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# QUANTUM LANGEVIN THEORY OF WHISPERING-GALLERY-MODE MICROSPHERE LASER*

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A quantum Langevin theory of whispering-gallery-mode microsphere laser theory is developed. The linear and nonlinear analysis are made for laser operation below and above the threshold. In these analysis, corresponding to the specific property of microsphere, the effect of inversion fluctuation is treated. The coherence functions of laser field are calculated, and the intensity, the amplitude fluctuation and the linewidth of the field are obtained, which are connected with the enhancement factor of whispering-gallery-mode microsphere. It is shown that the strong couple and strong pumping are useful for the amplification of intensity and the decrease of linewidth below the threshold. It is also shown that, for the laser action above threshold, the variances of photon number and the linewidth of internal field are related to the enhancement factor and the square of the enhancement factor, respectively.

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## I. INTRODUCTION

The optical resonances of a dielectric microsphere are termed morphology-dependent resonances (MDR's)[1] to emphasize that the resonance frequencies depend on the structure (size, shape and refractive index) of the microsphere, and also termed the whispering-gallery modes (WGM's) because of the similarity to whispering-gallery acoustic waves. WGM's have been studied extensively[2] for some reasons, for example, WGM's have extremely high $Q$ value, lead to decreasing threshold for lasing and several types of nonlinear optical processes, give rise to the effect of cavity quantum electrodynamics (QED), and have important applications in photonics or optoelectronics.[3,4]

The lasing from microspheres occurring at discrete wavelengths corresponding to WGM's is referred to as whispering-gallery-mode microsphere laser. WGM laser was observed first in a spherical millimetric crystal sample of $CaF_2 : Sm^{++}$ by Garrett, Kaiser and Bond.[5] Later, in the past decades, there have been a number of experimental papers on WGM laser from liquid or solid microspheres, such as ethanal 30-$\mu$m-radius droplets doped with rhodamine[6–8], dye-doped polystyrene solid spheres with diameters ranging from 10 to 92 $\mu$m[9] and 40–800 $\mu$m[10], microdroplets seeded fluorescent sol[11] or undoped polystyrene latex spheres[12] or nanoparticles[13], or highly scattering intralipid[14], double-layered dye-doped microdroplet consisting of a spherical dye-doped solid inner core covered with a dye-doped liquid-solution outer shell[15], three-dye-mixed-doped liquid microspheres[16], microdroplets containing a concentration-changing absorber[17], Nd: YAG spheres[18] with about 200 $\mu$m-diameter, Nd: glass sphere with a radius of 40–60 $\mu$m[19], pure silica sphere of 25–50 $\mu$m-radius[20], and so on. In these experiments, observations of threshold behavior, emission wavelength dependence and spatial distribution of the emission have provided strong evidence for the presence of laser emission linked with high-$Q$ WGM's. An important result is that the threshold pump level of the laser in the microsphere is much lower than that of a conventional laser.

From theoretical viewpoint, WGM lasers operate in a manner analogous to conventional linear Fabry-Perot cavity lasers except that the cavity modes provided by the mirror cavity of the linear cavity laser are replaced by WGM's. The gain of lasing is provided by the feedback coming from microspheres. Therefore, several authors have directly applied one-dimensional laser theory in many laser texts[21,22] to the WGM laser. For example, Lin *et al.*[23] exammined the effect of spatial hole burning on lasing 15.3-$\mu$m-diameter R6G-water solutions droplets, and indicated that although the lasing gain in R6G is nominally homogeneously broadened on a nanosecond time scale, the presence of spatial hole burning permits multimode oscillation simultaneously on many spherical cavity mode resonances. Mazumder *et al.*[17] used the stan-

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dard dye-laser theory[24] to explain the absorption-dependent blue shift of dye lasing in microdroplets. However, although there is analogy between WGM laser and usual linear cavity laser, a detailed comparison of WGM laser experiments and theory requires a complete WGM laser theory which combines usual laser theory with the Lorenz- Mie theory. Such WGM laser theory, to our knowledge, has not been made up to date, in spite of several related basic work reported[25−27]. This is just the motivation of our study.

The purpose of this paper is to develop a quantum Langven theory of WGM laser. Our work is the extension of the traditional laser quantum theory of Haken[28] and Lamb[29] and the one-dimensional output coupling laser quantum theory of Ujihara[30,31] and Baseia[32], and it emphasizes the lasing action associated with WGM in the microsphere. There are two main differences between the two theories: Compared with Haken's and Lamb's theory, in one-dimensional output coupling laser quantum theory, not only internal field but also external field can be treated. Moreover, no damping term and fluctuation term are introduced to the field, while thermal noise emerges automatically. In our present work, on the basis of the above quantum theories, considering the specific properties of the WGM microsphere, we further investigate the effect of inversion fluctuation, unlike the usual laser case in which the inversion fliuctuation is usually neglected. In addition, in our analyses, all results can be connected with the enhancement factor which can describe the properties of WGM microsphere. These two aspects are different from the usual quantum laser theories.

The paper is organized as follows: In Sec.Ⅱ, the quantization of light field in dielectric microsphere is performed. In Sec.Ⅲ we establish the quantum mechanical Langevin equation for a two-level model. In Sec.Ⅳ, we make a linear analysis for laser operation below threshold, the internal and external fields are both considered. In Sec.V, we investigate the laser action above threshold inside microsphere by calculating the amplitude and phase noise power spectrum and obtain the photon number variance and linewidth of laser. In Sec.6, we give a brief summary.

## Ⅱ. QUANTIZATION OF THE LIGHT FIELD IN A DIELECTRIC MICROSPHERE

The system consists of a microsphere of radius $a$ and dielectric constant $n^2$ with $N$ laser active atoms. According to the general quantization procedure of quantum mechancs, for our system, the quantization volume may be considered a large sphere of finite radius ($\Lambda \to \infty$) bounded by an ideally conducting sphere.

The electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ and their vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ can be written as[25−27]

$$\boldsymbol{E}(\boldsymbol{r},t)=\sum_s \frac{1}{i\sqrt{2}}(a_s^+ - a_s)\boldsymbol{e}_s(\boldsymbol{r}), \tag{1}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\sum_s \frac{1}{\sqrt{2}}(a_s^+ + a_s)\boldsymbol{b}_s(\boldsymbol{r}), \tag{2}$$

$$\boldsymbol{A}(\boldsymbol{r},t)=-\sum_s \frac{c}{\sqrt{2}\omega_s}(a_s^+ + a_s)\boldsymbol{e}_s(\boldsymbol{r}), \tag{3}$$

where $e_s(\boldsymbol{r})$, $b_s(\boldsymbol{r})$ are the solutions of the homogeneous Maxwell equations; $a_s$ and $a_s^+$ are annihilation and creation operators for frequencies $\omega_s$ of the light field and they are respectively take the form for TE wave and TM wave:

TE
$$\boldsymbol{e}_s(\boldsymbol{r})=\beta_E(l,m)j_l(nkr)\boldsymbol{X}_{lm}(\theta,\varphi), \qquad r<a, \tag{4a}$$
$$\boldsymbol{e}_s(\boldsymbol{r})=[\gamma_E^{(1)}(l,m)h_l^{(1)}(kr)+\gamma_E^{(2)}(l,m)h_l^{(2)}(kr)]\boldsymbol{X}_{lm}(\theta,\varphi), \qquad r>a, \tag{4b}$$
$$\boldsymbol{b}_s(\mathbf{r})=-\frac{c}{\omega_s}\bigtriangledown\times\boldsymbol{e}_s(\boldsymbol{r}); \tag{4c}$$

TM
$$\boldsymbol{b}_s(\boldsymbol{r})=\beta_M(l,m)j_l(nkr)\boldsymbol{X}_{lm}(\theta,\varphi), \qquad r<a, \tag{5a}$$
$$\boldsymbol{b}_s(\boldsymbol{r})=[\gamma_M^{(1)}(l,m)h_l^{(1)}(kr)+\gamma_M^{(2)}(l,m)h_l^{(2)}(kr)]\boldsymbol{X}_{lm}(\theta,\varphi), \qquad r>a, \tag{5b}$$
$$\boldsymbol{e}_s(\boldsymbol{r})=-\frac{c}{n^2(\boldsymbol{r})\omega_s}\bigtriangledown\times\boldsymbol{b}_s(\mathbf{r}), \tag{5c}$$

where $k=\omega_s/c$ is the vacuum wavenumber; $j_l$, $h_l^{(1)}$ and $h_l^{(2)}$ are respectively the spherical Bessel and Hankel functions, $\boldsymbol{X}_{lm}(\theta,\varphi)=[l(l+1)]^{-1/2}\boldsymbol{L}Y_{lm}$ is the angular momentum operator, $l$ is the orbital quantum number, $m$ is the azimuthal quantum number, $n^2(\boldsymbol{r})$ equals constant $n^2$ inside the microsphere and unity outside the microsphere.

Following the method of evaluating $M$ie scat-

tering coefficients, $\beta_E/\gamma_E^{(2)}$, $\gamma_E^{(1)}/\gamma_E^{(2)}$, $\beta_M/\gamma_M^{(2)}$ and $\gamma_M^{(1)}/\gamma_M^{(2)}$ can be derived using the boundary condition that the tangential field components at the sphere boundary are continous:

$$\frac{\beta_E}{\gamma_E^{(2)}} = p_E = \frac{h_l^{(1)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(2)}(x) - h_l^{(2)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x)}{\frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(1)}(x) - j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x)}, \tag{6}$$

$$\frac{\gamma_E^{(1)}}{\gamma_E^{(2)}} = q_E = \frac{\frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(2)}(x) - j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(2)}(x)}{j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x) - \frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(1)}(x)}, \tag{7}$$

$$\frac{\beta_M}{\gamma_M^{(2)}} = p_M = \frac{n^2[h_l^{(1)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(2)}(x) - h_l^{(2)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x)]}{\frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(1)}(x) - n^2 j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x)}, \tag{8}$$

$$\frac{\gamma_M^{(1)}}{\gamma_M^{(2)}} = q_M = \frac{\frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(2)}(x) - n^2 j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(2)}(x)}{n^2 j_l(y)\frac{\mathrm{d}}{\mathrm{d}x}\xi^{(1)}(x) - \frac{\mathrm{d}}{\mathrm{d}y}\Psi(y)h_l^{(1)}(x)}, \tag{9}$$

where $x = ka$, $y = nka$, $\Psi(y) = yj_l(y)$, $\xi^{(1)}(x) = xh_l^{(1)}(x)$, $\xi^{(2)}(x) = xh_l^{(2)}(x)$.

We choose the orthogonal relation of the eigenfunctions in the form

$$\int_{(r\le\Lambda)} \mathrm{d}^3 r \;\; n^2(\boldsymbol{r}) e_s(\boldsymbol{r})\cdot \boldsymbol{e}_{s'}(\boldsymbol{r}) = 4\pi\hbar\omega_s\delta_{ss'}, \tag{10a}$$

$$\int_{(r\le\Lambda)} \mathrm{d}^3 r \;\; b_s(\boldsymbol{r})\cdot \boldsymbol{b}_{s'}(\boldsymbol{r}) = 4\pi\hbar\omega_s\delta_{ss'}, \tag{10b}$$

and note that there exists the conservation of energy for each mode, thus obtain

$$|\gamma_E^{(1)}|^2 = |\gamma_E^{(2)}|^2 = |\gamma_M^{(1)}|^2 = |\gamma_M^{(2)}|^2 = \frac{2\pi\hbar c}{l(l+1)}\frac{k^3}{\Lambda}. \tag{11}$$

In addition, according to the asymptotic formulas for the positions, width and strengths of resonances in Mie scattering[33], we can express the coefficients $|\beta_E|^2$ and $|\beta_M|^2$ as Lorentzian type:

$$|\beta_E|^2 = \frac{\pi c}{\Lambda}K'\omega^3\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega_s-\omega)^2 + \left(\frac{\gamma}{2}\right)^2}, \tag{12}$$

where $K' = \frac{4\hbar}{\pi c^3}K$, $\frac{\gamma}{2}$ is the cavity width, $\omega$ is the allowed cavity resonance frequency, $K$ is the cavity enhancement factor[27] in resonant peak $\omega_s = \omega$, which can describe the properties of WGM microsphere, depending on the resonance frequency and size parameter $x = \frac{2\pi a}{\lambda}$.

Using the boundary condition at the shell of radius $\Lambda$, for TE case the allowed frequencies are

$$\omega_s = \left(\nu + \frac{l}{2}\right)\frac{\pi c}{\Lambda} - \frac{c}{\Lambda}\delta_E, \qquad \nu = 0, 1, 2, \ldots \tag{13a}$$

with

$$\exp(2i\delta_E) = \gamma_E^{(1)}/\gamma_E^{(2)}. \tag{13b}$$

Similarly, for TM wave,

$$\omega_s = \left(\nu + \frac{l}{2}\right)\frac{\pi c}{\Lambda} - \frac{c}{\Lambda}\delta_M, \qquad \nu = 0, 1, 2, \ldots \tag{14a}$$

with

$$\exp(2i\delta_M) = \gamma_M^{(1)}/\gamma_M^{(2)}, \tag{14b}$$

in the above, $\nu$ is the radial quantum number, the space of $\omega_s$ is $\Delta\omega_s = \pi c/\Lambda + O(\Lambda^{-2})$, in the $\Lambda\to\infty$, consequently

$$\sum_{\omega_s} \to \frac{\Lambda}{\pi c}\int d\omega_s. \tag{15}$$

In the following we only discuss the TE case although both TE and TM waves can be excited in a general case.

## Ⅲ. QUANTUM MECHANICAL LANGEVIN EQUATION OF MICROSPHERE LASER

We use the laser model of $N$ two-level atoms, with the the lower lasing level denoted by $|1\rangle_\mu$, upper lasing level by $|2\rangle_\mu$, and transition frequency $\overline{\omega}_\mu$ for the $\mu$th atoms.

In analogy to traditional quantum-machanical treatment of the laser[28,29], under the dipole approximation and the rotating wave approximation, we can obtain the Langevin equations for the field and the atom operators:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_s^+ = i\omega_s a_s^+ + \sum_\mu i g_{s,\mu}^* S_+, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}S_+ = (i\overline{\omega}_\mu - \gamma_\perp)S_+ - i\sum_s g_{s,\mu}a_s^+ S + \Gamma_{21,\mu}, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}S = \gamma_\parallel(S_0 - S) + 2iS_-\sum_s g_{s,\mu}a_s^+ - 2iS_+\sum_s g_{s,\mu}^* a_s + \Gamma_{22,\mu} - \Gamma_{11,\mu}. \tag{18}$$

Here $S_+$ and $S_-$ are the raising and lowering operators for the $\mu$th atom, representing the atomic dipole moment,

$$S_+ = (|2\rangle\langle 1|)_\mu, \tag{19a}$$

$$S_- = (|1\rangle\langle 2|)_\mu, \tag{19b}$$

and $S$ is the inversion operator for the $\mu$th atom, representing the atomic occupation number difference,

$$S = (|2\rangle\langle 2|)_\mu, -(|1\rangle\langle 1|)_\mu. \tag{20}$$

$\gamma_\perp$ represents the transverse decay rate of the dipole moment and $\gamma_\parallel$ is the longitudinal decay rate of the inversion which is the inverse of the longitudinal relaxation time $T$. $S_0$ is the unsaturated inversion of the $\mu$th atom which is built up by the pumping as well as by relaxiation processes. $\gamma_\perp$, $\gamma_\parallel$ and $S_0$ have been defined as

$$\gamma_\perp = \frac{1}{2}(w_{12} + w_{21} + \gamma_p), \tag{21a}$$

$$\gamma_\parallel = w_{12} + w_{21}, \tag{21b}$$

$$S_0 = \frac{w_{12} - w_{21}}{w_{12} + w_{21}}, \tag{21c}$$

where $w_{21}$ is the incoherent decay rate from level 2 to level 1, $w_{12}$ is the incoherent pumping rate from level 1 to level 2 and $\gamma_p$ is the rate of collisionally induced phase decay of the atoms.

The symbol $g_{s,\mu}$ is the coupling constant for the $s$th mode of field with the $\mu$th atom located at $\boldsymbol{r}_\mu$ :

$$g_{s,\mu} = -\frac{i}{\sqrt{2}\hbar}\frac{\overline{\omega}_\mu}{\omega_s}\boldsymbol{e}_s(\boldsymbol{r}_\mu)\cdot\boldsymbol{\theta}_{12,\mu}, \tag{22}$$

where $\boldsymbol{\theta}_{12,\mu}$ is the atom dipole moment matrix element for the $\mu$th atom.

$\Gamma_{21,\mu}$, $\Gamma_{22,\mu}$ and $\Gamma_{11,\mu}$ are respectively Langevin fluctuating force corresponding to atomic operators $S_+$, $(|2\rangle\langle 2|)_\mu$, $(|1\rangle\langle 1|)_\mu$. The correlation function between the fluctuating forces can be given by fluctuation-dissipation theorem[29]:

$$\begin{aligned}
&< \Gamma_{11,\mu}(t)\Gamma_{11,\mu'}(t') > = < \Gamma_{22,\mu}(t)\Gamma_{22,\mu'}(t') > \\
&= - < \Gamma_{11,\mu}(t)\Gamma_{22,\mu'}(t') > = - < \Gamma_{22,\mu}(t)\Gamma_{11,\mu'}(t') > \\
&= [w_{12} < (|1\rangle\langle 1|)_\mu > + w_{21} < (|2\rangle\langle 2|)_\mu >]\delta_{\mu\mu'}\delta(t-t') \\
&= D_\mu\delta_{\mu\mu'}\delta(t-t'), \qquad (23) \\
&< \Gamma_{12,\mu}(t)\Gamma_{12,\mu'}(t') > = < \Gamma_{21,\mu}(t)\Gamma_{21,\mu'}(t') > = 0, \qquad (24) \\
&< \Gamma_{12,\mu}(t)\Gamma_{21,\mu'}(t') > = [w_{21} < (|2\rangle\langle 2|)_\mu > \\
&+ (2\gamma_\perp - w_{12}) < (|1\rangle\langle 1|)_\mu >]\delta_{\mu\mu'}\delta(t-t') \\
&= D_{12,\mu}\delta_{\mu\mu'}\delta(t-t'), \qquad (25) \\
&< \Gamma_{21,\mu}(t)\Gamma_{12,\mu'}(t') > = [w_{12} < (|1\rangle\langle 1|)_\mu > \\
&+ (2\gamma_\perp - w_{21}) < (|2\rangle\langle 2|)_\mu >]\delta_{\mu\mu'}\delta(t-t') \\
&= D_{21,\mu}\delta_{\mu\mu'}\delta(t-t'), \qquad (26) \\
&< \Gamma_{22,\mu}(t)\Gamma_{21,\mu'}(t') > = < \Gamma_{11,\mu}(t)\Gamma_{21,\mu'}(t') > \\
&= -w_{21} < (|2\rangle\langle 1|)_\mu > \delta_{\mu\mu'}\delta(t-t'), \qquad (27) \\
&< \Gamma_{22,\mu}(t)\Gamma_{12,\mu'}(t') > = < \Gamma_{11,\mu}(t)\Gamma_{12,\mu'}(t') > \\
&= w_{12} < (|1\rangle\langle 2|)_\mu > \delta_{\mu\mu'}\delta(t-t'). \qquad (28)
\end{aligned}$$

For steady states, there are the following relation:

$$w_{12} < (|1\rangle\langle 1|)_\mu > - w_{21} < (|2\rangle\langle 2|)_\mu > = 0, \tag{29a}$$

$$< (|2\rangle\langle 1|)_\mu > = < (|1\rangle\langle 2|)_\mu > = 0, \tag{29b}$$

$$D_{12,\mu} = 2\gamma_\perp\frac{w_{21}}{w_{12}+w_{21}}, \tag{29c}$$

$$D_{21,\mu} = 2\gamma_\perp\frac{w_{12}}{w_{12}+w_{21}}, \tag{29d}$$

$$D_\mu = \frac{2w_{12}w_{21}}{w_{12}+w_{21}}. \tag{29e}$$

In the above the angular brackets denote the ensemble average with respect to the heatbaths of the atoms. We have assumed that the heatbaths for the atoms are statistically independent of each other.

In addition, the initial value of the field $a_s(0)$ has the following properties:

$$< a_s(0) > = < a_s^+(0) > = 0, \tag{30a}$$

$$< a_s^+(0)a_{s'}(0) > = \delta_{ss'}n_s, \tag{30b}$$

$$< a_s(0)a_{s'}^+(0) > = \delta_{ss'}(1+n_s), \tag{30c}$$

where the angular brackets denote the ensemble average with respect to the initial field; $n_s$ is the Plank distribution for the $s$th mode of the universe at the initial temperature. We have assumed statistical independence among the universal modes.

It is noted that, in Eq.(16), we have not introduced the damping term and fluctuating forces, in other words, we have neglected the fluctuating force coming from thermal radiation field. Thus thermal radiation field is completely determined by the initial field, such as vacuum field or coherent field.

## IV. LINEAR ANALYSIS

For the laser operation below threshold, we consider that the population inversion of atoms is kept constant. We assume the coherent interaction between the field and the atoms begin at $t$=0. We limit our analysis to atoms with identical properties, hence write $\overline{\omega}_\mu$=$\overline{\omega}$, $\boldsymbol{\theta}_{12,\mu} = \boldsymbol{\theta}_{12}$. For simplicity we assume $\boldsymbol{\theta}_{12}$ and $\boldsymbol{e}_s(\boldsymbol{r})$ are parallel.

### A. Inside the microsphere

For the field operator $a_s^+(t)$, we define the slowly varying variable $A_s^+(t)$ and the collective variable $A^+(t)$ as follows:

$$A_s^+(t) = a_s^+(t)\exp(-i\Omega t), \tag{31a}$$

$$A^+(t) = \sum_s g_s A_s^+(t), \tag{31b}$$

with

$$g_s = g_{s,\mu}/|f(\omega, \boldsymbol{r})|, \tag{31c}$$

and

$$|f(\omega, \boldsymbol{r})| = |j_l(nkr)\boldsymbol{X}_{lm}(\theta, \varphi)|, \tag{31d}$$

where $\Omega$ is the frequency of laser. The collective variable $A^+(t)$ represents the negative frequency part of the electric field, which can be shown from Eq.(1), Eq.(4a), Eq.(22) and Eqs.(31b–31d).

From Eqs.(16)–(18), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}A_s^+(t) = \sum_\mu g_{s,\mu}^* \int_0^t (\Omega - \omega_s)\exp[-i(\Omega - \omega_s)](t-t')]\widetilde{S_+}(t')\mathrm{d}t' + iA_s^+(0)(\omega_s - \Omega)\exp[-i(\Omega-\omega_s)t] + \sum_\mu ig_{s,\mu}^*\widetilde{S_+}(t), \tag{32}$$

where $\widetilde{S_+}(t)$ represents the slowly varying variable of the operator $S_+(t)$ and is given by

$$\widetilde{S_+}(t) = \alpha_0(t) + \alpha_A(t)A^+(t), \tag{33a}$$

which

$$\alpha_0(t) = \exp[-(\gamma_\perp - i\overline{\omega} + i\Omega)t]\{S_+(0) + \int_0^t \exp[(\gamma_\perp - i\overline{\omega})t']\Gamma_{21,\mu}(t')\mathrm{d}t'\}, \tag{33b}$$

$$\alpha_A(t) = \frac{-i}{\gamma_\perp - i\overline{\omega} + i\Omega}\{S_0 + \int_0^t \exp[(-\gamma_\parallel(t-t')]\Gamma_\mu(t')\mathrm{d}t'\}|f(\omega, \mathbf{r}_\mu)|. \tag{33c}$$

Using Eqs.(31b–31d), Eq.(22), Eq.(12) and Eq.(15), through algebra transformation, we can finally arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}A^+(t) = [-\frac{\gamma}{2} + i(\omega - \Omega)]A^+(t) + [\mu]\widetilde{S_+}(t) + \sum_s[\frac{\gamma}{2} - i(\omega - \omega_s)]g_s A_s^+(0)\exp[i(\omega_s - \Omega)t], \tag{34a}$$

where $[\mu]$ is a calculation sign for the index $\mu$, defined as

$$[\mu] = i\frac{\gamma}{2}\frac{\pi\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\sum_\mu |f(\omega, \boldsymbol{r}_\mu)|, \tag{34b}$$

with $\theta^2 = |\boldsymbol{\theta}_{12}|^2$. Thus we can write Eq.(34a) in the following form:

$$A^+(t) = \int_0^t F_{12}(t,t')[F_3(t') + F_4(t')]\mathrm{d}t' + F_1(t)F_c, \tag{35}$$

where

$$F_1(t) = \exp(ut)\exp[\alpha_{A1}\sum_\mu |f(\omega, \boldsymbol{r}_\mu)|^2 f_{\Gamma_\mu}(t)], \tag{36a}$$

with

$$u = -\frac{\gamma}{2} + i(\omega - \Omega) + \alpha_{A1}\overline{d}_0, \tag{36b}$$

$$\overline{d}_0 = \sum_\mu |f(\omega, \boldsymbol{r}_\mu)|^2 S_0, \tag{36c}$$

$$\alpha_{A1} = \frac{\gamma}{2}\frac{\pi\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\frac{1}{\gamma_\perp - i\overline{\omega} + i\Omega}, \tag{36d}$$

$$f_{\Gamma_\mu}(t) = \int_0^t \mathrm{d}\xi \int_0^\xi \exp[-\gamma_\parallel(\xi - t')]\Gamma_\mu(t')\mathrm{d}t', \tag{36e}$$

and

$$F_{12}(t,t') = \exp[u(t-t')]\exp\{\alpha_{A1} \cdot \sum_\mu |f(\omega, \mathbf{r}_\mu)|^2[f_{\Gamma_\mu}(t) - f_{\Gamma_\mu}(t')]\}, \tag{37}$$

$$F_3(t) = \sum_s[\frac{\gamma}{2} - i(\omega - \omega_s)]g_s A_s^+(0) \exp[i(\omega_s - \Omega)t], \tag{38}$$

$$F_c = \sum_s g_s A_s^+(0), \tag{39}$$

$$F_4(t) = [\mu]\alpha_0(t). \tag{40}$$

In the above, some terms have the following physical meaning:

The letter $u$ represents complex net gain and $f_{\Gamma_\mu}(t)$ is determined by the fluctuation of inversion, hence $F_1(t)$ and $F_{12}(t,t')$ represent the contributions of the gain and the inversion fluctuation coming from the incoherent pumping process and the spontaneous inversion process which is a stimulated emission process induced by the zero-point fluctuation of all modes in the active medium except for the lasing mode.

$F_3(t)$ and $F_c$ represent the contributions of the initial field which come from zero-point fluctuation of all modes.

In $F_4(t)$ term, $S_+(0)$ can be omitted, due to Eq.(29b). So $F_4(t)$ represents the dipole-moment fluctuation which originates from the spontaneous emission process.

If we neglect all fluctuating terms, we can obtain the semiclassical results. Now we let all fluctuating terms remain unchanged and calculate the quantum-mechanical coherence function. We obtain the following results which are shown in Appendix $A$:

$$< A^+(t)A(t') >= (J + J')\frac{1}{\alpha}\exp[u^*(t'-t)]\Big\{1 + 8(\mathrm{Re}\alpha_{A1})^2\sum_\mu |f(\omega, \boldsymbol{r}_\mu)|^2\frac{D_\mu}{\gamma_\parallel^2}\Big[\frac{1}{\alpha} - \frac{1}{\gamma_\parallel} + \frac{\alpha}{(\alpha + \gamma_\parallel)\gamma_\parallel}\Big]\Big\}, \tag{41a}$$

where $t' > t$,

$$J = \left(\frac{\gamma}{2}\right)^2 n(\omega)\frac{\pi\theta^2}{\hbar^2}\overline{\omega}^2\omega K', \tag{41b}$$

$$J' = \left(\frac{\gamma}{2}\right)^2 n(\omega)\left(\frac{\pi\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\right)^2 \sum_{\mu}|f(\omega,\boldsymbol{r}_\mu)|^2 D_{21,\mu}\frac{1}{\beta\beta^*}, \tag{41c}$$

$$\alpha = -(u+u^*) = 2\left(\frac{\gamma}{2} - \mathrm{Re}\alpha_{A1}\overline{d}_0\right), \tag{41d}$$

$$\beta = -\gamma_\perp - u - i(\Omega - \overline{\omega}). \tag{41e}$$

Here $n(\omega)$ is the photon number of initial field belonging to the cavity mode frequency $\omega$. It is easy to see that $J$ represents the contribution of the initial field for the field intensity, $J'$ comes from the spotaneous emitted photons and the term including $D_\mu$ represents the effect of the fluctuation of inversion. In the term $\exp[u^*(t'-t)]$, the absolute value of $u^*$ represents the linewidth of the field. The half of $\alpha$ is the real net gain.

It is worthwhile to emphasize that the $D_\mu$ term usually cannot be ignored for the WGM microsphere, which can be seen by the following order analysis.

For the laser operation below threshold, the real net gain is negative, i.e., $\mathrm{Re}\alpha_{A1}\overline{d}_0 < \frac{\gamma}{2}$, so from Eq.(41a), the $D_\mu$ term is less than $R$, here $R$ is defined by

$$R = 8\frac{\left(\frac{\gamma}{2}\right)^2}{S_0^2}\frac{D_\mu}{\gamma_\parallel^2}\left[\frac{1}{\alpha} - \frac{1}{\gamma_\parallel} + \frac{\alpha}{(\alpha+\gamma_\parallel)\gamma_\parallel}\right]. \tag{41f}$$

For microsphere, it has been shown that in Ref.[27], in weak-coupling limit, i.e., $K \ll \gamma\tau_0$, here $\tau_0$ is the spotaneous lifetime in vacuum, $\gamma_\parallel = K/\tau_0 \ll \gamma$ for the cavity resonance; in strong-couple limit, i.e., $K \gg \gamma\tau_0$, then $\gamma_\parallel = \frac{\gamma}{2}$, essentially independent of detuning. Therefore we know the decay $\gamma_\parallel$ is always less than $\frac{\gamma}{2}$ for the microsphere. From Eq.(29e), Eqs.(21a–21c), Eq.(41d), we see that:

(1) Weak-couple limit ($\gamma_\parallel \ll \frac{\gamma}{2}$, $\alpha \lesssim \frac{\gamma}{2}$),

(a) For strong pumping case ($w_{12} \gg w_{21}$), $\gamma_\parallel \to w_{12}$, $S_0 \to 1$, $D_\mu \to w_{21}$, therefore, $R \sim \frac{\frac{\gamma}{2}}{\gamma_\parallel}\frac{w_{21}}{\gamma_\parallel}$, which is a nondetemined finite value, depending on two ratios $\frac{\frac{\gamma}{2}}{\gamma_\parallel}$ and $\frac{w_{21}}{\gamma_\parallel}$.

(b) For weak pump ($w_{12} > w_{21}$), $S_0 < 1$, $R \sim \frac{\frac{\gamma}{2}}{\gamma_\parallel}\frac{w_{21}}{\gamma_\parallel}\frac{1}{S_0}$, which equals to a larger value (more than unity).

(2) Strong-couple limit ($\gamma_\parallel = \frac{\gamma}{2}$, $\alpha < \frac{\gamma}{2}$),

(a) For strong pump, $R \sim \frac{\frac{\gamma}{2}}{\alpha}\frac{w_{21}}{\gamma_\parallel}$ which is obviously a smaller value (less than unity) depending the ratios $\frac{\frac{\gamma}{2}}{\alpha}$ and $\frac{w_{21}}{\gamma_\parallel}$.

(b) For weak pumping case, $R \sim \frac{\frac{\gamma}{2}}{\alpha}\frac{w_{21}}{\gamma_\parallel}\frac{1}{S_0}$, which is a larger value (more than unity).

From the above analysis, it is shown that only strong couple and strong pumping case, the $D_\mu$ term has a smaller value. However, if $\gamma_\parallel \gg \frac{\gamma}{2}$, then for the above all cases, $R \to 0$. This is why the fluctuation of inversion can be neglected in the usual case $\gamma_\parallel \gg \frac{\gamma}{2}$.

Furthermore, From Eqs.(41b–41d) and Eq.(36d), we see that the intensity of the field increases as the enhancement factor $K$ increases, and the linewidth of field decreases with the $K$ factor. The strong couple (corresponding to larger $K$ value) make the system operate in an amplification region and has a narrow linewidth below threshold.

### B. Outside the microsphere

We define the collective variable of the electric field outside the microsphere similar to that inside the microsphere,

$$B^+(\boldsymbol{r},t) = \sum_s A_s^+(t)|\boldsymbol{e}_s(\boldsymbol{r})|, \qquad r > a. \tag{42a}$$

From Eq.(4b), here $|e_s(\boldsymbol{r})|$ can be written as

$$|\boldsymbol{e}_s(\boldsymbol{r})| = |\beta_E||h(\omega,\boldsymbol{r})|, \tag{42b}$$

and

$$|h(\omega,\boldsymbol{r})| = \left|\frac{q_E}{p_E}h_l^{(1)}(kr) + \frac{1}{p_E}h_l^{(2)}(kr)\right|, \tag{42c}$$

where $\beta_E$, $q_E$ and $p_E$ are given in Eqs.(12), (6) and (7). $B^+(\boldsymbol{r},t)$ represents the slowly varying negative frequency part of the field outside the microsphere.

From Eq.(32), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}B^+(\boldsymbol{r},t) = \left[-\frac{\gamma}{2} + i(\omega-\Omega)\right][\mu']\int_0^t \exp\left\{\left[i(\omega - \Omega) - \frac{\gamma}{2}\right](t-t')\right\}\widetilde{S_+}(t')\mathrm{d}t' + [\mu']\widetilde{S_+}(t) + i\sum_s A_s^+(0)|e_s(\boldsymbol{r})|(\omega_s - \Omega)\exp[i(\omega_s-\Omega)t], \tag{43}$$

where $[\mu']$ is a calculation sign, defined by

$$[\mu'] = -\frac{\gamma}{2}\frac{\pi|\boldsymbol{\theta}_{21}|}{\sqrt{2}\hbar}\overline{\omega}\omega^2 K'|h(\omega,\boldsymbol{r})|\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|, \tag{44}$$

and $\widetilde{S_+}(t)$ has been given in Eqs.(33a–33c).

By algebra transformation, we have the basic equation of the external field:

$$\frac{\mathrm{d}}{\mathrm{d}t}B^+(\boldsymbol{r},t)=\left[-\frac{\gamma}{2}+i(\omega-\Omega)\right]B^+(\boldsymbol{r},t)+[\mu']\widetilde{S_+}(t)+\sum_s\left[\frac{\gamma}{2}-i(\omega-\omega_s)\right]A_s^+(0)|\boldsymbol{e}_s(\boldsymbol{r})|\exp[i(\omega_s-\Omega)t]. \tag{45}$$

Such $B^+(\boldsymbol{r},t)$ can be expressed in the following form:

$$B^+=\int_0^t F'_{12}(t-t_1)[F'_3(t_1)+F'_4(t_1)+F'_5(t_1)]\mathrm{d}t_1+F'_1(t)F'_c, \tag{46}$$

where

$$F'_1(t)=\exp(u_1t), \tag{47a}$$

with

$$u_1=-\frac{\gamma}{2}+i(\omega-\Omega), \tag{47b}$$

and

$$F'_{12}(t-t_1)=\exp[u_1(t-t_1)], \tag{47c}$$

$$F'_3(t_1)=\sum_s\left[\frac{\gamma}{2}-i(\omega-\omega_s)\right]A_s^+(0)|\boldsymbol{e}_s(\boldsymbol{r})|\exp[i(\omega_s-\Omega)t_1], \tag{48}$$

$$F'_4(t_1)=[\mu']\alpha_0(t_1), \tag{49}$$

$$F'_5(t_1)=[\mu']\alpha_A(t_1)A^+(t_1), \tag{50}$$

$$F'_c=\sum_s A_s^+(0)|\boldsymbol{e}_s(\boldsymbol{r})|. \tag{51}$$

Here the physical meanings of $F'_3$ and $F'_4$ are the same as $F_3$ and $F_4$ in Part A. But $F_1$ and $F_{12}$ now repesent the free oscillation of the field, and $F'_5$ gives the contribution of the gain, inversion and total internal field.

By a lengthy calculation (the main derivation process is demonstrated in Appendix B), we finally obtain the coherence function $< B^+(t)B(t') >$ of the external field:

$$< B^+(t)B(t') >=\{N_1(1+D_3+\frac{D_4}{S_0^2})+N_2(D_1-D_3+\frac{D_2}{S_0}-\frac{D_4}{S_0^2})\}\exp[u^*(t'-t)], \tag{52}$$

where $t'>t$, and

$$N_1=\frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2(J+J')\frac{1}{\alpha}, \tag{53a}$$

$$N_2=\frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2(J+J')\frac{1}{\beta_3}, \tag{53b}$$

with

$$\beta_3=-u^*-u_1=\gamma-\alpha^*_{A1}\overline{d}_0, \tag{53c}$$

and

$$D_1=2\alpha^{*2}_{A1}\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^4\frac{D_\mu}{\gamma_\parallel^2}\left[-\frac{1}{\gamma_\parallel}+\frac{1}{\gamma_\parallel}\frac{\beta_2\beta_3}{(\beta_2-\gamma_\parallel)(\beta_3+\gamma_\parallel)}+\left(\frac{1}{\beta_3}-\frac{1}{\beta_2}\right)\right], \tag{54a}$$

$$D_2=2\alpha^*_{A1}a_1\frac{D_\mu}{\gamma_\parallel^2}\left[1-\frac{\beta_2\beta_3}{(\beta_2-\gamma_\parallel)(\beta_3+\gamma_\parallel)}\right], \tag{54b}$$

$$D_3=8(\mathrm{Re}\alpha_{A1})^2\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^2\frac{D_\mu}{\gamma_\parallel^2}\left[\frac{1}{\alpha}-\frac{1}{\gamma_\parallel}+\frac{\alpha}{(\alpha+\gamma_\parallel)\gamma_\parallel}\right], \tag{54c}$$

$$D_4=2\alpha_{A1}a_1\frac{D_\mu}{\gamma_\parallel^2}\left[\left(1-\frac{\alpha}{\alpha+\gamma_\parallel}\right)\left(1+\frac{\beta_2\beta_3}{(\beta_2-\gamma_\parallel)(\beta_3+\gamma_\parallel)}\right)\right]+2\alpha^*_{A1}a_1\frac{D_\mu}{\gamma_\parallel^2}\left[1+\frac{\alpha}{\alpha-\gamma_\parallel}-\left(1+\frac{\alpha}{\alpha+\gamma_\parallel}\right)\frac{\beta_2\beta_3}{(\beta_2-\gamma_\parallel)(\beta_3+\gamma_\parallel)}\right]+2a_1\frac{1}{\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^2}\frac{D_\mu}{\gamma_\parallel}\frac{\beta_2\beta_3}{(\beta_2-\gamma_\parallel)(\beta_3+\gamma_\parallel)}, \tag{54d}$$

with

$$\beta_2=-u_1^*+u^*=\alpha^*_{A1}\overline{d}_0,\qquad a_1=\frac{\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^4}{\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^2} \tag{54e}$$

In the above, the meanings of $J$, $J'$ and $u^*$ have been stated in Part A. $D_1$, $D_2$, $D_3$ and $D_4$ stand for the effect of the inversion fluctuation. If $D_1$, $D_2$, $D_3$ and $D_4$ are neglected, $< B^+(t)B(t') >$ has a simple form from Eqs.(52) and (41a):

$$< B^+(t)B(t') >=\frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2< A^+(t)A(t') >, \tag{55}$$

which is the relation between the internal and external field for the usual laser case and $\frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2$ is equivalent to the transmission coefficient. From Eq.(42c)

and Eqs.(6)–(7), we find $|h(\omega,\boldsymbol{r})|^2 \sim \frac{1}{K}$, i.e., large $K$ value will decrease the transmission of laser. Moreover, for the microsphere laser, $D_1$, $D_2$, $D_3$ and $D_4$ usually should be considered even though the effect of inversion fluctuation is smaller for strong couple and strong pumping (see the order analysis in part A). Especially for nondetuning case, considering $\alpha^*_{A1} \approx 0$ and $\beta_2 \approx 0$ due to $\overline{\omega} - \Omega \approx 0$ in Eq.(36d), we find $D_1 \approx 0$, $D_2 \approx 0$, and

$$D_4 \approx 2\alpha_{A1}a_1\frac{D_\mu}{\gamma_\parallel^2}\left(1-\frac{\alpha}{\alpha+\gamma_\parallel}\right). \tag{56}$$

Similar to the order analysis in part A, we see that $D_4 \approx 0$ for weak couple due to $\gamma_\parallel \ll \frac{\gamma}{2}$ and $\left(1-\frac{\alpha}{\alpha+\gamma_\parallel}\right) \approx 0$, and also $D_4 \approx 0$ for strong couple and strong pumping since $D_4 \sim \frac{w_{21}}{\gamma_\parallel} \ll 1$. Therefore, for nondetuning and strong couple and strong pumping we obtain

$$< B^+(t)B(t') >= \{N_1(1+D_3\}\exp[u^*(t'-t)], \tag{57}$$

which is the result corresponding to the minimum of inversion fluctuation effect.

## V. NONLINEAR ANALYSIS

For the laser operation above threshold, we should make a nonlinear analysis . In this section, we only investigate the quantum statistical properties of laser inside the microsphere, including the coherence functions of amplitude fluctuation of the field and the fluctuation of photon number of laser and the coherence functions of the phase fluctuation of the field and the linewidth of laser. As for the action outside the microsphere, similar analysis can be done except for using the basic equation of external field.

The Langevin equations Eqs.(16), (17) and (18) can be solved by the quasilinearization procedure[28,29]. In the approximation of a large photon number, the internal field operator $A^+$ and the inversion operator $S$ are expressed as

$$A^+ = (A_0+\Delta A)\exp(-i\Delta\Phi), \tag{58}$$

$$S = \overline{S} + \Delta S. \tag{59}$$

Here $\Delta A$, $\Delta\Phi$ and $\Delta S$ are Hermitian amplitude, phase and inversion fluctuating operators respectively. $A_0$ and $\overline{S}$ are respectively the average field amplitude and inversion (positive real).

From Eqs.(16)–(18) and (58)–(59), noting only the first-order fluctuation retained, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta A = -\frac{\gamma}{2}\Delta A + \mathrm{Re}[\mu]\alpha'_A\overline{S}\Delta A + \mathrm{Re}[\mu]\alpha'_A A_0\Delta S + F_r, \tag{60}$$

$$-A_0\frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi = (\omega-\Omega)\Delta A + \mathrm{Im}[\mu]\alpha'_A[\overline{S}\Delta A + A_0\Delta S + F_i, \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta S = -\gamma_\parallel\Delta S + 2\mathrm{Re}\alpha''_A[2\overline{S}\Delta A + A_0\Delta S] + \Gamma, \tag{62}$$

and

$$0 = -\frac{\gamma}{2} + \mathrm{Re}[\mu]\alpha'_A\overline{S}, \tag{63}$$

$$0 = (\omega-\Omega) + \mathrm{Im}[\mu]\alpha'_A\overline{S}, \tag{64}$$

$$0 = \gamma_\parallel(S_0-\overline{S}) - \frac{4\gamma_\perp}{\gamma_\perp^2+(\Omega-\overline{\omega})^2}|f(\omega,\boldsymbol{r}_\mu)|^2\overline{S}A_0^2, \tag{65}$$

where

$$\alpha'_A = -i\frac{1}{\gamma_\perp - i\overline{\omega}+i\Omega}|f(\omega,\boldsymbol{r}_\mu)|, \tag{66a}$$

$$\alpha''_A = -\frac{2}{\gamma_\perp - i\overline{\omega}+i\Omega}|f(\omega,\boldsymbol{r}_\mu)|^2A_0, \tag{66b}$$

$$F_r(t) = \mathrm{Re}[(F_3+F_4)\exp(i\Delta\Phi)],$$

$$F_i(t) = \mathrm{Im}[(F_3+F_4)\exp(i\Delta\Phi)], \tag{66c}$$

$$\Gamma = \Gamma_{22,\mu} - \Gamma_{11,\mu}. \tag{66d}$$

Equations (60)–(62) are the basic equations of the fluctuation of the field and Eqs.(63–65) give the average values of photon and inversion number.

By Fourier analysis of Eqs.(60)–(62), we have

$$\Delta A(\Omega) = \alpha_3\Gamma(\Omega) + \alpha_4F_r(\Omega), \tag{67}$$

$$-A_0i\Omega\Delta\Phi(\Omega) = \mathrm{Im}[\mu]\alpha'_AA_0\{(\alpha_1\alpha_{3+}\alpha_2)\Gamma(\Omega) + \alpha_1\alpha_4F_r(\Omega)\} + F_i(\Omega), \tag{68}$$

where

$$\alpha_1 = \frac{4\mathrm{Re}\alpha''_A\overline{S}}{i\Omega+\gamma_\parallel-2\mathrm{Re}\alpha''_AA_0}, \tag{69a}$$

$$\alpha_2 = \frac{1}{i\Omega+\gamma_\parallel-2\mathrm{Re}\alpha''_AA_0}, \tag{69b}$$

$$\alpha_3 = \frac{\mathrm{Re}[\mu]\alpha'_AA_0\alpha_2}{i\Omega-\mathrm{Re}[\mu]\alpha'_AA_0\alpha_1}, \tag{69c}$$

$$\alpha_4 = \frac{1}{i\Omega-\mathrm{Re}[\mu]\alpha'_AA_0\alpha_1}. \tag{69d}$$

In Appendix C, we derive the amplitude noise spectrum $< \Delta A^+(\Omega)\Delta A(\Omega) >$ and the relative photon number variance $< (\Delta n)^2 > / < n >$ of the laser

field.

$$< \Delta A^+(\Omega)\Delta A(\Omega) >= 4\alpha_3\alpha_3^* D_\mu + \frac{1}{4}\alpha_4\alpha_4^* J_0 + \frac{1}{2}\alpha_4\alpha_4^*[\mu][\mu]^* \frac{\gamma_\perp}{\gamma_\perp^2 + \Omega^2}, \quad (70)$$

$$\frac{< (\Delta n)^2 >}{< n >} = \eta_1 D_\mu + \eta_2 J_0 + \eta_3 J_0', \quad (71a)$$

with

$$J_0 = \left(\frac{\gamma}{2}\right)^2 [1 + 2n(\omega)] \frac{\pi\theta^2}{\hbar^2}\overline{\omega}^2 \omega K',$$

$$J_0' = 2[\mu][\mu]^* \frac{1}{\gamma_\perp}, \quad (71b)$$

$$\eta_1 = 8\frac{b^2}{b_1^2}\frac{\sqrt{\rho}}{\rho e_2}\cos\frac{\phi}{2}, \quad (71c)$$

$$\eta_2 = \frac{1}{2}(\rho + b_0^2)\frac{\sqrt{\rho}}{\rho e_2}\cos\frac{\phi}{2}, \quad (71d)$$

$$\eta_3 = \frac{1}{2}\frac{\frac{\sqrt{\rho}}{\rho e_2}[\cos\frac{\phi}{2}(\rho^2 + \rho\gamma_\perp^2 + b_0^2\gamma_\perp^2) + 2b_0^2\rho\cos\frac{3}{2}\phi] + \frac{b_0^2 - \gamma_\perp^2}{\gamma_\perp}}{\frac{\rho^2}{\gamma_\perp^2} + 2e_1 + \gamma_\perp^2}, \quad (71e)$$

$$b_0 = 2\gamma_\parallel - \gamma_\parallel\frac{\overline{S}}{S_0}, \quad (71f)$$

$$b = \frac{1}{\overline{S}}\gamma\gamma_\parallel(S_0 - \overline{S}), \quad (71g)$$

$$b_1 = \frac{2}{A_0}\gamma_\parallel(S_0 - \overline{S}), \quad (71h)$$

and

$$e_1 = \frac{1}{2}(2b - b_0^2), \quad (71i)$$

$$e_2 = b_0\sqrt{4b - b_0^2}, \quad (71j)$$

$$\rho = \sqrt{e_1^2 + e_2^2}, \qquad \phi = \arctan\frac{e_2}{e_1}. \quad (71k)$$

The meanings of the terms on the right hand side of Eqs.(70), (71a) are obvious, which come from the inversion fluctuation, the initial field and the spontaneous emission, respectively.

In the above, we have assumed that, at $t$=0, the gain is much larger than the loss, i.e. $\mathrm{Re}\alpha_A' S_0 \gg \frac{\gamma}{2}$, which means $S_0 \gg \overline{S}$ from Eq.(63). Hence $b_0 > 0$, $b > 0$, and $2b - b_0^2 > 0$ due to $\gamma_\parallel \leq \frac{\gamma}{2}$ for the microsphere.

We make an analysis for the three terms on the right side of Eq.(71a). For doing this, we first get the expression of the average value $A_0$ and $\overline{S}$ from Eqs.(63)–(65):

$$A_0^2 = \frac{1}{4}\frac{\pi\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\gamma_\parallel N S_0, \quad (72)$$

$$\overline{S} = \frac{\gamma_\perp}{\frac{\pi\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\sum_\mu |f(\omega, \boldsymbol{r}_\mu)|^2}. \quad (73)$$

Here $N$ is the total number of active atoms. $A_0^2$ represents the average photon number, which increases with the $K$ factor, and $\overline{S}$ is the average population inversion, i.e., the inversion at the threshold, which decreases with the $K$ factor. These results agree with those in the semiclassical treatment for microsphere laser.

Due to $S_0 \gg \overline{S}$, $b \gg b_0^2$, $e_1 \approx b$, $e_2 \approx 2b_0\sqrt{b}$, $\rho \approx b$, $\phi \approx 0$, from Eq.(71f–71h), Eq.(29e) and Eqs.(72–73), the first term of Eq.(71a) can be expressed as

$$\eta_1 D_\mu = A_0^2\frac{\gamma}{\gamma_\parallel}\frac{w_{21}}{\gamma_\parallel}\frac{w_{12}}{w_{12} + w_{21}}\frac{1}{S_0}\frac{1}{\overline{S}}, \quad (74)$$

which indicates $\eta_1 D_\mu \propto \frac{1}{\overline{S}^2} \propto K^2$; for strong couple ($\gamma_\parallel = \frac{\gamma}{2}$) and strong pumping ($\gamma_\parallel \gg w_{21}$), $\eta_1 D_\mu$ approaches a smaller value. Similarly, we get $\eta_2 \approx \frac{1}{8\gamma_\parallel}$, and

$$\eta_2 J_0 = A_0^2\left(\frac{\gamma}{2}\right)^2\frac{1}{\gamma_\parallel^2}\frac{1}{N}\frac{1}{S_0}[1 + n(\omega)], \quad (75)$$

which implies $\eta_2 J_0 \propto \frac{1}{\overline{S}} \propto K$, we also have $\eta_3 \approx \frac{1}{8\gamma_\parallel}$, and

$$\eta_3 J_0' = \frac{1}{1 + 2n(\omega)}\frac{1}{\overline{S}}(\eta_2 J_0), \quad (76)$$

therefore $\eta_3 J_0' \propto \frac{1}{\overline{S}^2} \propto K^2$.

On the other hand, we obtain the noise power spectrum of the field phase for nondetuning case, from Eq.(68) and (64),

$$\Omega^2 < \Delta\Phi^+(\Omega)\Delta\Phi(\Omega) >= \frac{1}{A_0^2} < F_i^+(\Omega)F_i(\Omega) > . \quad (77)$$

Furthermore we get

$$< \frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi^+(t)\frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi(t') >= \frac{1}{4A_0^2}J_0\delta(t - t') + \frac{1}{4A_0^2}[\mu][\mu]^*\exp[-\gamma_\perp|t - t'|], \quad (78)$$

and the linewidth of the laser

$$\Delta\Omega = \frac{1}{4A_0^2}(J_0 + J_0'). \quad (79)$$

Equation (79) shows that the linewidth come from the contribution of initial field and spotaneous emission, the effect of the inversion fluctuation can be neglected for nondetuning case. $\Delta\Omega \propto \frac{1}{A_0^2}$, i.e., the linewidth decreases with the average photon number. Equation (79) can be expressed in the following form:

$$\Delta\Omega = \frac{2(\frac{\gamma}{2})^2[1 + n(\omega)]}{\gamma_\parallel N S_0}\left\{1 + \frac{1}{1 + 2n(\omega)}\frac{1}{\overline{S}}\right\}. \quad (80)$$

In this formula, the second term is usually much larger than the first term, so $\Delta\Omega \propto \frac{1}{S} \propto K$. For strong couple ($\gamma_\parallel = \frac{\gamma}{2}$) and strong pumping ($S_0 = 1$), if $N\overline{S} \gg 1$, then $\Delta\Omega \ll \frac{\gamma}{2}$, i.e., the linewidth of the laser is much less than the cavity width.

## VI. CONCLUSION

In summary, we have developed the quantum Langevin theory of whispering-gallery-mode microsphere laser, and also made the linear analysis for laser operation below threshold. In this analysis, we have not neglected the inversion fluctuation term and calculated the coherence functions of the internal and external fields. We have discussed the dependence of intensity and the linewidth of laser on the enhancement factor of the microsphere and the pumping level. It is shown that the strong couple and strong pumping are useful for the amplification of intensity and decrease of linewidth below threshold. We have also made a nonlinear analysis for the laser action above threshold inside the microsphere. We have calculated the amplitude and phase noise power spectrum, and derived the expression of the relative photon number variance and the linewidth of laser. We have made a detailed discussion on the relation between these results and the enhancement factor. It is also shown that, for the laser action above threshold, the variances of photon number and linewidth of the internal field are related to the enhancement factor and the square of the enhancement factor respectively. The general equation and specific results presented in this paper may be useful for some theoretical and experimental problem of WGM microsphere laser.

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## Appendix A

From Eq.(35), considering that

$$< F_3(t_1)F_4^+(t_2) >=< F_4(t_1)F_3^+(t_2) >=< F_4(t_1)F_c >=< F_cF_4^+(t_2) >= 0, \tag{A.1}$$

we have

$$\begin{aligned} < A^+(t)A(t') > &= \int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F_{12}(t,t_1)F_{12}^+(t',t_2) > [< F_3(t_1)F_3^+(t_2) > \\ &+ < F_4(t_1)F_4^+(t_2) >] + \int_0^t \mathrm{d}t_1 < F_{12}(t,t_1)F_1^+(t') >< F_3(t_1)F_c^+ > \\ &+ \int_0^{t'} \mathrm{d}t_2 < F_1(t)F_{12}^+(t',t_2) >< F_cF_3^+(t_2) > + < F_1(t)F_1^+(t') >< F_cF_c^+ >, \end{aligned} \tag{A.2}$$

From Eq.(37), it follows

$$\begin{aligned} &< F_{12}(t,t_1)F_{12}^+(t',t_2) >= \exp[u(t-t_1)+u^*(t'-t_2)] \\ &\times < \exp\left\{\alpha_{A1}\sum_\mu |f(\omega,\boldsymbol{r}_\mu)|^2[f_{\Gamma_\mu}(t) - f_{\Gamma_\mu}(t_1)] + \alpha_{A1}^*\sum_\mu |f(\omega,\boldsymbol{r}_\mu)|^2[f_{\Gamma_\mu}(t') - f_{\Gamma_\mu}(t_2)] > \right\} \\ &\approx \exp[u(t-t_1)+u^*(t'-t_2)]\left\{1 + \frac{1}{2} < \left(\alpha_{A1}\sum_\mu |f(\omega,\boldsymbol{r}_\mu)|^2[f_{\Gamma_\mu}(t) - f_{\Gamma_\mu}(t_1)]\right.\right. \\ &\left.\left. + \alpha_{A1}^*\sum_\mu |f(\omega,\boldsymbol{r}_\mu)|^2[f_{\Gamma_\mu}(t') - f_{\Gamma_\mu}(t_2)]\right)^2 > \right\}. \end{aligned} \tag{A.3}$$

Here the exponential term containing the inversion fluctuation is expanded to quadratic term, which is valid for the strong couple and strong pumping case (see the order analysis in Sec.IV).

From Eq.(36e) and Eq.(23), by integration, we get

$$< f_{\Gamma_\mu}(t)f_{\Gamma_\mu}(t) >= \frac{4D_\mu}{\gamma_\parallel^2}\left\{\frac{1}{2\gamma_\parallel}[1-\exp(-2\gamma_\parallel t)] - \frac{2}{\gamma_\parallel}[1-\exp(-\gamma_\parallel t)] + t\right\}, \tag{A.4}$$

$$<f_{\Gamma_\mu}(t_1)f_{\Gamma_\mu}(t_1)>=\frac{4D_\mu}{\gamma_\parallel^2}\left\{\frac{1}{2\gamma_\parallel}[1-\exp(-2\gamma_\parallel t_1)]-\frac{2}{\gamma_\parallel}[1-\exp(-\gamma_\parallel t_1)]+t_1\right\}, \tag{A.5}$$

$$<f_{\Gamma_\mu}(t)f_{\Gamma_\mu}(t_1)>=\frac{4D_\mu}{\gamma_\parallel^2}\left\{\frac{1}{2\gamma_\parallel}[\exp(-\gamma_\parallel(t-t_1))-\exp(-\gamma_\parallel(t+t_1))] \tag{A.6}$$

$$-\frac{1}{\gamma_\parallel}[1-\exp(-\gamma_\parallel t_1)]-\frac{1}{\gamma_\parallel}[\exp(-\gamma_\parallel(t-t_1))-\exp(-\gamma_\parallel t)]+t_1\right\}$$

$$<f_{\Gamma_\mu}(t_1)f_{\Gamma_\mu}(t)>=<f_{\Gamma_\mu}(t)f_{\Gamma_\mu}(t_1)>. \tag{A.7}$$

Equations (38), (31c), (22), (12), (15) give the following relation:

$$<F_3(t_1)F_3^+(t_2)>=\left(\frac{\gamma}{2}\right)^2 n(\omega)\frac{\pi\theta^2}{\hbar^2}\overline{\omega}^2\omega K'\delta(t_1-t_2)=J\delta(t_1-t_2). \tag{A.8}$$

Thus we finally obtain

$$\int_0^t \mathrm{d}t_1\int_0^{t'}\mathrm{d}t_2<F_{12}(t,t_1)F_{12}^+(t',t_2)><F_3(t_1)F_3^+(t_2)>$$

$$=J\frac{1}{\alpha}\exp[u^*(t'-t)]\left\{1+8(\mathrm{Re}\alpha_{A1})^2\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^2\frac{D_\mu}{\gamma_\parallel^2}\left[\frac{1}{\alpha}-\frac{1}{\gamma_\parallel}+\frac{\alpha}{(\alpha+\gamma_\parallel)\gamma_\parallel}\right]\right\}. \tag{A.9}$$

In the above we have neglected terms that vanish for large $t$ or $t^{'}$, such as $\exp(u^*t)$. In the following we also do this. From Eqs.(40) and (33b), we get

$$<F_4(t_1)F_4^+(t_2)>=[\mu][\mu]^*\exp[i\Omega(t_2-t_1)]\int_0^{t_1}\mathrm{d}T'\int_0^{t_2}\mathrm{d}T''\exp[-(\gamma_\perp-i\overline{\omega})(t_1-T')]$$

$$\times\exp[-(\gamma_\perp+i\overline{\omega})(t_2-T'')]<\Gamma_{21}(T')\Gamma_{12}(T'')>. \tag{A.10}$$

By integration, we have

$$\int_0^t \mathrm{d}t_1\int_0^{t'}\mathrm{d}t_2<F_{12}(t,t_1)F_{12}^+(t',t_2)>[<F_4(t_1)F_4^+(t_2)>=J'\frac{1}{\alpha}\exp[u^*(t'-t)]\left\{1+8(\mathrm{Re}\alpha_{A1})^2\right.$$

$$\left.\times\sum_\mu|f(\omega,\boldsymbol{r}_\mu)|^2\frac{D_\mu}{\gamma_\parallel^2}\left[\frac{1}{\alpha}-\frac{1}{\gamma_\parallel}+\frac{\alpha}{(\alpha+\gamma_\parallel)\gamma_\parallel}\right]\right\}. \tag{A.11}$$

From Eq.(36a), we easy obtain

$$<F_1(t)F_1^+(t')>\propto\exp(ut)\exp(u^*t')\approx 0. \tag{A.12}$$

Equations (38), (39), (31c), (22), (12) and (15) lead to

$$<F_3(t_1)F_c^+>=\sum_s\left[\frac{\gamma}{2}-i(\omega-\omega_s)\right]g_sg_s^*n_s\exp[i(\omega_s-\Omega)t_1],$$

$$=\left(\frac{\gamma}{2}\right)^2 n(\omega)\frac{\theta^2}{2\hbar^2}\overline{\omega}^2\omega K'\int_{-\infty}^{\infty}\frac{\exp(i\xi t_1)}{\frac{\gamma}{2}-i\xi}\mathrm{d}\xi\exp[i(\omega-\Omega)t_1]=0. \tag{A.13}$$

Similarly, we have

$$<F_cF_3^+(t_2)>=0. \tag{A.14}$$

Finally we obtain Eq.(41a).

## Appendix B

In the following derivation process, we neglect all terms that vanish for large $t$ or $t'$, and all terms that do not represent the lasing ocsillation.

Considering the zero terms similar to Eq.(A1) and Eqs.(A12)–(A14), we obtain from Eq.(46)

$$\begin{aligned} < B^+(t)B(t') > = & \int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) > [< F'_3(t_1)F'^+_3(t_2) > \\ & + < F'_4(t_1)F'^+_4(t_2) > + < F'_3(t_1)F'^+_5(t_2) > + < F'_4(t_1)F'^+_5(t_2) > \\ & + < F'_5(t_1)F'^+_3(t_2) > + < F'_5(t_1)F'^+_4(t_2) >], \end{aligned} \tag{B.1}$$

From Eqs.(48), (42b) and (12), we get

$$< F'_3(t_1)F'^+_3(t_2) > = J_1\delta(t_1-t_2). \tag{B.2}$$

with

$$J_1 = 2\left(\frac{\gamma}{2}\right)^2 \pi n(\omega)\omega^3 K'|h(\omega,\boldsymbol{r})|^2. \tag{B.3}$$

Thus

$$\int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) >< F'_3(t_1)F'^+_3(t_2) > = \frac{J_1}{\alpha_1}\exp[u_1^*(t'-t)], \tag{B.4}$$

with

$$\alpha_1 = -(u_1+u_1^*). \tag{B.5}$$

Due to Eqs.(49) and (33b), we have

$$\int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) >< F'_4(t_1)F'^+_4(t_2) > = \frac{J'_1}{\alpha_1}\exp[u_1^*(t'-t)], \tag{B.6}$$

with

$$J'_1 = [\mu'][\mu']^* D_{21,\mu}\frac{1}{\beta_1\beta_1^*}, \tag{B.7}$$

$$\beta_1 = -u_1 - \gamma_\perp - i(\Omega - \overline{\omega}). \tag{B.8}$$

From Eqs.(50), (35), (33c), (49), (33b), we obtain

$$\begin{aligned} \int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 & < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) >< F'_3(t_1)F'^+_5(t_2) > \\ & = Q[S_0(1+D_1)+D_2]\exp[u^*(t'-t)], \end{aligned} \tag{B.9}$$

with

$$Q = J_2[\mu'']^*\frac{1}{\beta_2\beta_3}, \tag{B.10}$$

$$[\mu''] = [\mu']\frac{-i}{\gamma_\perp - i\overline{\omega} + i\Omega}|f(\omega,\boldsymbol{r}_\mu)|, \tag{B.11}$$

But we see that

$$\int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) >< F'_5(t_1)F'^+_3(t_2) > \propto \exp[u_1^*(t'-t)]. \tag{B.12}$$

From Eqs.(50), (35), (33c), (49) and (33b), we also obtain

$$\begin{aligned} \int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 & < F'_{12}(t-t_1)F'^+_{12}(t'-t_2) >< F'_4(t_1)F'^+_5(t_2) > \\ & = Q'[S_0(1+D_1)+D_2]\exp[u^*(t'-t)], \end{aligned} \tag{B.13}$$

with

$$Q' = [\mu'']^*[\mu'][\mu]^* D_{21,\mu}\frac{1}{\beta_1\beta_2\beta_3\beta^*}. \tag{B.14}$$

However we find

$$\int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^{+}_{12}(t'-t_2) >< F'_5(t_1)F'^{+}_4(t_2) > \propto \exp[u_1^*(t'-t)]. \tag{B.15}$$

From Eqs.(50), (35) and (33c), we can derive

$$\begin{aligned}\int_0^t \mathrm{d}t_1 \int_0^{t'} \mathrm{d}t_2 < F'_{12}(t-t_1)F'^{+}_{12}(t'-t_2) >< F'_5(t_1)F'^{+}_5(t_2) > \\ = (Q_1+Q'_1)[S_0^2(1+D_3)+D_4]\exp[u^*(t'-t)],\end{aligned} \tag{B.16}$$

with

$$Q_1 = J[\mu'][\mu']^* \frac{1}{\beta_2\beta_2^*}\left[\frac{1}{\alpha}-\frac{1}{\alpha+\beta_2^*}\right], \tag{B.17}$$

$$Q'_1 = [\mu][\mu]^*[\mu'][\mu']^* D_{21,\mu}\frac{1}{\beta\beta^*\beta_2\beta_2^*}\left[\frac{1}{\alpha}-\frac{1}{\alpha+\beta_2^*}\right]. \tag{B.18}$$

We express $Q$, $Q'$, $Q_1$, $Q'_1$ by using $J$ and $J'$ :

$$QS_0 = \frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2\frac{1}{\beta_3}J, \tag{B.19}$$

$$QS_0 = \frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2\frac{1}{\beta_3}J', \tag{B.20}$$

$$Q_1S_0^2 = \frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2\left(1-\frac{\alpha}{\beta_3}\right)\frac{J}{\alpha}, \tag{B.21}$$

$$Q'_1S_0^2 = \frac{2\hbar^2}{\theta^2}|h(\omega,\boldsymbol{r})|^2\left(1-\frac{\alpha}{\beta_3}\right)\frac{J'}{\alpha}, \tag{B.22}$$

Finally, from Eq.(B1) we get Eq.(52).

## Appendix C

We adopt Fourier transform in the following form

$$f(\Omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\exp(-i\Omega t)\mathrm{d}t, \tag{C.1}$$

so that

$$\begin{aligned}< \Gamma_{22}^+(\Omega)\Gamma_{22}(\Omega') > &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}t'' < \Gamma_{22}^+(t')\Gamma_{22}(t'') > \exp(i\Omega t')\exp(-i\Omega' t'') \\ &= D_\mu\delta(\Omega-\Omega'),\end{aligned} \tag{C.2}$$

and

$$< \Gamma^+(\Omega)\Gamma(\Omega) >= 4D_\mu. \tag{C.3}$$

To calculate $< F_r^+(\Omega)F_r(\Omega') >$, we consider

$$< F_r^+(t')F_r(t'') >= \frac{1}{4}\{< F_3(t')F_3^+(t'') > + < F_3^+(t')F_3(t'') > + < F_4(t')F_4^+(t'') > + < F_4^+(t')F_4(t'') >\}. \tag{C.4}$$

From Eqs.(A8) and (30b), (30c), we have

$$< F_3(t')F_3^+(t'') > + < F_3^+(t')F_3(t'') >= J_0\delta(t'-t''). \tag{C.5}$$

Again from Eq(A10), and by integration, we get

$$\begin{aligned}< F_r^+(\Omega)F_r(\Omega') > &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}t'' < F_r^+(t')F_r(t'') > \exp(i\Omega t')\exp(-i\Omega' t'') \\ &= \frac{1}{4}J_0\delta(\Omega-\Omega') + \frac{1}{2}[\mu][\mu]^*\frac{\gamma_\perp}{\gamma_\perp^2+\Omega^2}\delta(\Omega-\Omega').\end{aligned} \tag{C.6}$$

It is easily shown that

$$< F_i^+(\Omega)F_i(\Omega) >=< F_r^+(\Omega)F_r(\Omega) >, \tag{C.7}$$

thus we get Eq. (70).

From Fourier transform of $\Delta A^+(\Omega)$ and $\Delta A(\Omega)$, we have

$$< \Delta A^+(t)\Delta A(t) >= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\Omega < \Delta A^+(\Omega)\Delta A(\Omega) > . \tag{C.8}$$

Considering

$$< (\Delta n)^2 >=< [(A_0 + \Delta A)^2 - A_0^2]^2 >\approx 4 < n >< \Delta A^2 >, \tag{C.9}$$

we can express the relative photon number variance as

$$\frac{< (\Delta n)^2 >}{< n >} = 4 < (\Delta A)^2 > . \tag{C.10}$$

Using Eqs.(70), (C8) and (C10), by integration, we obtain Eqs.(71a–71k).

From Eqs.(77) and (C6), and

$$< \frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi^+(t)\frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi(t') >= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\Omega\{\Omega^2 < \Delta\Phi^+(\Omega)\Delta\Phi(\Omega) > \exp[-i\Omega(t-t')] >, \tag{C.11}$$

it is easy to get Eq.(78).

The width of laser can be calculated by

$$\Delta\Omega = \frac{< [\Delta\Phi(\tau) - \Delta\Phi(0)]^2 >}{\tau}, \tag{C.12}$$

where

$$\Delta\Phi(\tau) - \Delta\Phi(0) = \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi(t)\mathrm{d}t. \tag{C.13}$$

Thus we obtain

$$< [\Delta\Phi(\tau) - \Delta\Phi(0)]^2 >= \int_0^\tau \mathrm{d}t' \int_0^\tau \mathrm{d}t'' < \frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi^+(t')\frac{\mathrm{d}}{\mathrm{d}t}\Delta\Phi(t'') > . \tag{C.14}$$

Substituting Eq.(78) into Eq(C14), by integration, we obtain Eq.(79).

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