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# SYMMETRIES AND CONSERVED QUANTITIES FOR SYSTEMS OF GENERALIZED CLASSICAL MECHANICS*

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In this paper, the symmetries and the conserved quantities for systems of generalized classical mechanics are studied. First, the generalized Noether's theorem and the generalized Noether's inverse theorem of the systems are given, which are based upon the invariant properties of the canonical action with respect to the action of the infinitesimal transformation of $r$-parameter finite group of transformation; second, the Lie symmetries and conserved quantities of the systems are studied in accordance with the Lie's theory of the invariance of differential equations under the transformation of infinitesimal groups; and finally, the inner connection between the two kinds of symmetries of systems is discussed.

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## I. INTRODUCTION

The conservation laws (or first integrals) of the mechanical systems are always of mathematical importance and at the same time, they are regarded as the manifestation of some profound physical principles. The fact that the existence of a conserved quantity means that an inner dynamical symmetry in the mechanical system plays an irreplaceable role in the physical interpretation of motion. The modern methods in finding conservation laws are mainly two ways[1] with each having a concept of its own: the first one is based upon the invariant properties of Hamiltonian action with respect to the action of the infinitesimal transformation of the finite transformation group, i.e., Noether symmetries and conserved quantities; the second one is based upon the Lie's theory of the invariance of differential equations under the action of infinitesimal groups.

The differential equations of many physical problems appear to be the variational problems. The theory of the generalized classical mechanics, initiated by Ostrogradsky and Jacobi in the years from 1848 to 1858, has been gaining much development during the recent fifty years, and many important results have been made.[2−6] The Lagrange equations with high-order derivative have been studied in physics, especially in mechanics and field theory, for example, in the electromagnetic theory with second order derivatives. In mathematics, some good results have been made in the descriptions of modern geometric methods, for example, in Ref.[2]. This paper studies the symmetries and conserved quantities, including the Noether symmetries and conserved quantities, the Lie symmetries and conserved quantities.

## II. EQUATIONS OF MOTION FOR SYSTEM OF GENERALIZED CLASSICAL MECHANICS

Consider a system of generalized classical mechanics. Suppose that the configuration of the system is determined by $n$ generalized coordinates $q_s (s = 1, \cdots, n)$, and the Lagrangian $L$ is a function of time $t$, generalized coordinates $q_s (s = 1, \cdots, n)$ and derivatives of $q_s$ from first order to $k$th order. The condition that the integral

$$\int_{t_1}^{t_2} L(t, q_s, \dot{q}_s, \cdots, \overset{(k)}{q}_s)\mathrm{d}t, \tag{1}$$

takes its extremum values is the generalized Euler-Lagrange equations

$$\sum_{j=0}^{k}(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial \overset{(j)}{q}_s} = 0 \qquad (s = 1, \cdots, n). \tag{2}$$

Introducing the generalized momenta and Hamiltonian function

$$p_{s/m} = \sum_{j=0}^{k-m}(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial \overset{(j+m)}{q}_s} \quad (s = 1, \cdots, n; m = 1, \cdots, k), \tag{3}$$

$$H\big(\overset{(m-1)}{q}_s, p_{s/m}, t\big) = \sum_{s=1}^{m}\sum_{m=1}^{k} p_{s/m}\overset{(m)}{q}_s - L, \tag{4}$$

then the equations of motion (2) can be written in the canonical form

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$$\frac{\mathrm{d}}{\mathrm{d}t}\overset{(m-1)}{q_s}=\frac{\partial H}{\partial p_{s/m}},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_{s/m}=-\frac{\partial H}{\partial \overset{(m-1)}{q_s}}\quad (s=1,\cdots,n;m=1,\cdots,k). \tag{5}$$

Let

$$a^\mu=\begin{cases}\overset{(m-1)}{q_s} & (\mu=(2m-2)n+s;m=1,\cdots,k;s=1,\cdots,n);\\ p_{s/m} & (\mu=(2m-1)n+s;m=1,\cdots,k;s=1,\cdots,n),\end{cases} \tag{6}$$

$$R_\mu=\begin{cases}p_{s/m} & (\mu=(2m-2)n+s;m=1,\cdots,k;s=1,\cdots,n);\\ 0 & (\mu=(2m-1)n+s;m=1,\cdots,k;s=1,\cdots,n),\end{cases} \tag{7}$$

then Eqs.(5) can be further expressed in the covariant algebra form

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}=0 \qquad \mu,\nu=1,\cdots,2kn. \tag{8}$$

The contravariant algebra form equivalent to Eqs.(8) is

$$\dot{a}^\mu-\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0 \qquad \mu,\nu=1,\cdots,2kn, \tag{9}$$

where

$$\omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu},\quad (\omega_{\mu\nu})=(\omega^{\mu\nu})^{-1},$$
$$(\omega^{\mu\nu})=\begin{bmatrix}\Omega_1 & & \\ & \ddots & \\ & & \Omega_k\end{bmatrix} \tag{10}$$

and

$$\Omega_1=\cdots=\Omega_k=\begin{bmatrix}0_{n\times n} & I_{n\times n}\\ -I_{n\times n} & 0_{n\times n}\end{bmatrix}, \tag{11}$$

where $0_{n\times n}$ is the zero matrix with order $n$, and $I_{n\times n}$ is the unit matrix with order $n$.

## Ⅲ. NOETHER SYMMETRIES FOR A SYSTEM OF GENERALIZED CLASSICAL MECHANICS

### A. Noether quasi-symmetric transformations and generalized Killing equations

Introduce the infinitesimal transformations of $r$ parameter finite transformation group

$$t^*=t+\Delta t,a^{\mu^*}=a^\mu+\Delta a^\mu\quad (\mu=1,\cdots,2kn) \tag{12}$$

and their expansion formulae are

$$t^*=t+\varepsilon_\alpha\xi_0^\alpha(t,\boldsymbol{a}),\quad a^{\mu^*}=a^\mu+\epsilon_\alpha\xi_\mu^\alpha(t,\boldsymbol{a})$$
$$(\mu=1,\cdots,2kn;\alpha=1,\cdots,r), \tag{13}$$

where $\varepsilon_\alpha$ are infinitesimal parameters, $\xi_0^\alpha$ and $\xi_\mu^\alpha$ are generators of the infinitesimal transformations.

The canonical action of the system is

$$S(\gamma)=\int_{t_1}^{t_2}\left(\sum_{s=1}^{n}\sum_{m=1}^{k}p_{s/m}\overset{(m)}{q_s}-H\right)\mathrm{d}t$$
$$=\int_{t_1}^{t_2}(R_\mu\mathrm{d}a^\mu-H\mathrm{d}t). \tag{14}$$

Under the infinitesimal transformations (13), the phase trajectory $\gamma$ is transformed to the nearby one $\gamma^*$, and the difference of the action (14) before and after transformation, accurate to the first-order infinitesimals, is[1]

$$\Delta S=\int_{t_1}^{t_2}\left\{(R_\mu\dot{a}^\mu-H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)-\frac{\partial H}{\partial t}\Delta t+\left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}\right)\Delta a^\mu+R_\mu\Delta\dot{a}^\mu\right\}\mathrm{d}t. \tag{15}$$

According to the relation between the simultaneous variation and non-simultaneous variation,[1] we have

$$\Delta S=\int_{t_1}^{t_2}\varepsilon_\alpha\left\{\frac{\mathrm{d}}{\mathrm{d}t}(R_\mu\xi_\mu^\alpha-H\xi_0^\alpha)+\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}\right]\bar{\xi}_\mu^\alpha\right\}\mathrm{d}t, \tag{16}$$

where $\bar{\xi}_\mu^\alpha=\xi_\mu^\alpha-\dot{a}^\mu\xi_0^\alpha$.

**Definition 1** If for each of the transformations (13) always holds

$$\Delta S=-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\mathrm{d}t, \tag{17}$$

the transformations (13) are called quasi-symmetrical transformations in Noether's sense of the generalized classical mechanical system, where $G = G(t, \boldsymbol{a})$, with $\Delta G = \varepsilon_\alpha G^\alpha$. If $G^\alpha = 0$, the transformations (13) are called Noether symmetrical transformations of the system.

According to definition 1 and formula (16), we can easily obtain the following theorem.

**Theorem 1** If the infinitesimal transformations (13) satisfy the conditions

$$R_\mu \dot{\xi}^\alpha_\mu + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\xi^\alpha_\mu - \frac{\partial H}{\partial t}\xi^\alpha_0 - H\dot{\xi}^\alpha_0 = -\dot{G}^\alpha \qquad (\alpha = 1, \cdots, r), \tag{18}$$

then the transformations (13) are the Noether quasi-symmetric transformations of the generalized classical mechanical system.

If $r = 1$, formulae (18) give the Noether equality

$$R_\mu \dot{\xi}_\mu + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\xi_\mu - \frac{\partial H}{\partial t}\xi_0 - H\dot{\xi}_0 = -\dot{G}. \tag{19}$$

Expanding $r$ equations of (18), and making the terms containing $\dot{a}^\mu$ and terms not containing $\dot{a}^\mu$ vanish respectively, we have

$$\begin{aligned} &R_\mu \frac{\partial \xi^\alpha_\mu}{\partial a^\nu} + \frac{\partial R_\nu}{\partial a^\mu}\xi^\alpha_\mu - H\frac{\partial \xi^\alpha_0}{\partial a^\nu} = -\frac{\partial G^\alpha}{\partial a^\nu} \qquad (\nu = 1, \cdots, 2kn), \\ &R_\mu \frac{\partial \xi^\alpha_\mu}{\partial t} - \frac{\partial H}{\partial a^\mu}\xi^\alpha_\mu - \frac{\partial H}{\partial t}\xi^\alpha_0 - H\frac{\partial \xi^\alpha_0}{\partial t} = -\frac{\partial G^\alpha}{\partial t}. \end{aligned} \tag{20}$$

Equations (20) are called generalized Killing equations of the generalized classical mechanical system.

### B. Generalized Noether theorem

**Theorem 2** For the generalized classical mechanical system (2), if the infinitesimal transformations (13) of $r$ parameter finite group of transformation are Noether quasi-symmetric of the system, then the system has $r$ linear independent conserved quantities

$$I^\alpha = R_\mu \xi^\alpha_\mu - H\xi^\alpha_0 + G^\alpha = C_\alpha \quad (\alpha = 1, \cdots, r). \tag{21}$$

**Proof** Since the infinitesimal transformations (13) are the generalized Noether quasi-symmetric transformations of the system, according to definition 1, we have

$$\Delta S = -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\mathrm{d}t.$$

Substituting Eq.(16) into the above formula, we obtain

$$\int_{t_1}^{t_2} \varepsilon_\alpha \left\{ \frac{\mathrm{d}}{\mathrm{d}t}(R_\mu \xi^\alpha_\mu - H\xi^\alpha_0 + G^\alpha) + \left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right]\bar{\xi}^\alpha_\mu \right\} \mathrm{d}t = 0.$$

Substituting Eqs.(8) into the above formula, and considering that the integral interval $[t_1, t_2]$ is arbitrary and $\varepsilon_\alpha$ are independent, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(R_\mu \xi^\alpha_\mu - H\xi^\alpha_0 + G^\alpha) = 0.$$

Integrating this equation, we obtain the conserved quantities (21).

Analogously, we can prove the following theorem.

**Theorem 3** For the generalized classical mechanical system (2), if the infinitesimal transformations (13) of $r$ parameter finite group of transformation are Noether symmetric transformation of the system, then the system has $r$ linear independent conserved quantities

$$I^\alpha = R_\mu \xi^\alpha_\mu - H\xi^\alpha_0 = C_\alpha \qquad (\alpha = 1, \cdots, r). \tag{22}$$

Theorems 2 and 3 may be called generalized Noether theorems of the generalized classical mechanical system. If we can find out the Noether symmetrical transformations or the Noether quasi-symmetrical

transformations of the system by theorem 1 or the generalized killing equations (20), then we can find out the conserved quantities of the system by theorem 2 or theorem 3.

### C. Generalized Noether inverse theorem

Suppose that the generalized classical mechanical system (2) have $r$ linear independent conserved quantities

$$I^\alpha = I^\alpha(t, \boldsymbol{a}) = C_\alpha \qquad (\alpha = 1, \cdots, r). \tag{23}$$

Multiplying each side of Eqs.(8) by $\xi_\mu^\alpha$ and making summation for $\mu$, and at the same time differentiating Eq.(23) with respect to $t$, then adding both results, we obtain

$$\frac{\partial I^\alpha}{\partial t} + \frac{\partial I^\alpha}{\partial a^\nu}\dot{a}^\nu + \left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right]\bar{\xi}_\mu^\alpha = 0. \tag{24}$$

Obviously, in Eqs.(24), the term which contains $\dot{a}^\mu\dot{a}^\nu$ is null. According to the fact that the coefficients of $\dot{a}^\nu$ in Eqs.(24) are null, we obtain

$$\frac{\partial I^\alpha}{\partial a^\nu} + \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\xi_\mu^\alpha + \frac{\partial H}{\partial a^\nu}\xi_0^\alpha = 0 \qquad (\nu = 1, \cdots, 2kn; \alpha = 1, \cdots, r). \tag{25}$$

Considering formulae (10), we can obtain from Eqs.(25)

$$\xi_\mu^\alpha = \omega^{\mu\nu}\left(\frac{\partial I^\alpha}{\partial a^\nu} + \frac{\partial H}{\partial a^\nu}\xi_0^\alpha\right) \qquad (\mu = 1, \cdots, 2kn). \tag{26}$$

For making the transformations be the Noether quasi-symmetric transformations, it is necessary that the integrals (23) equal to the conserved quantities (21), that is,

$$R_\mu\xi_\mu^\alpha - H\xi_0^\alpha + G^\alpha = I^\alpha. \tag{27}$$

Then, we have

**Theorem 4** If we know $r$ linear independent conserved quantities (23) of the generalized classical mechanical system (2), then the infinitesimal transformations determined by the formulae (13), (26) and (27) must be the Noether quasi-symmetric transformations of the system.

For making the transformations be the Noether symmetric transformations, it is necessary that the integrals (23) equal to the conserved quantities (22), that is,

$$R_\mu\xi_\mu^\alpha - H\xi_0^\alpha = I^\alpha. \tag{28}$$

By using $(2kn+1)r$ equations (26), (28), we can determine $(2kn+1)r$ generators $\xi_0^\alpha, \xi_\mu^\alpha$. Then, we have

**Theorem 5** If there exist $r$ linear independent conserved quantities (23) of the generalized classical mechanical system (2), then the infinitesimal transformations determined by formulae (13), (26) and (28) must be the Noether symmetric transformations of the system.

Theorems 4 and 5 are called generalized Noether inverse theorems of the generalized classical mechanical system. Using these theorems, we can find the Noether symmetric transformations and the Noether quasi-symmetric transformations relative to the given conserved quantities of the generalized classical mechanical system.

## IV. LIE SYMMETRIES FOR SYSTEMS OF GENERALIZED CLASSICAL MECHANICS

### A. Direct problem of Lie symmetries

Take the infinitesimal transformations

$$\begin{aligned} &t^* = t + \Delta t = t + \varepsilon\xi_0(t, \boldsymbol{a}), \\ &a^{\mu^*} = a^\mu + \Delta a^\mu = a^\mu + \varepsilon\xi_\mu(t, \boldsymbol{a}) \\ &(\mu = 1, \cdots, 2kn). \end{aligned} \tag{29}$$

Introducing the infinitesimal generator vector

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu} \tag{30}$$

and its first extended vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_\mu - \dot{a}^\mu\dot{\xi}_0)\frac{\partial}{\partial\dot{a}^\mu}, \tag{31}$$

the invariance of Eqs.(9) under the infinitesimal transformation (29) leads up to the satisfaction of the following determining equations:

$$\begin{aligned} &\dot{\xi}_\mu - \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\dot{\xi}_0 = X^{(0)}\left(\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right) \\ &(\mu = 1, \cdots, 2kn). \end{aligned} \tag{32}$$

**Definition 2** If the generators $\xi_0, \xi_\mu$ of the infinitesimal transformations (29) satisfy the determining equations (32), then the corresponding symmetries can be called the Lie symmetries of the generalized classical mechanical system.

Then, we have

**Theorem 6** For the generators $\xi_0, \xi_\mu$ of the infinitesimal transformations (29) satisfying the determining Eqs.(32), if there exists a gauge function $G = G(t, \boldsymbol{a})$ satisfying the structure equation

$$(R_\mu \dot{a}^\mu - H)\dot{\xi}_0 + X^{(1)}(R_\mu \dot{a}^\mu - H) + \dot{G} = 0, \tag{33}$$

then the generalized classical mechanical system has a Lie symmetric conserved quantity

$$I = R_\mu \xi_\mu - H\xi_0 + G = \text{const.} \tag{34}$$

**Proof**

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} =& \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu \xi_\mu + R_\mu \dot{\xi}_\mu - \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial a^\nu}\dot{a}^\nu\right)\xi_0 - H\dot{\xi}_0 - (R_\mu \dot{a}^\mu - H)\dot{\xi}_0 \\ & - \xi_\nu \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - R_\mu(\dot{\xi}_\mu - \dot{a}^\mu \dot{\xi}_0) + \xi_0 \frac{\partial H}{\partial t} + \xi_\mu \frac{\partial H}{\partial a^\mu} \\ =& -\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right]\xi_\mu - \frac{\partial H}{\partial a^\nu}\dot{a}^\nu \xi_0. \\ =& 0.\end{aligned}$$

**Theorem 7** The structure equation (33) is equivalent to the Noether equality (19).

**Theorem 8** The structure Eq.(33) has led to the following generalized Killing equations:

$$\begin{aligned} & R_\mu \frac{\partial \xi_\mu}{\partial a^\nu} + \frac{\partial R_\nu}{\partial a^\mu}\xi_\mu - H\frac{\partial \xi_0}{\partial a^\nu} = -\frac{\partial G}{\partial a^\nu} \qquad (\nu = 1, \cdots, 2kn), \\ & R_\mu \frac{\partial \xi_\mu}{\partial t} - \frac{\partial H}{\partial a^\mu}\xi_\mu - \frac{\partial H}{\partial t}\xi_0 - H\frac{\partial \xi_0}{\partial t} = -\frac{\partial G}{\partial t}. \end{aligned} \tag{35}$$

**Theorem 9** For a generalized classical mechanical system, the conserved quantity relative to the Lie symmetry has the same form of the conserved quantity relative to the Noether symmetry.

### B. Inverse problem of Lie symmetries

For the generalized classical mechanical system, the inverse problem of Lie symmetries is to find out the corresponding Lie symmetries according to the known conserved quantity

The solution of the inverse problem of Lie symmetries of the system is as follows: firstly, according to Noether theory, we can find out the corresponding Noether symmetries by the known integral; secondly, we must verify whether the Noether symmetries obtained above are Lie symmetries. If the generators we obtained satisfy the determining equations of Lie symmetries, then the symmetries are the Lie symmetries of the generalized classical mechanical system.

## V. RELATIONS BETWEEN NOETHER SYMMETRIES AND LIE SYMMETRIES OF GENERALIZED CLASSICAL MECHANICAL SYSTEM

For the generalized classical mechanical system (2), the relations between the Noether symmetries and the Lie symmetries have the following results.

**Theorem 10** For the generalized classical mechanical system (2), if the infinitesimal transformations (29) are Lie symmetric, and $\dot{G}$ in Eq.(33) is null or is total derivative of some function of $t, \boldsymbol{a}$, then the corresponding symmetries are Noether's; otherwise, they are not Noether's.

**Proof** Theorem 10 can be deduced from theorem 7.

**Theorem 11** For the generalized classical mechanical system (2), if the infinitesimal transformations (29) are Lie symmetric, and the generalized Killing equations (35) are solvable, then the corresponding symmetries are Noether's; otherwise, they are not Noether's.

Theorems 10 and 11 answer the question whether the Lie symmetries of the generalized classical mechanical system are Noether symmetries or not.

**Theorem 12** For the generalized classical me-

chanical system (2), if the infinitesimal transformations (29) are Noether symmetric, and the infinitesimal generators $\xi_0, \xi_\mu$ satisfy the determining equations (32), then the corresponding symmetries are Lie's; otherwise, they are not Lie's.

Theorem 12 answers the question whether the Noether symmetries of the generalized classical mechanical system are Lie symmetries or not.

## VI. EXAMPLES

**Example 1** Suppose the Lagrangian of a generalized classical mechanical system[1] is

$$L = \frac{1}{2}\alpha_1\dot{q}^2 + \frac{1}{2}\alpha_2\ddot{q}^2 \quad (\alpha_1 > 0, \alpha_2 > 0). \tag{36}$$

We study its symmetries and conserved quantities.

According to formule (3), (4), (6) and (7), we can easily obtain

$$H = a^2a^3 - \frac{1}{2}\alpha_1(a^3)^2 + \frac{1}{2\alpha_2}(a^4)^2,$$
$$\omega^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{37}$$

Equations (9) give

$$\dot{a}^1 = a^3, \qquad \dot{a}^2 = 0, \qquad \dot{a}^3 = \frac{1}{\alpha_2}a^4, \qquad \dot{a}^4 = \alpha_1 a^3 - a^2. \tag{38}$$

The Noether symmetric condition (18) gives

$$a^2\dot{\xi}_1 + a^4\dot{\xi}_3 + (\dot{a}^1 - a^3)\xi_2 - (a^2 - \alpha_1 a^3)\xi_3 + \left(\dot{a}^3 - \frac{1}{\alpha_2}a^4\right)\xi_4 - -H\dot{\xi}_0 = -\dot{G}. \tag{39}$$

Using Eq.(39), we can obtain the following generators of the infinitesimal transformations and gauge functions:

$$\xi_0 = -1, \qquad \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0, \qquad G = 0, \tag{40}$$
$$\xi_0 = 0, \qquad \xi_1 = 1, \qquad \xi_2 = \xi_3 = \xi_4 = 0, \qquad G = 0, \tag{41}$$
$$\xi_0 = 0, \qquad \xi_1 = t, \qquad \xi_2 = \alpha_1, \qquad \xi_3 = 1, \qquad \xi_4 = 0, \qquad G = -\alpha_1 a^1, \tag{42}$$
$$\xi_0 = 0, \xi_1 = \sqrt{\alpha_2/\alpha_1}\mathrm{e}^{\sqrt{\alpha_1 t/\alpha_2}}, \xi_2 = 0, \xi_3 = \mathrm{e}^{\sqrt{\alpha_1 t/\alpha_2}}, \xi_4 = \sqrt{\alpha_1\alpha_2}\mathrm{e}^{\sqrt{\alpha_1 t/\alpha_2}}, G = -a^3\sqrt{\alpha_1\alpha_2}\mathrm{e}^{\sqrt{\alpha_1 t/\alpha_2}}. \tag{43}$$

Here, formulae (40), (41) correspond to the Noether symmetric transformations of the system, and formulae (42), (43) correspond to the Noether quasi-symmetric transformations.

Secondly, let us find out the conserved quantities. Applying Eqs.(21) to formulae (40)–(43), we can obtain the following conserved quantities:

$$I^1 = a^2a^3 - \frac{1}{2}\alpha_1(a^3)^2 + \frac{1}{2\alpha_2}(a^4)^2 = C_1, \tag{44}$$
$$I^2 = a^2 = C_2, \tag{45}$$
$$I^3 = a^2t + a^4 - \alpha_1 a^1 = C_3, \tag{46}$$
$$I^4 = \mathrm{e}^{\sqrt{\alpha_1 t/\alpha_2}}\left(a^2\sqrt{\frac{\alpha_2}{\alpha_1}} + a^4 - a^3\sqrt{\alpha_1\alpha_2}\right) = C_4. \tag{47}$$

Finally, we study the Lie symmetries of the system. The determining equations (32) of Lie symmetries give

$$\dot{\xi}_1 - a^3\dot{\xi}_0 = \xi_3, \quad \dot{\xi}_2 = 0, \ \alpha_2\dot{\xi}_3 - a^4\dot{\xi}_0 = \xi_4, \quad \dot{\xi}_4 + (a^2 - \alpha_1 a^3)\dot{\xi}_0 = -\xi_2 + \alpha_1\xi_3. \tag{48}$$

It is easy to verify that the infinitesimal generators $\xi_0, \xi_\mu(\mu = 1, \cdots, 4)$ given by formulae (40)–(43) can all satisfy the determining equations (48), and therefore, these generators are all corresponding to the Lie symmetries of the system under study.

**Example 2** Suppose the Lagrangian of a generalized classical mechanical system is

$$L = \frac{1}{2}\ddot{q}^2 - \frac{1}{b}\dot{q}\mathrm{arctan}bt. \tag{49}$$

Let us try to study the Noether symmetries and the Lie symmetries of the system.

$$\text{Let } a^1 = q, a^2 = p_{1/1}, a^3 = \dot{q}, a^4 = p_{1/2}. \tag{50}$$

The generalized momenta and Hamiltonian functions are

$$p_{1/1} = -\frac{1}{b}\mathrm{arc\,tan}bt - \dddot{q}, \qquad p_{1/2} = \ddot{q}, \tag{51}$$
$$H = a^2a^3 + \frac{1}{2}(a^4)^2 + \frac{1}{b}a^3\mathrm{arc\,tan}bt. \tag{52}$$

The generalized Killing equations (20) give

$$
\begin{aligned}
&a^2\frac{\partial\xi_1}{\partial a^1}+a^4\frac{\partial\xi_3}{\partial a^1}+\xi_2-H\frac{\partial\xi_0}{\partial a^1}=-\frac{\partial G}{\partial a^1},\quad a^2\frac{\partial\xi_1}{\partial a^2}+a^4\frac{\partial\xi_3}{\partial a^2}-H\frac{\partial\xi_0}{\partial a^2}=-\frac{\partial G}{\partial a^2},\\
&a^2\frac{\partial\xi_1}{\partial a^3}+a^4\frac{\partial\xi_3}{\partial a^3}+\xi_4-H\frac{\partial\xi_0}{\partial a^3}=-\frac{\partial G}{\partial a^3},\quad a^2\frac{\partial\xi_1}{\partial a^4}+a^4\frac{\partial\xi_3}{\partial a^4}-H\frac{\partial\xi_0}{\partial a^4}=-\frac{\partial G}{\partial a^4},\\
&a^2\frac{\partial\xi_1}{\partial t}+a^4\frac{\partial\xi_3}{\partial t}-a^3\xi_2-\left(a^2+\frac{1}{b}\mathrm{arc\,tan}bt\right)\xi_3-a^4\xi_4-\frac{a^3}{1+b^2t^2}\xi_0-H\frac{\partial\xi_0}{\partial t}=-\frac{\partial G}{\partial t}.
\end{aligned}
\tag{53}
$$

The determining equations (32) of Lie symmetries give

$$
\dot\xi_1-a^3\dot\xi_0=\xi_3,\quad \dot\xi_2=0,\quad \dot\xi_3-a^4\dot\xi_0=\xi_4,\quad \dot\xi_4+\left(a^2+\frac{1}{b}\mathrm{arc\,tan}bt\right)\dot\xi_0=-\xi_2-\frac{1}{1+b^2t^2}\xi_0. \tag{54}
$$

Equations (53) have a solution as follows:

$$
\xi_0=1,\quad \xi_1=a^3,\quad \xi_2=0,\quad \xi_3=a^4,\quad \xi_4=-a^2-\frac{1}{b}\mathrm{arc\,tan}bt,\quad G=-\frac{1}{2}(a^4)^2+\frac{1}{b}a^3\mathrm{arc\,tan}bt. \tag{55}
$$

The generators (55) correspond to the Noether symmetries of the system. Substituting Eqs.(55) into the determining equations (54), we can easily verify that they are satisfied, and therefore, the generators (55) are also corresponding to the Lie symmetries of the system by theorem 12.

We can find a solution of Eqs.(54) as follows:

$$
\begin{aligned}
&\xi_0=t,\qquad \xi_1=3a^1-\frac{1}{4b^4}[(b^2t^2-1)\ln(1+b^2t^2)+4bt\mathrm{arc\,tan}bt-(3b^2t^2+1)],\\
&\xi_2=0,\qquad \xi_3=2a^3-\frac{1}{2b^3}[bt\ln(1+b^2t^2)-2bt+2\mathrm{arc\,tan}bt],\quad \xi_4=a^4-\frac{1}{2b^2}\ln(1+b^2t^2).
\end{aligned}
\tag{56}
$$

The generators (56) correspond to the Lie symmetries of the system. Substituting Eqs.(56) into the generalized Killing equations (53), we have

$$
\begin{aligned}
&\frac{\partial G}{\partial a^1}=-3a^2,\qquad \frac{\partial G}{\partial a^2}=0,\qquad \frac{\partial G}{\partial a^3}=-3a^4+\frac{1}{2b^2}\ln(1+b^2t^2),\qquad \frac{\partial G}{\partial a^4}=0,\\
&\frac{\partial G}{\partial t}=3a^2a^3+\frac{3}{2}(a^4)^2+\frac{1}{2b^4}\mathrm{arc\,tan}bt[6b^3a^3-bt\,\ln(1+b^2t^2)\\
&\qquad\qquad +2bt-2\mathrm{arc\,tan}bt]+\frac{a^3t}{1+b^2t^2}.
\end{aligned}
\tag{57}
$$

Obviously, Eqs.(57) have no solution. By theorem 11, the symmetries corresponding to the generators (56) are not Noether's.

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