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EXACT SOLUTION FOR SUPER-JAYNES-CUMMINGS MODEL

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Based on the linear quantum transformation theory, we further propose a supersymmetric unitary transformation, whose operator can be constructed by the supersymmetric generators of Lie superalgebra. Therefore, by making use of the supersymmetric unitary transformation, we can obtain the eigenspectrum, the eigenstates and the super-partition function for the Super-Jaynes-Cummings model when the coupling constants are Grrassmann valued.

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I. INTRODUCTION

It is well known that the Jaynes-Cummings (JC) model[1] plays an important role in the study of a single two-level atom interacting in rotating wave approximation with a single quantized radiation resonance field mode. Some exactly solvable generalizations of JC model, which are nonlinear both in bosonic and spin variables, are considered, and exact wave functions as well as energy level of these model are evaluated.[2,3] Recently, considerable interest has been paid to $U(1/1)$ superalgebra and its boson-fermion realization[4−6] because $U(1/1)$ algebra happens to be the underlying symmetry of the JC model. In Ref.[4] Kochetov made use of the $U(1/1)$ coherent state to obtain a path integral representation of the partition function of the JC model, while in Ref.[6], Professor Fan obtained the normally ordered boson-fermion realization of the JC Hamiltonian with Grassmann valued coupling form by using the technique of integration within an ordered product(IWOP). However, in this paper, based on the linear quantum transformation (LQT) theory[7−10], we will further present a method, called a supersymmetric unitary transformation, which provides a new approach for calculating the eigenvalue, the eigenstates and the super-partition function of the super-Jaynes-Cummings (SJC) model when the coupling constants are Grassmann valued. In the following derivation, one can see clearly that this method is not only simple but also general because it can be applied to the study of the supersymmetric system whether its coupling constants are Grassmann or ordinary c-valued numbers. So, we believe that it will play an active role in studying the supersymmetric quantum mechanics.

Let b and f be boson and fermion operators, respectively,

$$[b, b^+] = 1, \ \{f, f^+\} = 1, \ [b, f] = 0, \ f^2 = f^{+^2} = 0, \tag{1}$$

we will give the exact solution of the following SJC model whose Hamiltonian reads[5]

$$\widehat{H} = 2\omega_1 b^+ b + 2\omega_2 f^+ f + \lambda b f^+ + b^+ f\overline{\lambda}, \tag{2}$$

where the coupling constants $(\lambda, \overline{\lambda})$ are considered as Grassmann valued, and they have properties of Grassmann number, $\lambda^2 = \overline{\lambda}^2 = 0, \{\lambda, f\} = 0$. The Hamiltonian (2) is an element of the spectrum generating unitary superalgebra $U(1/1)$.[5]

II. SUPERSYMMETRIC UNITARY TRANSFORMATION

Let $|0\rangle$ be a vacuum state of fermion, then $f|0\rangle = 0, f^+|0\rangle = 1$, if we denote the states $|0\rangle$ and $|1\rangle$ with the matrix representations, respectively, $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we know that

$$f = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$f^+ = |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3}$$

therefore

$$f^+ f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_z + 1). \tag{4}$$

Now defining the supersymmetric transformation generators as follows:

$$N = b^+ b + f^+ f = \begin{pmatrix} bb^+ & 0 \\ 0 & b^+ b \end{pmatrix}, \tag{5}$$

$$Q = a^+ f = \begin{pmatrix} 0 & 0 \\ a^+ & 0 \end{pmatrix}, \ Q^+ = f^+ a = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}. \tag{6}$$

It is easy to see that (N, Q, Q^+) form the supersymmetric generators and have Lie superalgebra properties, i.e.,

$$\begin{aligned} &Q^2 = {Q^+}^2 = 0, \quad [Q^+, Q] = N\sigma_z, \quad [N, Q] = [N, Q^+] = 0, \\ &\{Q, \sigma_z\} = \{Q^+, \sigma_z\} = 0, \quad N = \{Q, Q^+\}, \quad (Q^+ - Q)^2 = -N, \end{aligned} \tag{7}$$

in which $\{\}$ denotes the anticommutation bracket. With the help of Eqs.(5) and (6), Eq.(2) can be rewritten as

$$\widehat{H} = 2\omega_1 N + (\omega_2 - \omega_1)\sigma_z + \lambda Q^+ + Q\overline{\lambda} + (\omega_2 - \omega_1). \tag{8}$$

Now, by the aid of the supersymmetric transformation generators mentioned above, we may construct a supersymmetric unitary transformation operator so that Hamiltonian (8) can be diagonalized.The supersymmetric unitary transformation operator is defined as

$$T = \exp\left\{-\frac{\theta}{2}\frac{1}{\sqrt{\lambda\overline{\lambda}N}}(\lambda Q^+ - Q\overline{\lambda})\right\}, \tag{9}$$

where $N^{-1/2}$ is

$$N^{-1/2} = \begin{pmatrix} \dfrac{1}{\sqrt{bb^+}} & 0 \\ 0 & \dfrac{1}{\sqrt{b^+b}} \end{pmatrix}. \tag{10}$$

Using the formulas[11], $bf(b^+b) = f(b^+b+1)b$, $b^+f(bb^+) = f(b^+b)b^+$, one can easily verify the following commutation relation

$$[N^{-1/2}, Q] = [N^{-1/2}, Q^+] = 0. \tag{11}$$

Therefore, from Eqs. (7), (11) and the formula $(\lambda Q^+ - Q\overline{\lambda})^2 = -\lambda\overline{\lambda}N$, Eq.(9) can be expanded to the following form

$$T = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\frac{1}{\sqrt{\lambda\overline{\lambda}N}}(\lambda Q^+ - Q\overline{\lambda}). \tag{12}$$

Now, consider the following supersymmetric unitary transformation

$$\widehat{H}' = T^{-1}\widehat{H}T.$$

From Eqs.(7),(11) and (12), we can calculate the following result

$$\begin{aligned} T^{-1}(\lambda Q^+ + Q\overline{\lambda})T &= \cos\theta(\lambda Q^+ + Q\overline{\lambda}) \\ &\quad + \sin\theta\frac{\lambda\overline{\lambda}}{\sqrt{\lambda\overline{\lambda}N}}[Q^+, Q], \\ T^{-1}\sigma_z T = \sigma_z T^2 &= \cos\theta\sigma_z \\ &\quad - \sin\theta\frac{1}{\sqrt{\lambda\overline{\lambda}N}}(\lambda Q^+ + Q\overline{\lambda}). \end{aligned}$$

So, we have

$$\begin{aligned} \widehat{H}' = T^{-1}\widehat{H}T &= 2\omega_1 N + \left[\cos\theta - (\omega_2 - \omega_1)\sin\theta\frac{1}{\sqrt{\lambda\overline{\lambda}N}}\right](\lambda Q^+ + Q\overline{\lambda}) \\ &\quad + \left[\sqrt{\lambda\overline{\lambda}}\sin\theta + (\omega_2 - \omega_1)\frac{\cos\theta}{\sqrt{N}}\right]\frac{1}{\sqrt{N}}[Q^+, Q] + (\omega_2 - \omega_1). \end{aligned}$$

If we let

$$\frac{\sqrt{\lambda\overline{\lambda}}}{\omega_2 - \omega_1} = \frac{1}{\sqrt{N}}\tan\theta \tag{13}$$

(note that Eq. (13) should be understood in the sense of the eigenvalue and eigenvalue equation for the operator N), we can obtain the diagonalized Hamiltonian as follows:

$$\widehat{H}' = T^{-1}\widehat{H}T = 2\omega_1 N + \frac{\omega_2 - \omega_1}{\cos\theta}\sigma_z + (\omega_2 - \omega_1). \tag{14}$$

The corresponding eigenstates of $\widehat{H}'$ read

$$\begin{aligned} \psi'_{1,n+1} &= \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} \equiv |\uparrow, n\rangle, \\ \psi'_{2,n+1} &= \begin{pmatrix} 0 \\ |n+1\rangle \end{pmatrix} \equiv |\downarrow, n+1\rangle. \end{aligned} \tag{15}$$

When

$$\frac{1}{\sqrt{n+1}}\tan\theta = \frac{\sqrt{\lambda\overline{\lambda}}}{\omega_2 - \omega_1}, \tag{16}$$

we get

$$\cos\theta = \frac{\omega_2 - \omega_1}{\sqrt{\lambda\overline{\lambda}(n+1) + (\omega_2 - \omega_1)^2}}. \tag{17}$$

Obviously, the eigenvalue equation should be as follows:

$$\begin{aligned} \widehat{H}'\psi'_{1,n+1} &= \left[2\omega_1 n + 2\omega_2 + \frac{\lambda\overline{\lambda}(n+1)}{2(\omega_1 - \omega_2)}\right]\psi'_{1,n+1}, \\ \widehat{H}'\psi'_{2,n+1} &= \left[2\omega_1(n+1) + \frac{\lambda\overline{\lambda}(n+1)}{2(\omega_1 - \omega_2)}\right]\psi'_{2,n+1}, \end{aligned} \tag{18}$$

where we have fully used the properties of the Grassmann number $\lambda^2 = \overline{\lambda}^2 = 0$.

Finally, we obtain the eigenstates of $\widehat{H}$

$$\begin{aligned}\psi_{1,n+1} =& T\psi'_{1,n+1} = \cos\frac{\theta}{2}\begin{pmatrix}|n\rangle\\0\end{pmatrix} + \sqrt{\frac{\overline{\lambda}}{\lambda}}\sin\frac{\theta}{2}\begin{pmatrix}0\\|n+1\rangle\end{pmatrix},\\ \psi_{2,n+1} =& T\psi'_{2,n+1} = \cos\frac{\theta}{2}\begin{pmatrix}0\\|n+1\rangle\end{pmatrix} - \sqrt{\frac{\lambda}{\overline{\lambda}}}\sin\frac{\theta}{2}\begin{pmatrix}|n\rangle\\0\end{pmatrix}.\end{aligned} \tag{19}$$

It should be pointed out that the state $|\downarrow, 0\rangle$ which is not included in Eqs.(19) is another eigenstate of $\widehat{H}$.

III. SUPER-PARTITION FUNCTION

In order to calculate the super-partition function Z of the SJC model (2) in boson-fermion Fock space, we first write down the matrix representation of the Hamiltonian $\widehat{H}$. From Eq.(4), we get the corresponding energy eigenvalue as follows:

$$E_0 = 0 \quad (\text{for } n = 0), \tag{20}$$

$$E^{n>0} = 2\omega_1(n+I) + (\omega_2-\omega_1)\left[1+\frac{\lambda\overline{\lambda}(n+I)}{2(\omega_1-\omega_2)^2}\right](2I-1) + (\omega_2-\omega_1)$$
$$(\text{for any} \quad n>0, \quad I=0,1). \tag{21}$$

Therefore, we have

$$H_n = \begin{pmatrix}\langle n,0|H|n,0\rangle & 0\\ 0 & \langle n-1,1|H|n-1,1\rangle\end{pmatrix} = \begin{pmatrix}2\omega_1 n + \dfrac{n\lambda\overline{\lambda}}{2(\omega_1-\omega_2)} & 0\\ 0 & 2\omega_1 n + 2(\omega_2-\omega_1) + \dfrac{n\lambda\overline{\lambda}}{2(\omega_1-\omega_2)}\end{pmatrix}, \tag{22}$$

$$H_0 = 0. \tag{23}$$

Using the formula of super-partition function in Ref.[11], $Z = \mathrm{str}\exp(-\beta H)$, where str represents supertrace, and from Eqs. (22) and (23), we obtain

$$\begin{aligned}Z =& \mathrm{str}\exp(-\beta H) = \sum_{n=0}^{\infty}\mathrm{str}\exp(-\beta H_n)\\ =& 1 + \sum_{n=1}^{\infty}\exp(-\beta\omega_1 n)[1-\exp(-2\beta(\omega_2-\omega_1))]\exp\left(-\frac{\beta n\lambda\overline{\lambda}}{2(\omega_1-\omega_2)}\right)\\ =& 1 + \sum_{n=1}^{\infty}\exp(-2\beta\omega_1 n)[1-\exp(-2\beta(\omega_2-\omega_1))]\left[1+\frac{n\beta\lambda\overline{\lambda}}{2(\omega_2-\omega_1)}\right]\\ =& 1 + \sum_{n=1}^{\infty}[\exp(-2\beta\omega_1 n) - \exp(-2\beta\omega_1 n - 2\beta(\omega_2-\omega_1))]\\ &+ \frac{\beta\lambda\overline{\lambda}}{2(\omega_2-\omega_1)}\sum_{n=1}^{\infty} n\exp(-2\beta\omega_1 n)[1-\exp(-2\beta(\omega_2-\omega_1))].\end{aligned}$$

Note:

$$1 + \sum_{n=1}^{\infty}[\exp(-2\beta\omega_1 n) - \exp(-2\beta\omega_1 n - 2\beta(\omega_2-\omega_1))] = (1-\exp(-2\beta\omega_1))^{-1} - \sum_{n=1}^{\infty}\exp(-2\beta\omega_1(n-1))\exp(-2\beta\omega_2)$$
$$= \frac{1-\exp(-2\beta\omega_2)}{1-\exp(-2\beta\omega_1)},$$

and

$$\sum_{n=1}^{\infty} n\exp(-2\beta\omega_1 n) = \frac{\exp(-2\beta\omega_1)}{(1-\exp(-2\beta\omega_1))^2},$$

we finally obtain the super-partition function as follows:

$$Z = \frac{1-\exp(-2\beta\omega_2)}{1-\exp(-2\beta\omega_1)} + \frac{\beta\lambda\overline{\lambda}(\exp(-2\beta\omega_1)-\exp(-2\beta\omega_2))}{2(\omega_2-\omega_1)(1-\exp(-2\beta\omega_1))^2}. \tag{24}$$

In summary, we have used the supersymmetric unitary transformation to derive the exact, explicit and compact expression of the eigenvalue, the eigenstates and the super-partition function for SJC model whose Hamiltonian is an element of the spectrum generating unitary superalgebra $U(1/1)$. As we have discussed above, we can also see clearly that this method can be applied to the study of a general supersymmetric system with n-mode boson and m-mode fermion coupling Hamiltonian which is an element of the spectrum generating unitary superalgebra $U(n/m)$.[12] We shall return to this topic in future work.

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