equations when p=1, that is, $\Phi_\alpha(r,0)$ is equal to φ_0^α, while $\Phi(r,1)$ gives the solution of the nonlinear problem. Thus if we find all derivatives of $\Phi_\alpha(r,p)$ at p=0, the solution of the homotopy equations (15)–(18) can be expressed by the Taylor's expansion with respect to parameter p in the inclusion and host regions

$$\Phi_1^\alpha(r,p) = \Phi_\alpha(r,0) + \sum_{k=1}^{\infty} p^k \Phi_\alpha^{[k]}(r,0)/k!,$$

where the superscript $[k]$ denotes the k-th derivative with respect to parameter p. Clearly, let p=1 the solution of the original equations (12)–(14) can be written as

$$\varphi_1^\alpha(r) = \varphi_0^\alpha(r) + \sum_{k=1}^{\infty} \Phi_\alpha^{[k]}(r,0)/k! \qquad \alpha = i,m, \tag{19}$$

where $\varphi_0^\alpha(r) = \Phi_\alpha(r,0)$.

In order to obtain the analytical solution of the original problem, we shall determine the $\Phi_\alpha^{[k]}(r,0)$ from the homotopy equations (15)–(18). To simplify the calculations, let us choose the applied field as the initial solution, and let the homotopy function at p=0 and its derivatives with respect to r be denoted respectively by the subscripts 0 and r. Taking the derivative of Eq.(16) with respect to p, the first-order derivative of the homotopy function at p=0 in the host region obeys the ODE as follows:

$$r^{-2}(r^2\,\Phi_{0mrr}^{[1]} + r\,\Phi_{0mr}^{[1]} - \Phi_{0m}^{[1]}) = 0,$$

where the solution is given by $\Phi_{0m}^{[0]} = c_1/r$. Similarly, using Eq.(15) in inclusion region, the first-order derivative of the homotopy function at p=0, $\Phi_{oi}^{[1]}$, is given by $\Phi_{0i}^{[1]} = d_1 r$. Consider the continuity potential on the surface of the inclusion, $\Phi_{0i}^{[1]} - \Phi_{0m}^{[1]} = 0|_{r=1}$, obtained from Eq.(17), we can determine $c_1 = d_1$. Applying the first-order of the homotopy function to Eq.(18), i.e., the continuity of flow, we get the constant c_1

$$\begin{aligned} c_1 = d_1 =&[(\sigma_1 - \sigma_m)E_0 + (\chi_i - \chi_m)E_0^3]/[\sigma_i + \sigma_m \\ &+ (2\chi_m + 3\chi_i)E_0^2]. \end{aligned}$$

By similar process, the second-order derivatives of the homotopy equations in the host and inclusion regions can be derived from Eqs.(15) and (16), respectively.

$$\begin{aligned} &r^2\,\Phi_{0mrr}^{[2]} + r\,\Phi_{0mr}^{[2]} - \Phi_{0m}^{[2]} \\ =&-16r^{-3}\chi_m c_1^2 E_0/[\sigma_m + 5\chi_m E_0^2/2] \end{aligned} \tag{20}$$

and

$$r^{-2}(\sigma_i + 5\chi_i E_0^2/2)(r^2\,\Phi_{0irr}^{[2]} + r\,\Phi_{0ir}^{[2]} - \Phi_{0i}^{[2]}) = 0. \tag{21}$$

Consider the continuity boundary conditions of the potential and flow on the surface of the inclusion, solving Eqs.(20) and (21), the second-order derivatives of the homotopy functions in the host and inclusion regions are, respectively,

$$\Phi_{0m}^{[2]} = A/r - ar^{-3}/8 \tag{22}$$

and

$$\Phi_{0i}^{[2]} = Br, \tag{23}$$

where $a = 16\chi_m E_0 c_1^2/(\sigma_m + 5\chi_m E_0^2/2)$,

$$A = [3a\sigma_m/8 + a\sigma_i/8 + \chi_m(7aE_0^2/8 - 4E_0c_1^2) + \chi_i(3aE_0^2/8 + 6E_0c_1^2)]/[\sigma_i + \sigma_m + (2\chi_m + 3\chi_i)E_0^2],$$
$$B = A - a/8.$$

Higher-order derivatives of the homotopy function can be obtained in the same way, aided by computer algebra. In this paper, we only determine the derivatives of homotopy function up to the second-order for the first mode. Thus, the lower approximate solutions of potential are derived by the homotopy method,

$$\varphi_1^i(r,\theta) = -A_1E_0r\cos\theta \qquad (24)$$

and

$$\varphi_1^m(r,\theta) = E_0(-r + B_1r^{-1} - C_1r^{-3})\cos\theta, \qquad (25)$$

where
$A_1 = 1 - d_2 + d_3/16 - d_4/2,$
$B_1 = d_2 + d_4/2,$
$C_1 = d_3/16,$

$$d_2 = [\sigma_i - \sigma_m + (\chi_i - \chi_m)E_0^2]/[\sigma_i + \sigma_m + (2\chi_m + 3\chi_i)E_0^2],$$
$$d_3 = 16\chi_mE_0^2d_2^2/(\sigma_m + 5\chi_mE_0^2/2),$$
$$d_4 = [3d_3\sigma_m/8 + d_3\sigma_i/8 + \chi_mE_0^2(7d_3/8 - 4d_2^2) + \chi_iE_0^2(3d_3/8 + 6d_2^2)]/[\sigma_i + \sigma_m + (2\chi_m + 3\chi_i)E_0^2].$$

To test the homotopy method, the validity of the formlism is examined by comparing the approximate solutions with the numerical solutions in the weakly and strongly nonlinear regions. The mode functions Eqs.(12)–(14) are caluclated for various values of parameter $\chi|E_0|^2/\sigma(> 1 \text{ or } < 1)$ by means of four-order Rung-Kutta algorithm. In Fig.1, the mode function $\varphi_1(r)$ derived from homotopy equation are compared with the numerical solution calculated from Eqs.(12)–(14). For convenience of comparsion between the homotopy method and numerical method, we plot the results of the quantity $G = \varphi_1(r)/\text{Max}(|\varphi_1(r)|)$ in Fig.1. As clearly seen from Fig.1, the homotopy method gives an excellent agreement with the numerical method for various values of parameter $\chi|E_0|^2/\sigma$. The comparison shows that the homotopy method is valid for solving the weakly and strongly nonlinear composite media.

IV. DISCUSSIONS AND CONCLUSIONS

In this work, applying the mode expansion to the nonlinear electrostatic potential, the homotopy method is used to solve mainly the electrostatic boundary-value problems of the strongly nonlinear meia. To avoid the mathematical complexity, we have studied the case of cylindrical inclusion embedded in host region. In order to test the validity of the homotopy method, we have compared the solutions derived by homotopy method with numer solutions, and have obtained a good agreement. Therefore, based on the approximate analytical solutions of the potential obtained by homotopy method, we conclude that the method can be used to study various nonlinear composite media. In particular, for strongly nonlinear composite media ($J = \chi|E|^2E$), we can also derive the approximate formula of electrostatic potential by assuming $\sigma_i = \sigma_m$=0 in Eqs.(24) and (25). In the discussion we should remark that although the homotopy method is used to solve cylindrical inclusion in two-dimension, the method can be generalized to three-dimensional problem by dealing with the more complicated Legendre functions. In next work, we shall disucss the effective conductivity of strongly nonlinear composite media.

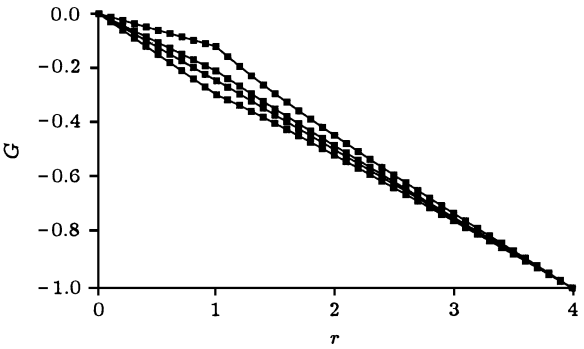


Fig.1. For cylindrical inclusion with radius unity, the results of the quantity $G = \varphi_1(r)/\text{Max}(|\varphi_1(r)|)$ obtained by homotopy method (–) and numerical method (■) are compared for various values of parameter $\chi_i|E_0|^2/\sigma_i$. From top to bottom, the parameter $\chi_i|E_0|^2/\sigma_i$=0.2 ($\chi_i > \chi_m$), 315($\chi_i > \chi_m$),0.04($\chi_i < \chi_m$),25($\chi_i < \chi_m$).

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