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# ROTATING RINDLER SPACE TIME WITH CONSTANT ANGULAR VELOCITY*

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A new space time metric is derived from Kerr metric if its mass and location approach to infinite in an appropriate way. The new space-time is an infinitesimal neighborhood nearby one of the two horizon poles of an infinite Kerr black hole. In other words, it is the second order infinitesimal neighborhood nearby one of the two horizon poles of a Kerr black hole. It is flat and has event horizon and infinite red shift surface. We prove that it is a rotating Rindler space time with constant angular velocity.

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## I. INTRODUCTION

The space time metric of the region sufficiently close to the horizon of a Kerr black hole has been obtained by Suen *et al.*[1] and is called the Rindler approximation to Kerr. Unfortunately, using their method, the space time metric of the region sufficiently close to the horizon pole of a Kerr black hole cannot be obtained.

By using a limit process, vacuum $C$ metric when $m = 0$ is derived from the metric of a Schwarzschild black hole[2] and vacuum $C$ metric is derived from the metric of two superposed Schwarzschild black holes.[3] In this paper, a new space time metric is derived from Kerr metric by using the limit process, and the character of the new space time is discussed. It is the second order infinitesimal neighborhood nearby one of the two horizon poles of Kerr black hole. It is flat and has event horizon and infinite red shift surface. We will prove that it is a rotating Rindler space time with constant angular velocity.

## II. DERIVATION OF THE NEW METRIC

The metric of a Kerr black hole located at the origin of the Weyl coordinate system is[4]

$$\begin{aligned}\mathrm{d}s^2 =&\frac{\lambda^2[(\widetilde{R}+\widetilde{R}')^2-4m^2]+a^2(\widetilde{R}-\widetilde{R}')^2}{\lambda^2(\widetilde{R}+\widetilde{R}'+2m)^2+a^2(\widetilde{R}-\widetilde{R}')^2}\left\{\mathrm{d}\tilde{t}+\frac{am(\widetilde{R}+\widetilde{R}'+2m)[4\lambda^2-(\widetilde{R}-\widetilde{R}')^2]}{\lambda^2[(\widetilde{R}+\widetilde{R}')^2-4m^2]+a^2(\widetilde{R}-\widetilde{R}')^2}\mathrm{d}\tilde{\varphi}\right\}^2\\ &-\frac{\lambda^2(\widetilde{R}+\widetilde{R}'+2m)^2+a^2(\widetilde{R}-\widetilde{R}')^2}{\lambda^2[(\widetilde{R}+\widetilde{R}')^2-4m^2]+a^2(\widetilde{R}-\widetilde{R}')^2}\tilde{\rho}^2\mathrm{d}\tilde{\varphi}^2-\frac{\lambda^2(\widetilde{R}+\widetilde{R}'+2m)^2+a^2(\widetilde{R}-\widetilde{R}')^2}{4\lambda^2\widetilde{R}\widetilde{R}'}(\mathrm{d}\tilde{\rho}^2+\mathrm{d}\tilde{z}^2),\end{aligned}\tag{1}$$

where

$$\widetilde{R}=[\tilde{\rho}^2+(\tilde{z}+\lambda)^2]^{1/2},\quad \widetilde{R}'=[\tilde{\rho}^2+(\tilde{z}-\lambda)^2]^{1/2};\tag{2}$$

$$\lambda=(m^2-a^2)^{1/2}=m(1-k^2)^{1/2}=mp,$$
$$k=a/m,\quad p=(1-k^2)^{1/2},\tag{3}$$

and $m, a$ are mass and angular momentum per unit mass of the black hole, respectively. The ranges of the coordinate values are

$$0\le\tilde{\rho}<\infty,\quad -\infty<\tilde{t},\tilde{z}<\infty,\qquad 0\le\tilde{\rho}\le 2\pi.\tag{4}$$

It is assumed that

$$0<a<m,\quad 0<k,p<1,\quad 0<\lambda<m,\tag{5}$$

since the black hole is not an extreme one. We notice

$$\widetilde{R}>0,\quad \widetilde{R}'>0.\tag{6}$$

A Kerr black hole at the origin of Weyl coordinate system $(\tilde{t},\tilde{\rho},\tilde{z},\tilde{\varphi})$ is shown in Fig.1(a).

First, we introduce a translation $H$ along $\tilde{z}$-axis as follows:

$$\hat{t}=\tilde{t},\quad \hat{\rho}=\tilde{\rho},\quad \hat{z}=\tilde{z}+H,\quad \hat{\varphi}=\tilde{\varphi},\tag{7}$$

and

$$\hat{R}=[\hat{\rho}^2+(\hat{z}-H+\lambda)^2]^{1/2}=\widetilde{R},$$
$$\widehat{R}'=[\hat{\rho}^2+(\hat{z}-H-\lambda)^2]^{1/2}=\widetilde{R}'.\tag{8}$$

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In order to get a reasonable result, we set

$$H > \lambda. \tag{9}$$

A Kerr black hole at $\hat{z} = H$ of the Weyl coordinate system $(\hat{t}, \hat{\rho}, \hat{z}, \hat{\varphi})$ is shown in Fig.1(b).

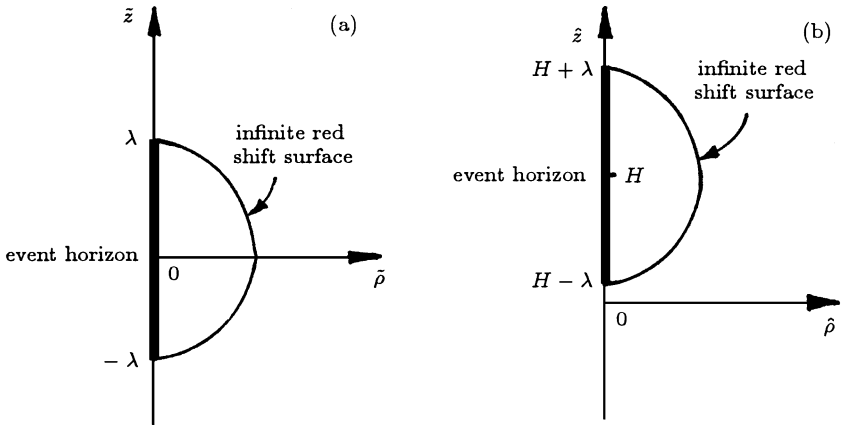



**Fig.1.** A Kerr black hole in Weyl coordinate system. (a) A Kerr black hole at the origin of Weyl coordinate system $(\tilde{t}, \tilde{\rho}, \tilde{z}, \tilde{\varphi})$; (b) A Kerr black hole at $z = H$ in Weyl coordinate system $(\hat{t}, \hat{\rho}, \hat{z}, \hat{\varphi})$.

Next, we introduce the following scale transformation

$$\begin{aligned}
&t = \frac{1}{8mB}\hat{t} = \frac{1}{8mB}\tilde{t},\\
&\rho = 8mB\hat{\rho} = 8mB\tilde{\rho},\\
&z = 8mB\hat{z} = 8mB(\tilde{z} + H),\\
&\varphi = \hat{\varphi} = \tilde{\varphi}.\\
&R = \left[\rho^2 + \left(z - \frac{1}{2B}\right)^2\right]^{1/2} = 8mB\widehat{R} = 8mB\widetilde{R},\\
&R' = \left[\rho^2 + \left(z - \frac{1}{2B} - 16mB\lambda\right)^2\right]^{1/2}\\
&\quad = 8mB\widehat{R} = 8mB\widetilde{R}',
\end{aligned} \tag{10, 11}$$

where $B$ is a new parameter defined as

$$\frac{1}{2B} = [4m(H - \lambda)]^{1/2}. \tag{12}$$

Substituting Eqs.(10) and (11) into Eq.(1), we get

$$\begin{aligned}
\mathrm{d}s^2 = {}& \frac{\lambda^2[(R+R')^2 - (16m^2B)^2] + a^2(R-R')^2}{\lambda^2(R+R'+16m^2B)^2 + a^2(R-R')^2}\Big\{8mB\mathrm{d}t\\
&+ \frac{ma(R+R'+16m^2B)[(16mB\lambda)^2 - (R-R')^2]}{8mB\{\lambda^2[(R+R')^2 - (16m^2B)^2] + a^2(R-R')^2\}}\mathrm{d}\varphi\Big\}^2\\
&- \frac{\lambda^2(R+R'+16m^2B)^2 + a^2(R-R')^2}{\lambda^2[(R+R')^2 - (16m^2B)^2] + a^2(R-R')^2}\frac{\rho^2}{(8mB)^2}\mathrm{d}\varphi^2\\
&- \frac{\lambda^2(R+R'+16m^2B)^2 + a^2(R-R')^2}{4\lambda^2RR'}\frac{1}{(8mB)^2}(\mathrm{d}\rho^2 + \mathrm{d}z^2).
\end{aligned} \tag{13}$$

Notice that the translation and scale transformations are all constants and do not change the nature of the space time. That is to say, metric (13) is no other than a Kerr metric in Weyl form.

Now, we set $m$, $a$, $H$, and $\lambda$ approach to infinity, keep $B$, $k$, $\rho$ and $z$ to be finite. Taking the limit process to $R$ and $R'$, we get

$$\lim R = R, \quad \lim R' = 16m\lambda B - \left(z - \frac{1}{2B}\right). \tag{14}$$

By applying the limit process to metric (13), it reduces to

$$\begin{aligned}\mathrm{d}s^2 =&\frac{4B\left[(1-2k^2)R-\left(z-\frac{1}{2B}\right)\right]}{p(1+p)}\mathrm{d}t^2+2\frac{k}{p}\left[R+\left(z-\frac{1}{2B}\right)\right]\mathrm{d}t\mathrm{d}\varphi\\&-\frac{(1+p)}{4pB}\left[R+\left(z-\frac{1}{2B}\right)\right]\mathrm{d}\varphi^2-\frac{(1+p)}{8pBR}(\mathrm{d}\rho^2+\mathrm{d}z^2)\\=&g_{00}\mathrm{d}t^2+2g_{23}\mathrm{d}t\mathrm{d}\varphi+g_{33}\mathrm{d}\varphi^2+g_{11}\mathrm{d}\rho^2+g_{22}\mathrm{d}z^2,\end{aligned}\tag{15}$$

where

$$\begin{aligned}&g_{00}=\frac{4B\left[(1-2k^2)R-\left(z-\frac{1}{2B}\right)\right]}{p(1+p)},\quad g_{03}=\frac{k}{p}\left[R+\left(z-\frac{1}{2B}\right)\right],\\&g_{33}=\frac{-(1+p)}{4pB}\left[R+\left(z-\frac{1}{2B}\right)\right],\quad g_{11}=g_{22}=\frac{-(1+p)}{8pBR}.\end{aligned}\tag{16}$$

Equation (15) is the new metric we obtained.

## Ⅲ. DISCUSSION

1. It is easily proved that metric (15) satisfies the following equations

$$R_{\mu\nu}=0,\quad R^{\mu}_{\nu\rho\sigma}=0,\qquad \mu,\nu,\rho,\sigma=1,2,3,4.\tag{17}$$

Therefore, metric (15) is a solution of Einstein vacuum field equations and the space time is flat.

2. On the basis of the limit process, we see that $m$ and $a$ approaching to infinity means that the black hole is infinite; that $\rho$, $z$ keeping to be finite means $\hat{\rho}$, $\hat{z}$ approaching to infinitesimal and that $B$ keeping to be finite means $H$-$\lambda$ approaching to infinitesimal, that is to say, the origin of Weyl coordinate system $(\hat{t},\hat{\rho},\hat{z},\hat{\varphi})$ is infinitely close to the horizon pole of the Kerr black hole. Therefore, it can be concluded that space time (15) is an infinitesimal neighborhood nearby one of the two horizon poles of an infinite Kerr black hole. In other words, it is the second order infinitesimal neighborhood nearby one of the two horizon poles of an ordinary Kerr black hole.

3. The infinite redshift surface and event horizon are determined by the two equations[5]

$$g_{00}=0,\quad g_{00}-\frac{g_{03}^2}{g_{33}}=0,\tag{18}$$

respectively. Solving Eqs.(18), we get that the coordinates of the infinite redshift surface satisfy the relaton

$$\rho=\pm\frac{2k(1-k^2)^{1/2}}{1-2k^2}\left(z-\frac{1}{2B}\right),\tag{19}$$

and the coordinates of the event horizon are

$$\rho=0,\quad z\geq\frac{1}{2B}.\tag{20}$$

On the basis of Eq.(19), the relations between the infinite redshift surface and parameter $k$ are

$$\rho=\frac{2k(1-k^2)^{1/2}}{1-2k^2}\left(z-\frac{1}{2B}\right),\quad \frac{1}{2B}\leq z<\infty,\quad \text{when}\quad 0<2k^2<1,\tag{21}$$

$$0\leq\rho<\infty,\quad z=\frac{1}{2B},\quad \text{when}\quad 2k^2=1,\tag{22}$$

$$\rho=\frac{2k(1-k^2)^{1/2}}{1-2k^2}\left(\frac{1}{2B}-z\right),\quad -\infty<z\leq\frac{1}{2B},\quad \text{when}\quad 1<2k^2<2.\tag{23}$$

These cases are shown in Fig.2.

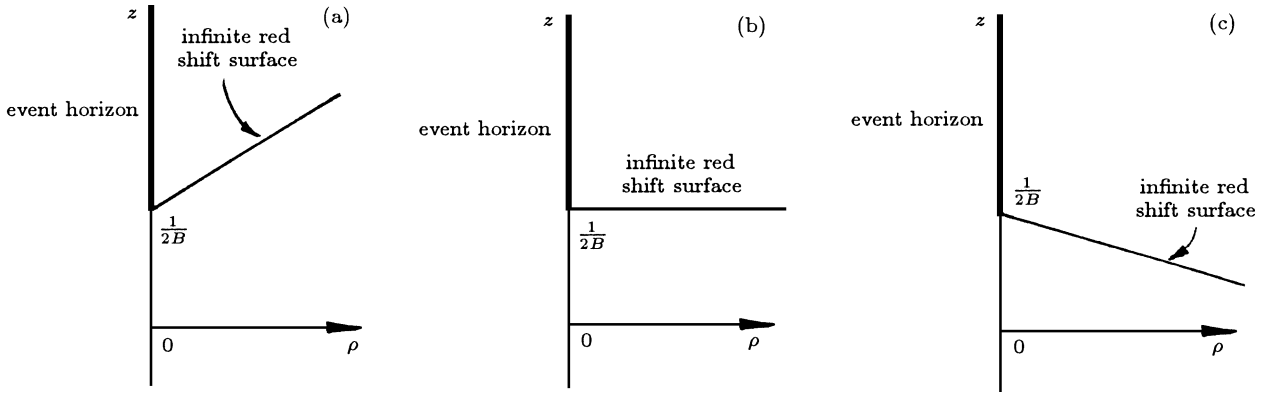



**Fig.2.** The infinite redshift surface and event horizon of space time (15). (a) When $0 < 2k^2 < 1$; (b) When $2k^2 = 1$; (c) When $1 < 2k^2 < 2$.

4. In order to find the nature of metric (15), we introduce the following coordinate transformation

$$\phi = \varphi - \frac{4kB}{1+p}t. \tag{24}$$

Substituting Eq.(24) into Eq.(15), we get

$$\mathrm{d}s^2 = \frac{4Bp}{1+p}\left[R - \left(z - \frac{1}{2B}\right)\right]\mathrm{d}t^2 - \frac{1+p}{4Bp}\frac{1}{R - \left(z - \frac{1}{2B}\right)}\rho^2\mathrm{d}\phi^2 - \frac{1+p}{8BpR}(\mathrm{d}\rho^2 + \mathrm{d}z^2). \tag{25}$$

Metric (25) is static. On the basis of Eqs.(24) and (25), we see that the space time (15) is a rotating static space time with constant angular velocity $\frac{4kB}{1+p}$. Furthermore, we introduce a constant scale transformation as follows:

$$T = \alpha t, \quad r = \frac{1}{\alpha}\rho, \quad Z = \frac{z}{\alpha}, \quad \psi = \phi, \quad A = \alpha B, \quad \alpha = \left[\frac{2p}{1+p}\right]^{1/2}. \tag{26}$$

$$\overline{R} = \left[r^2 + \left(Z - \frac{1}{2A}\right)^2\right]^{1/2} = \alpha R. \tag{27}$$

Substituting Eqs.(26) and (27) into Eq.(25), we get

$$\mathrm{d}s^2 = 2A\left[\overline{R} - \left(Z - \frac{1}{2A}\right)\right]\mathrm{d}T^2 - \frac{1}{2A\left[\overline{R} - \left(Z - \frac{1}{2A}\right)\right]}r^2\mathrm{d}\psi^2 - \frac{1}{8A\overline{R}}(\mathrm{d}r^2 + \mathrm{d}Z^2). \tag{28}$$

It is well known, Eq.(28) is the standard Rindler metric in Weyl form.[6]

From the above discussion, it can be concluded that space time (15) is a rotating Rindler space time with constant angular velocity.

## ACKNOWLEDGMENT

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