Volume 9, Number 6
1009-1963/2000/09(06)/0424-08
CHINESE PHYSICS
June, 2000
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TRANSIENT LIFETIME SPECTRUM OF TWO CHARGED PARTICLES IN A PAUL TRAP*

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(Received 12 November 1999; revised manuscript received 17 February 2000)

The transient properties of laser-cooled two charged particles in a Paul trap are studied numerically. We find the existence of characteristic lifetime of an attractor, which is thought to be an important feature of the transient dynamics of the weakly dissipative system. The theoretical analysis shows that it is caused by pseudo-periodic orbit which is the residual sign of periodic orbit of the focus-saddle bifurcation. Study of dissipative coupled standard maps shows that this is a general conclusion for weakly dissipative system.

PACC: 3280P; 4250; 0545

I. INTRODUCTION

Dynamical behavior of the laser-cooled ions in a Paul trap has, over recent years, attracted much attention in research which has led to significant results. Diedrich *et al.* [1] and Blumel *et al.*[2] reported the observation of a phase transition for laser-cooled ions in a Paul trap. Hoffnagle *et al.*[3] examined experimentally and numerically two trapped Ba^+ ions and found that the ordered state, quasi-periodic with two frequencies, become chaotic when a bifurcation introduced a third frequency at a critical value of a control parameter. In 1990, they pointed out[4] that the route to chaos is due solely to ion-ion collisions. In recent years they observed frequency locking of two microspheres (diameter 10 μm) in a Paul trap, in close agreement with theoretical predictions.[5,6] They also been indicated that the transition from transient to sustained chaos for two ions in a Paul trap near the edge of the stability region is due to a boundary crisis.[7] At the same time, Blumel *et al.* presented experimental and theoretical results on the behavior of ion crystals close to the Mathieu instability.[8] They proved the existence of an instability region I embedded in the single-ion Mathieu stability region M of Paul-trap control parameters a and q.[9] Their analytical calculations and numerical simulations show that cooling can result in the melting of Coulomb crystals stored in a Paul trap.[10] Recently our group has also been engaged in studies of the trapped ions system and found the bistable and multi-attractor coexistence phenomenon.[11] The basins of attraction have a riddled structure.[11,12] By studying the transient behavior we point out that permanent chaos does not exist and every initial condition is always attracted to an attractor which is a frequency-locking state.[11]

In this paper, we pay our attention to the transient process of the system. It is found that although the transient lifetimes of every initial condition are quite different, the transient lifetimes of some initial conditions from which the system settles into the same attractor are confined to a small region on the time axis. In other words, there exists a certain transient lifetime distribution for each attractor, which we call the transient lifetime spectrum. We also find the phenomenon in coupled dissipative standard maps. The rest of the paper is organized as follows: In section II, the periodic attractors and transient lifetime spectrum are introduced and discussed. In section III, on the basis of a discussion of the dynamics of coupled dissipative standard maps, we point out that in some aspects, dynamical behavior of the two systems is similar. Section IV summarizes results and analyses the mechanism of transient lifetime spectrum.

II. PERIODIC ATTRACTORS AND TRANSIENT LIFETIME SPECTRUM

For two trapped ions having the same charge e and mass m the scaled classical equations of motion for the relative motion of an ion pair can be written as dimensionless equations[3]

*Project supported by the National Natural Science Foundation of China (Grant No. 19835020).

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \Gamma\frac{\mathrm{d}r}{\mathrm{d}\tau} - r\left[\frac{\alpha}{(r^2+z^2)^{3/2}} + q\cos 2\tau\right] = 0, \tag{1}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + \Gamma\frac{\mathrm{d}z}{\mathrm{d}\tau} + z\left[-\frac{\alpha}{(r^2+z^2)^{3/2}} + 2q\cos 2\tau\right] = 0, \tag{2}$$

where (z, r) refers to the trap's axial and radial coordinates. The scaled time variable is $\tau = \Omega t/2$, where Ω is the micro-motion angular frequency. This pair of coupled, nonlinear differential equations describing the motion of two ions in a Paul trap is referred to as the Mathieu-Coulomb equations[7] for the special case of V_{dc}=0. Several interactions are involved here. The Coulomb interaction, which is a nonlinear one, will cause the two particles to collide and begin a more unstable motion when they are close enough to each other. It is this force that is the cause of transient chaotic motion of the particles.[4] The trap force, whose intensity is represented by q, is a periodic driving force with Ω as its angular frequency. This term indicates that the system can exchange energy with the outside continuously. The damping force Γ due to laser radiation pressure determines that the system is not a Hamiltonian system but a dissipative one. Since the equations of motion come from the classical model and depend only on dimensionless combinations of charge and mass, the dynamics is essentially the same for particles of any size.

A. Multi-attractor coexistence

Equations (1)and (2) were investigated by using numerical integration. To simplify the problem we fixed the intensity of the Coulomb interaction at α=16.2. By changing values of the other two parameters Γ and q, we find a variety of final states, which are frequency-locked attractors with different winding numbers. For a certain q, the number of attractors increases with the decrease of damping parameter Γ. For instance, when q=0.76 and $\Gamma = 9\times10^{-3}$, three attractors of periodic-1 limit cycle, 1/2 frequency-locked state (FL) and 1/8 FL coexist (the fraction is the winding number in (r, v_{r}) subspace). When Γ decreases to 4×10^{-4}, 24 other attractors emerge apart from the above three, some of them with winding numbers 2/15, 1/10. The number of attractors as a function of damping parameter is shown in Fig.1. For each parameter Γ, we choose 400 initial conditions. The final state is automatically judged by the computer. It can be seen that the relationship between the number of attractors and damping Γ shows an exponential law although deviation still exists. This means that the number of attractors increases greatly when the damping decreases. From the inverse ratio relation between the damping parameter and average transient lifetime,[11] we know that the average transient lifetime duration determines the number of attractors. The longer the average lifetime, the larger the number of attractors. The main reason of deviation from the exponential law in Fig.1 is the limited number of samples in statistics, that is, 400 initial values is too small. The difficulty of using more samples comes from the limited amount of computation time, because the transient process of every initial condition is very long. If more initial conditions are tested, some attractors with relatively small basins of attraction will be found. Presumably the curve $N-\Gamma$ would be close to a good exponential decay if a very large number of samples ($N > 10^4$) were chosen.

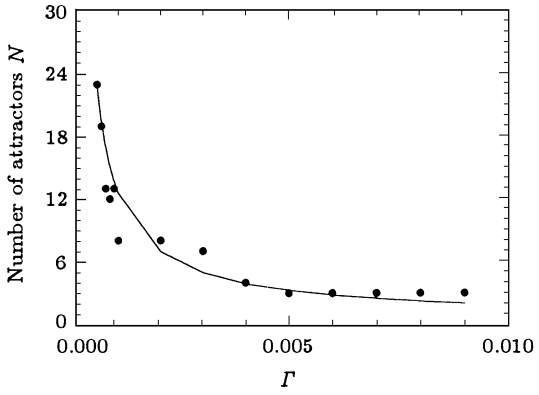


Fig.1. The number of attractors as a function of damping Γ, when q=0.76. For each Γ, 400 initial conditions are chosen.

B. Transient lifetime spectrum of attractors

The result of Fig.1 tells us that under the conditions of small damping, multi-attractors coexist. If we pay attention to the transient lifetime of every attractor, an interesting phenomenon is observed. Every attractor has its own average transient lifetime,

named T_i. For a certain damping Γ, the locations of T_i on the time axis are fixed and separate from each other. Figure 2(a) is plotted for the case of q=0.76 and $\Gamma = 7 \times 10^{-4}$ which expresses this fact clearly. The initial conditions are selected randomly and uniformly in a box of (-5.0, 5.0) for z, $\dot{z}$, r and $\dot{r}$. The horizontal axis is the transient lifetime and the vertical axis the number of initial conditions under which particles arrive at a FL state after the corresponding transient lifetime, so the curve in the figure is a distribution of the number of initial conditions as a function of transient lifetime. In order to explain the peaks in Fig.2(a), we plot the other figures of Fig.2. Figure 2(b) is the number of initial conditions being attracted to 5/37 FL as a function of transient lifetime. Similarly, Fig.2(c) is the corresponding curve related to attractor 7/52 FL and Fig.2(d) is related to 1/15 FL. Comparing Fig.2(a) with the other three, it is clear that the first sharp peak corresponds to all initial conditions whose final states are on the 5/37 FL attractor; the second peak corresponds to that of the 7/52 attractor and the fourth to the 1/15 attractor. Besides the three main attractors there are ten others. They altogether only have a share of 14.75% of all initial conditions and their corresponding peaks can be seen directly from Fig.2(a). Our numerical results indicate that they have the same property as that of the former three. We thus conclude that every attractor has an independent $N - \tau$ relation (where N means the number of initial conditions arriving at the same attractor). We call the collection of the relations the transient lifetime spectrum of the attractors. It can be seen from Fig.2 that the transient lifetime spectrum is discrete, so the average transient lifetime of every attractor is fixed on the time axis for the same damping, which we call the characteristic lifetime of an attractor. When $\Gamma = 7 \times 10^{-4}$, for instance, the characteristic lifetimes of the three attractors, 5/37, 7/52 and 1/15, are 0.64×10^5, 0.96×10^5 and 1.69×10^5, respectively.

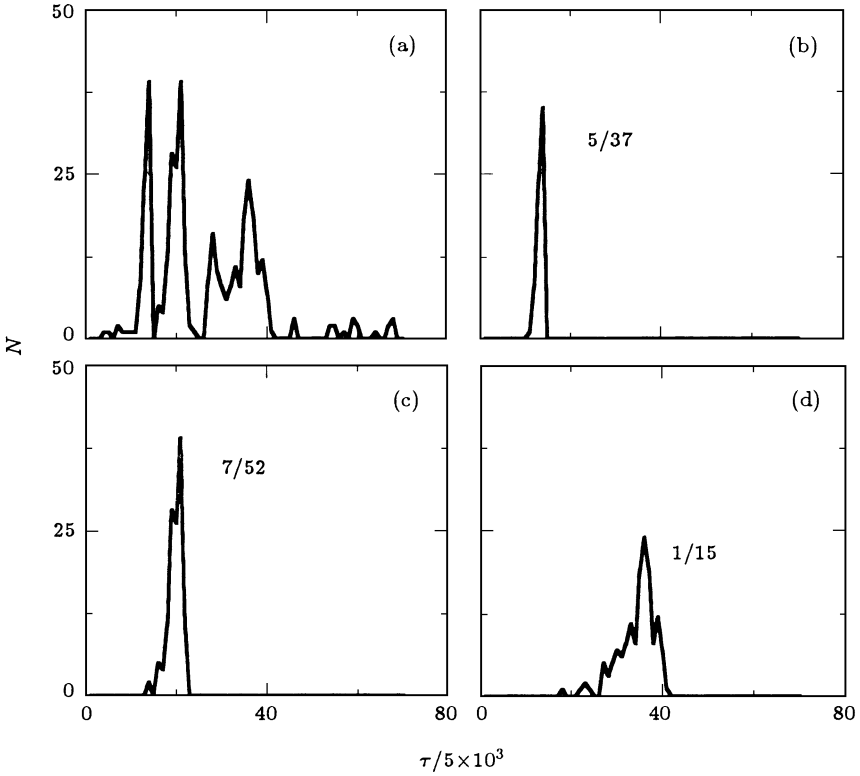


Fig.2. The distribution of the number of initial conditions as a function of transient lifetime-the transient lifetime spectrum when q=0.76 and $\Gamma = 7 \times 10^{-4}$. (a) The transient lifetime spectrum of the 13 attractors; (b), (c) and (d) the transient lifetime spectrum of 5/37 FL, 7/52 FL and 1/15 FL, respectively.

On the other hand, numerical results indicate that the characteristic lifetime will change with the change of damping Γ. The tendency of the change is that when Γ is decreased, the characteristic lifetime increases exponentially, which is the same as that shown in Fig.5 of Ref.[11]. For the above three attractors, when Γ is increased to 1.56×10^{-3}, their characteristic lifetimes decrease to $0.61 \times 10^5, 0.89 \times 10^5$ and 1.56×10^5, respectively.

C. Transient lifetime spectrum is an intrinsic property of the weakly dissipative system

The existence of the transient lifetime spectrum and characteristic lifetimes of attractors are the essential transient features of the system as long as the damping is small. To verify this conjecture, we investigate the effect of initial kinetic and potential energies of the particles on the average transient lifetime. First, initial velocities (kinetic energy) are fixed at v_z=10 and v_r=0 and initial coordinates (potential energy) z, r are changed from 1 to 20 by steps of unity. For the 400 initial conditions, 13 attractors can be found when $\Gamma = 7 \times 10^{-4}$ and q=0.76. Their average transient lifetimes and percentages are listed in Table 1.

Table 1. Number of initial conditions=400, (v_z, v_r)=(10,0), $\Gamma = 7 \times 10^{-4}$.

Q^*	Number	Average lifetime	Percentage/ %
1	3	91275	0.75
2	5	205182	1.25
8	7	89310	1.75
15	148	168974	37
16	3	452627	0.75
24	12	281745	3
31	5	335219	1.25
36	1	189428	0.25
37	68	63546	17
40	1	148166	0.25
41	1	670814	0.25
52	125	95781	31.25
82	21	137330	5.25

* in (r, v_r) subspace, winding number=P/Q.

Then, changing the velocities from v_z=10 to 30 and v_r=0 to 30, and keeping the other parameters and initial conditions unchanged, we obtain Table 2. Comparing the two tables, we see several new attractors appear and some original attractors disappear in Table 2 and the total number of attractors is 13 in Table 1 and 15 in Table 2. Despite the differences, the average transient lifetimes of the main attractors (the high percentage attractors, for instance, 5/37, 1/15, 7/52 FL) in both tables are similar, and the percentage of the attractors has no clear difference in the two tables. Furthermore, by changing the region of initial coordinates (r, z) from 42 to 80 by step 2, and fixing velocities at v_z=30 and v_r=30, the initial potential energy is changed. Under these 400 initial conditions we can also obtain data similar to those in Table 1 or in Table 2. The above results show that the average transient lifetime of attractors is not related to the initial energy of the particles. The transient lifetime spectrum and the characteristic lifetime of attractors is an intrinsic property of the weakly dissipative nonlinear system, as we have conjectured.

Table 2. Number of initial conditions=400, (v_z, v_r)=(30,30), $\Gamma = 7 \times 10^{-4}$.

Q^*	Number	Average lifetime	Percentage/ %
1	4	174892	1
2	4	180831	1
8	11	98720	2.75
15	150	173015	37.5
16	4	458566	1
24	14	263288	3.5
31	3	333219	0.75
37	57	62572	14.25
46	1	291332	0.25
51	3	34384	0.75
52	124	94063	31
56	1	398862	0.25
72	1	227564	0.25
82	22	136941	5.5
96	1	38135	0.25

* in (r, v_r) subspace, winding number=P/Q.

III. NOMADIC PHENOMENON

The transient processes starting from different initial conditions and finally settling into similar stable periodic orbit are almost similar. In the transient process, some intermittent periodic behavior can be dis-

cerned. The system seems to pass by some periodic orbits, which we call pseudo-periodic orbits, and stays on one of the pseudo-periodic orbits for some time and then depart from it for another one and so on, until settles into a stable periodic orbit. This nomadic process can be expressed more clearly by a time bifurcation diagram shown in Fig.3. The phenomenon is in some way just like the intermittent chaos. We think the phenomenon is related to saddle-node bifurcation of the stable periodic orbit, and manifests the bifurcation. As pseudo-periodic motion is a main part of the transient process, the characteristic lifetime of an attractor is determined by it.

To see if it is a general phenomenon of weakly dissipative system, we study the transient process of dissipative coupled standard maps, which has the same dynamical properties as laser-cooled two particles. The dissipative coupled standard maps

$$\begin{aligned} I_{n+1} &= (1-v)I_n + K_i\sin(\theta_n) + \mu_i\sin(\theta_n+\varphi_n),\\ \theta_{n+1} &= \theta_n + I_{n+1},\\ J_{n+1} &= (1-v)J_n + K_j\sin(\varphi_n) + \mu_j\sin(\theta_n+\varphi_n),\\ \varphi_{n+1} &= \varphi_n + J_{n+1}, \end{aligned} \tag{3}$$

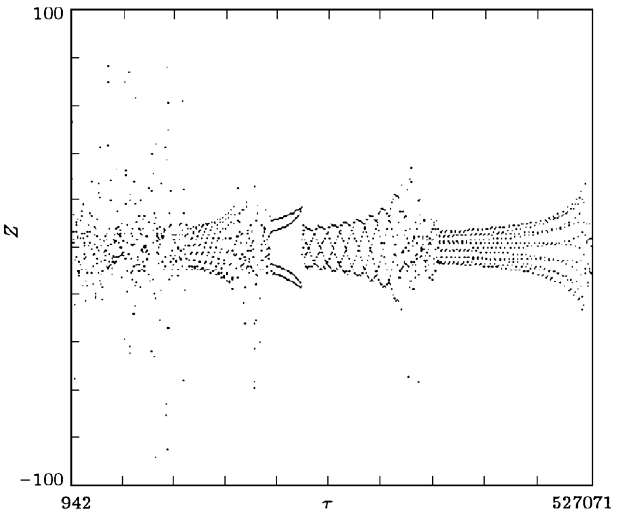


Fig.3. A time bifurcation diagram of variable z on the Poincaré section.

lives in a four-dimensional phase space, where K_i and K_j are nonlinear parameters (also called stochasticity parameters), μ_i and μ_j are coupling parameters, and v is a dissipative parameter. In the case of v=0, this is a mapping of more than two dimensions, KAM curves cannot isolate the stochastic layers. Stochastic layers lie along an inter-connected web of resonance such that initial conditions in any part of the web can ultimately diffuse to all parts of it. The process is known as Arnold diffusion.[13] In the case of μ_i=μ_j=0, the dynamics of a single dissipative standard map was studied recently by the authors of Ref.[14], where coexistence of more than 100 attractors is the main feature. The dynamical behavior becomes more complicated for the dissipative coupled standard maps,Eq.(3). Due to coupling there exists diffusion on one hand; on the other hand due to dissipation the tendency of attraction also exists. The fact that the two factors, diffusion and attraction, exist at the same time induces competition between them in the dynamical behavior of Eq.(3). Let us consider two extreme situations. First, for no dissipation, v=0, diffusion phenomenon only exists from coupling, and the value of the coupling parameter determines the velocity of diffusion. For the initial condition near the region of chaotic web, the system lies in a permanent chaotic state. Second, for no coupling, μ_i=μ_j=μ=0 , particles will be attracted directly to attractors. The transient lifetime changes with the change of dissipative parameter v.When v=0.02, for example, an initial condition is attracted to the period-1 attractor after iterating 1000 times, while for v=0.0003, the same initial condition is attracted after iterating about 7×10^4 times. The time evolution of a trajectory in (J,φ) subspace under these two extreme situations is shown in Fig.4 and Fig.5. Figure 4 shows the diffusion phenomenon when v=0, μ_i=μ_j=μ=0.02. The chaotic motion will be sustained. The evolution of the same initial condition for μ=0 and v=0.0003 is shown in Fig.5. It is clear that the particle is attracted directly. For a relatively larger coupling parameter value, diffusion is dominant and the system has a long transient process. While for a relatively larger dissipative parameter value, attraction is dominant and the system undergoes a relatively shorter transient process. The result of numerical simulation indicates that for a strong coupled (μ_i=μ_j= μ=0.02.), weak dissipative (v=0.0002–0.0004) and nonlinear parameter $K_i = K_j = K = 1.2$ standard maps system, competition between the two processes is the most fierce. In the transient process, the system frequently lingers near some pseudo-periodic orbit for a moment and then departs .

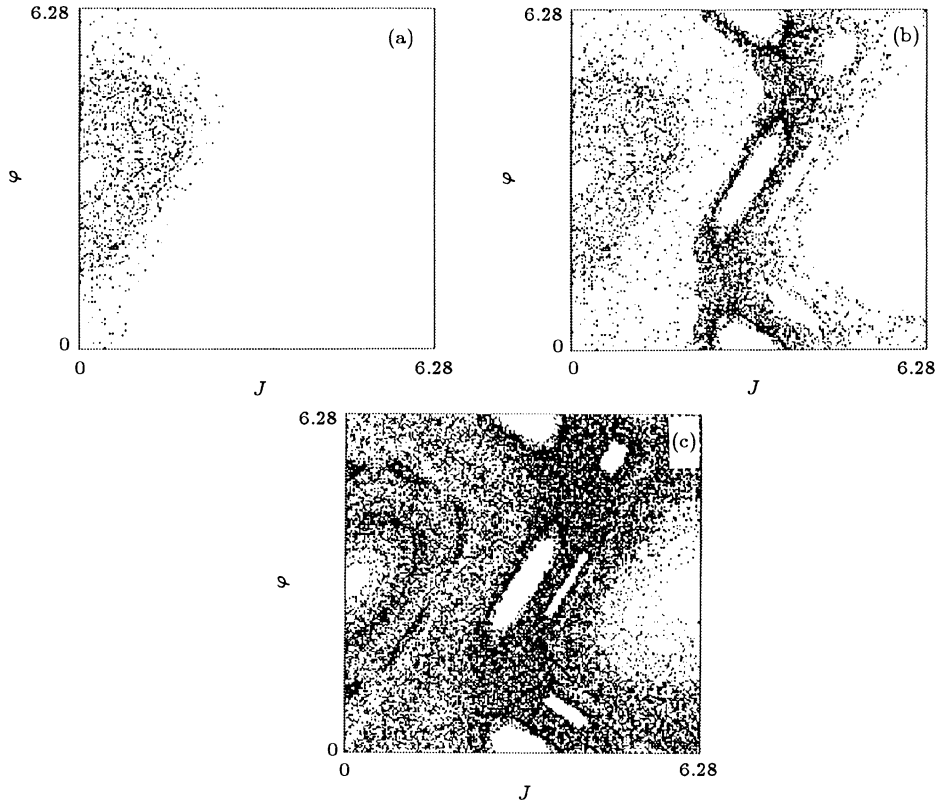


Fig.4. Phase diagrams in (J, φ) subspace without dissipation (v=0); (a) iteration number n=3000; (b) n=10000; (c) n=45000.

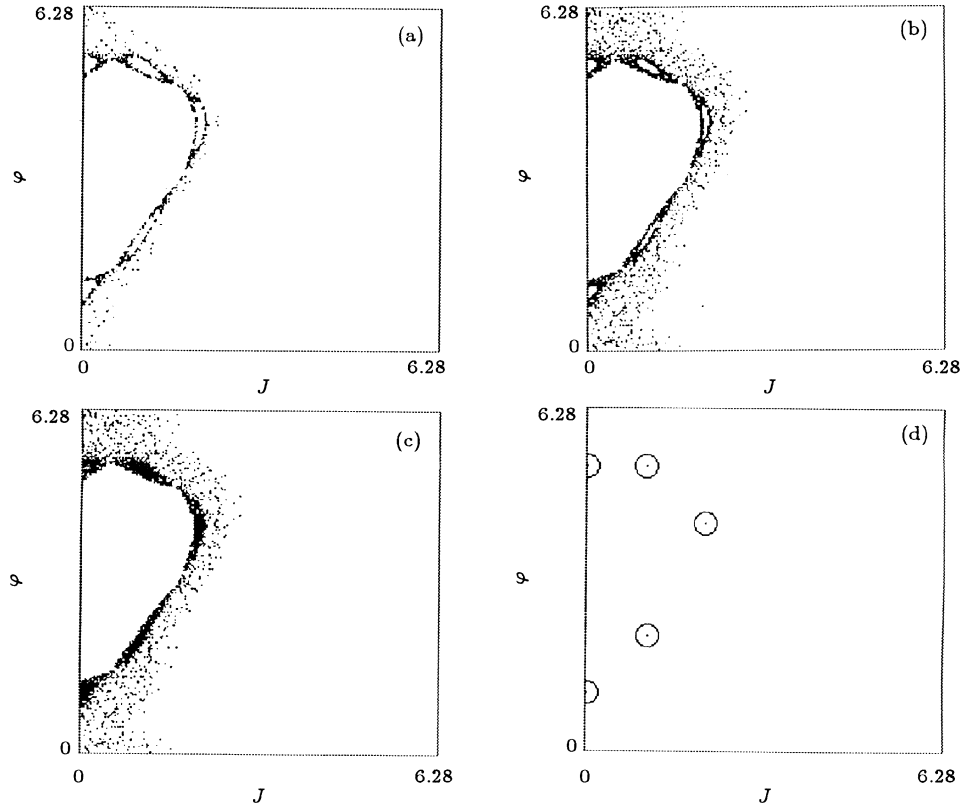


Fig.5. Phase diagrams in (J, φ) subspace without coupling (μ=0): (a), (b) and (c) is phase diagrams when iteration number n=2000, 6500 and 20000, respectively. (d) The particle is attracted to the period-5.

Similar to the noncoupled case, the number of attractors in this case is very large. Finally, the system can settle in one of these stable attractors. A typical process is shown in Fig.6, which are phase diagrams in (J,φ) subspace. For different iteration numbers, the system stays near different peusodo-periodic orbits and then departs. It can be seen clearly from the figure that after passing by several pseudo-periodic orbits, the system is finally attracted to the period-5 attractor and remains stable. The iterating number in this case is much larger than that in Fig.5.

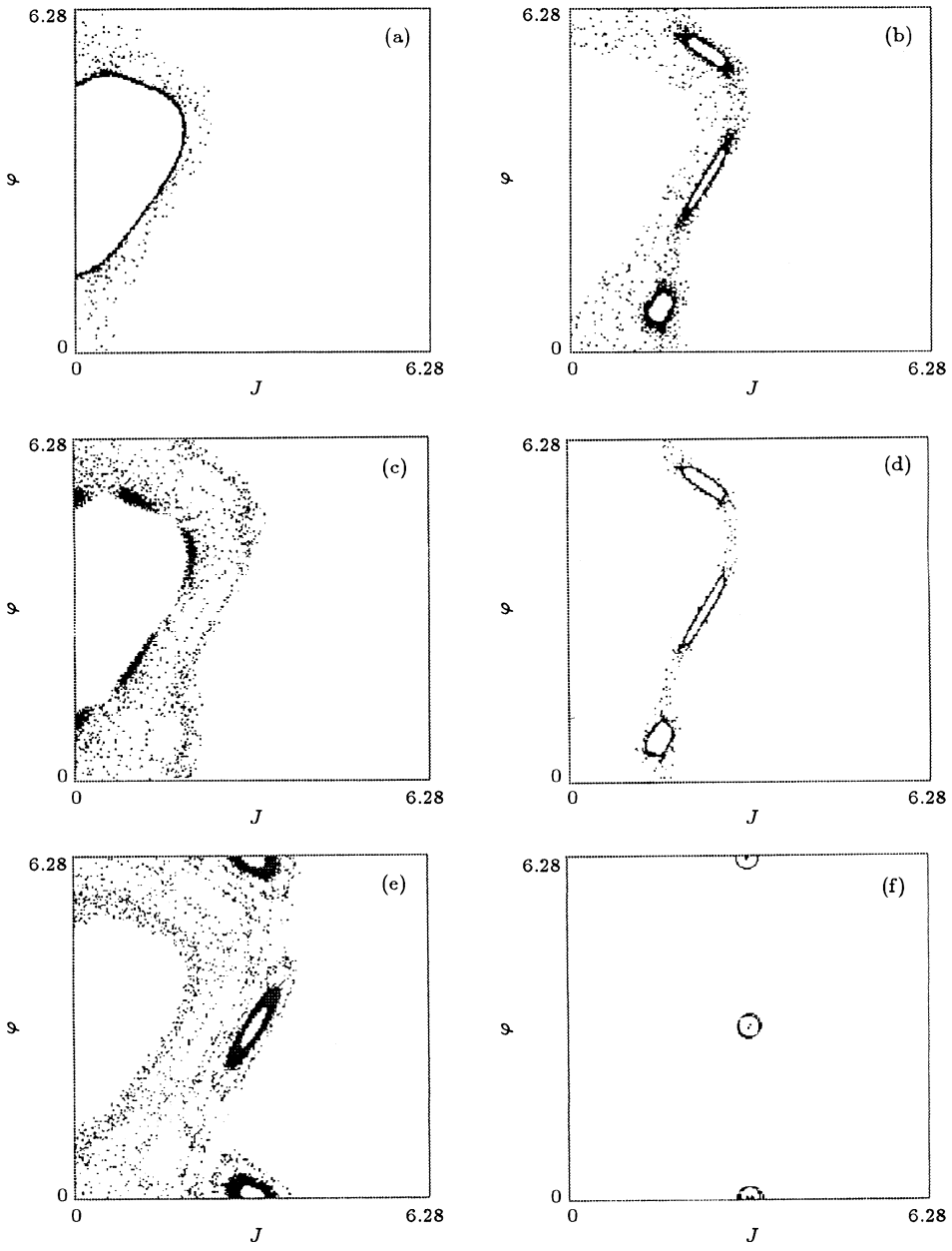


Fig.6. A transient process of a strong coupled, weak dissipative standard maps in (J,φ) subspace. For different iteration numbers, Figs. (a),(b),(c),(d) and (e),the system stays near different orbits.(f) It finally falls onto the period-5,Fig.6(f).

The transient process of the coupled dissipative standard map also manifests characteristic transient time spectrum similar to the previous discussion. Figure 7 is the transient time distribution of coupled dissipative standard map for processes starting from 400 randomly selected initial conditions. It is clear that the peaks in Fig.7(a) correspond of the characteristic times of the attractors. Figures 7(b), (c) and (d) are separate diagrams of Fig.7(a), corresponding to the distribution statistics of transient processes which finally converge to 2p,3p and 8p orbits, respectively. These results demonstrate that the phenomenon is

general for weakly dissipative systems.

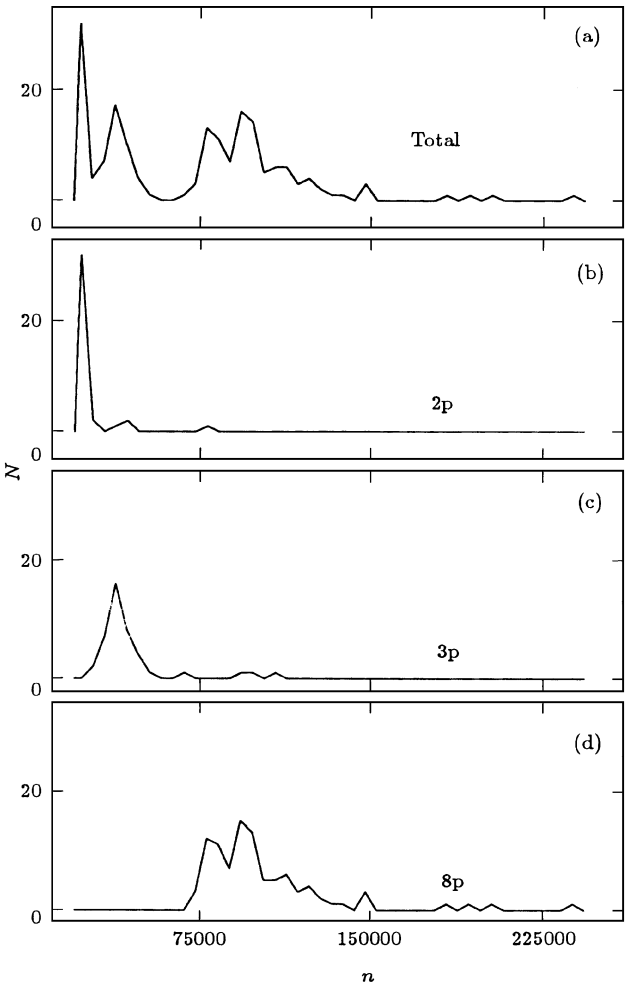


Fig.7. The distribution of transient time for coupled dissipative standard maps.

IV. DISCUSSION AND CONCLUSION

The feature of a transient time spectrum reveals that most of the transient trajectories which finally settle into a certain FL periodic orbit must experience the same processions. Our recent studies on the general dynamical feature of transition from a Hamiltonian system to a dissipative system find that periodic attractors of the dissipative system stem from elliptic periodic orbits of the Hamiltonian system, and as the dissipative parameter ε increases, the periodic attractors annihilate by collision with unstable periodic orbits. If the dissipative parameter ε is not far from this critical value ε_c, the pseudo-periodic exists near the periodic orbits. The lingering time is proportional to $(\varepsilon - \varepsilon_c)^{-1/2}$. This determines the transient times of attractors.

In brief, we have studied the transient properties of the laser cooled particle pair in a Paul trap. The existence of a transient lifetime spectrum is a crucial feature of the transient dynamics of the system. "Periodic islands" appear on the time axis as they appear in phase space. We think this is caused by pseudo-periodic orbit in the transient process and may be a general feature of weakly dissipative systems with multi-attractor coexistence.

ACKNOWLEDGMENT

The authors thank H.W. Yin for discussions.

REFERENCES

[1] F.Diedrich, E.Peik, J.M.Chen, W.Quint, H.Walther, *Phys. Rev. Lett.,* **59**(1987), 2931 .

[2] R. Blumel, J. M. Chen, E. Peik, W. Quint, W. Schleich, Y. R. Shen, H. Walther, *Nature*, **334**(1988), 309 .

[3] J. Hoffnagle, R. G. DeVoe, L. Reyna, R. G. Brewer, *Phys. Rev. Lett.,* **61** (1988), 255 .

[4] R.G.Brewer, J.Hoffnagle, R.G.DeVoe, L.Reyna, W. Henshaw, *Nature*, **344** (1990), 305 .

[5] J. Hoffnagle, R. G. Brewer, *Phys. Rev. Lett.,* **71** (1993), 1828 .

[6] J. Hoffnagle, R. G. Brewer, *Science,* **265** (1994), 213 .

[7] J. Hoffnagle, R. G. Brewer, *Phys. Rev.,* **A50** (1994), 4157.

[8] R. Blumel, C. Kappler, W. Quint, H. Walther, *Phys. Rev.,* **A40** (1989), 808 .

[9] J. W. Emmert, M. Moore, R. Blumel, *Phys. Rev.,* **A48** (1993), 1757.

[10] R. Blumel, *Phys. Rev.,* **A51** (1995), 620 .

[11] J. L. Shen, H. W. Yin , J. H. Dai , H. J. Zhang, *Phys. Rev.,* **A55** (1997), 2159.

[12] J. L. Shen, H. W. Yin, J. H. Dai, H. J. Zhang, *Chin. Phys. Lett.,* **13**(1996), 81.

[13] B. P. Wood, A. J. Lichtenerg, M. A. Lieberberman, *Phys. Rev.,* **A42**(1990), 5885

[14] Ulrike Feudel, Celso Grebogi, Brian R.Hunt, James A.Yorke, *Phys. Rev.,* **E54**(1996), 71.