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RESONANT INTERACTION BETWEEN A PAUL-TRAPPED ION AND A STANDING WAVE LASER*

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An ultracold two-level ion experiencing the standing wave of a resonant laser in a Paul trap is investigated in the Lamb-Dicke limit and weak excitation regime, with full consideration of the time-dependence of the trapping potential. The analytical forms of the wave functions of the system can be described with our approach, and the time evolution of the pseudo-energy of the system as well as the squeezing property of the quadrature components is studied in comparison with the treatment of harmonic oscillator model.

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I. INTRODUCTION

Nonclassical characteristics of a trapped ultracold ion has attracted considerable interest during the last few years, especially in the field of quantum optics and precision spectroscopy. With sophisticated laser technique, a trapped ion in a Paul trap has been cooled into the motional ground state[1], and various nonclassical motional states, such as Fock, coherent, squeezed and Schrödinger-cat states, etc., have been successfully generated.[2] Particularly, a two-qubit controlled-NOT gate with a $^9Be^+$ ion was recently implemented in the Pual trap[3], which has been regarded as the first step of the long way to the achievement of an actual quantum computation in the near future.

As the model of an ultracold ion strongly coupled to a harmonic oscillator in the limit where coherent interaction can dominate over dissipative processes is formally similar to the cavity quantum electrodynamics (QED), the ion trap problem was generally treated by means of Jaynes-Cummings model(JCM)[4], a well known treatment for the interaction between matter and light in the field of quantum optics. In such a model, the ion is simplified to be two levels, the ion's motion, under the action of the trap potential, is quantized as a harmonic oscillator, and the cooling laser is taken the classical form of a standing or traveling wave. Taking advantage of the analogy between cavity QED and a trapped ion interacting with a laser, some techniques developed in the framework of cavity QED can be immediately transcribed to the ion-trap system in the Lamb-Dicke limit(LDL). Further investigation[5] told us that the situation beyond LDL for the ion-trap system just corresponds to the nonlinear JCM which is also a well-developed subject in the quantum optics.

We note that the secular approximation is used in the above treatment, in which the time-dependent potential of the Paul trap was replaced by a pseudopotential. Thus the potential of the Paul trap could be quantum mechanically treated as a harmonic oscillator. As the effects of the micromotion in experiments are clearly visible as additional resonances in excitation spectra[6], and it has been shown that the time-dependent potential influences the cooling dynamics, the cooling rates, and the final temperature[7], an exact investigation excluding the secular approximation in the above treatment is needed. In this case, the solution of the problem will not be able to get help from the JCM.

In a frame moving with the ion, the effect of the time-dependent potential could be taken into the position operator determined by a quantum Mathieu equation.[7] Thus the problem was transformed into an effective form similar to the Hamiltonian of JCM in the rotating wave approximation, and the laser cooling mechanism for a single ion could be presented in principle from this treatment. We also note that the treatment in Ref.[7] was used again by Bardroff *et al.*[8] to map the dynamics of the motion of a trapped ion onto its internal dynamics, and meanwhile the technique proposed by Vogel *et al.*[5] was also introduced to make an endoscopic study both within and beyond

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the LDL. Obviously, the above treatments focused on the investigation of the optical process influenced by the time-dependent trapping potential, which could be tested experimentally by observing the quantum jump of the system. In this work we adopt an alternative method, decoupling the internal and (external) motional states of the system, to discuss a special case of the interaction of a trapped ion with the laser, that is, a two-level ultracold ion in a Paul trap experiencing a resonant standing wave laser. The main advantage of this treatment is that we can obtain the analytical forms of the wave functions of the system, which enables us to compare with former treatments of time-dependent harmonic oscillator(TDHO) model[9,10] and discover the contribution of the laser to the stability of the trapped ion.

Ⅱ. SOLUTION

Consider the laser to be a standing wave which modifies the motion of the ion in x direction through the photon recoil accompanying each absorption and emission of laser photons. Thus the motion in y and z directions does not depend on the laser field and just present the regular center-of-mass(COM) behavior. Therefore, in what follows, we will concentrate on the one-dimensional problem of the motion of the ion along the x direciton.

For a Paul-trapped ultracold ion in the standing wave of a resonant laser,the typical Hamiltonian is

$$H = \frac{\hbar}{2}\omega_a\sigma_z - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(t)x^2 + \frac{\hbar\Omega}{2}\sin(kx+\phi)(\sigma_+ e^{-i\omega_a t} + \sigma_- e^{i\omega_a t}), \quad (1)$$

where ω_a is the transition frequency between two levels of the ion, and is also the frequency of the resonant laser; $\omega^2(t) = \frac{\nu^2}{4}[a + 2q\cos(\nu t)]$ with ν being the micromotion frequency, and a and q the dimensionless constants depending on the specific geometry and applied voltage to the trap, m is the mass of the ion, k is the wave vector of the standing wave and Ω the Rabi frequency; σ_i $(i = \pm, z)$ are Pauli operators, and the position of the ion in the standing wave is denoted by ϕ. As the trap frequency is supposed to be much greater than the atomic decay rate (strong confinement limit), the effect of the atomic decay is neglected.

Performing a unitary transformation $U = \exp(-i\omega_a\sigma_z t/2)$, we obtain

$$H^I = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(t)x^2 + \frac{\hbar\Omega}{2}\sigma_x\sin(kx+\phi). \quad (2)$$

In the LDL and weak excitation regime, Eq.(2) could be rewritten in a dimensionless form

$$H_I = -\frac{1}{2}\frac{\partial^2}{\partial\xi^2} + \frac{1}{2}\omega_0^2(t)\xi^2 + (A\xi + B)\sigma_x, \quad (3)$$

where $H_I = H^I/\hbar\nu$, $\omega_0(t) = \omega(t)/\nu$, $\xi = x/a_0 = \sqrt{\frac{m\nu}{\hbar}}x$, $\Omega_0 = \Omega/\nu$, $A = \Omega_0\eta\cos\phi/2$, $B = \Omega_0\sin\phi/2$ and $\eta = ka_0$.

We suppose that the statevector is of the general form $\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_e\\ \Phi_g\end{pmatrix}$, where Φ_e and Φ_g are motional states entangled respectively with the internal up and down states of the ion. For convenience of deduction, let $\Psi_1 = \Phi_e + \Phi_g$ and $\Psi_2 = \Phi_e - \Phi_g$, and combining Ψ with Eq.(3) yields the Schrödinger equations

$$i\frac{\partial}{\partial t}\Psi_1 = -\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\Psi_1 + \frac{1}{2}\omega_0^2(t)\xi^2\Psi_1 + (A\xi + B)\Psi_1, \quad (4)$$

$$i\frac{\partial}{\partial t}\Psi_2 = -\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\Psi_2 + \frac{1}{2}\omega_0^2(t)\xi^2\Psi_2 - (A\xi + B)\Psi_2. \quad (5)$$

We could obtain, by means of the results in the Appendix, Ψ_1 and Ψ_2 from the above equations

$$\Psi_i^n(\xi,t) = (2^n n!\sqrt{\pi})^{-1/2}\sqrt{\frac{D(0)}{D(t)}}\exp\left[-\frac{1}{2}\left(\frac{\xi}{D(t)} + \gamma_i(t)\right)^2\right]\exp[i(\alpha\xi^2 + \beta_i\xi)]$$
$$H_n\left(\frac{\xi}{D(t)} + \gamma_i(t)\right)\exp\left[-i\left(n+\frac{1}{2}\right)\int^t\frac{d\sigma}{D^2(\sigma)}\right]\exp\left[-i\int^t G_i(\sigma)d\sigma\right], \quad (6)$$

where i=1,2 and n=0,1,2,$\cdots$, $\alpha = \dot{D}(t)/2D(t)$, $\beta_1 = -D(t)\dot{\gamma}_1(t)$, $\beta_2 = -D(t)\dot{\gamma}_2(t)$ with the dot denoting the time derivation,

$$G_1(t) = B + \beta_1^2/2 + [\omega_0^2(t) + \dot{\alpha} + 2\alpha^2]\gamma_1^2 D^2(t) - (A + 2\alpha\beta_1 + \dot{\beta}_1)\gamma_1(t)D(t),$$
$$G_2(t) = -B + \beta_2^2/2 + [\omega_0^2(t) + \dot{\alpha} + 2\alpha^2]\gamma_2^2 D^2(t) - (-A + 2\alpha\beta_2 + \dot{\beta}_2)\gamma_2(t)D(t),$$

and $D(t)$, $\gamma_1(t)$ and $\gamma_2(t)$ are determined by the following equations, respectively

$$D^3(t)\ddot{D}(t) + \omega_0^2(t)D^4(t) = 1, \quad (7)$$

$$\gamma_1(t) = D^3(t)[A - 2\dot{D}(t)\dot{\gamma}_1(t) - D(t)\ddot{\gamma}_1(t)], \quad (8)$$

$$\gamma_2(t) = D^3(t)[-A - 2\dot{D}(t)\dot{\gamma}_2(t) - D(t)\ddot{\gamma}_2(t)], \quad (9)$$

To use the statevector of Eq.(6), we have to solve $D(t)$, $\gamma_1(t)$ and $\gamma_2(t)$ from Eqs.(7),(8) and (9). Equation (7) is a familiar nonlinear equation presented as an auxiliary one in various solutions of TDHO.[9−11] It could be solved bymeans of function series expansion (FSE)[10,11], in which $D(t)$ is taken as a series form

$$D(t) = \sum_{k=0}^{\infty} C_k \cos^k \nu t, \quad (10)$$

where $C_0 = (4/a)^{\frac{1}{4}}$ and

$$C_k = \frac{1}{k^2\nu^2 C_0^3 - aC_0^3}[-\nu^2 F_1^k + \nu^2 F_2^{k+2} + aF_4^k/4 + qF_3^{k-1}/2], \quad (11)$$

with $F_1^k = \sum'_{i+j+l+p=k} p^2 C_i C_j C_l C_p$, $F_2^k = \sum_{i+j+l+p=k} p(p-1)C_i C_j C_l C_p$, $F_3^k = \sum_{i+j+l+p=k} C_i C_j C_l C_p$, $F_4^k = \sum'_{i+j+l+p=k} C_i C_j C_l C_p$, in which $\sum'$ represents the summation excluding the term of $C_0^3 C_k$. As they have been investigated in Refs.[10,11], the validity of solution and the physical meaning of the convergence of FSE for the problem of the Paul trap will not be discussed here. For Eqs.(8) and (9), we set

$$P_i(t) = \dot{\gamma}_i(t), \quad (12)$$

$$\dot{P}_i(t) = [\pm A - 2\dot{D}(t)P_i(t) - \gamma_i(t)/D^3(t)]/D(t), \quad (13)$$

where $\pm$ correspond to $i = 1$ and 2 respectively. Thus $\gamma_i(t)$ (i=1,2) could be numerically solved by fourth-order Runge-Kutta integration.

Ⅲ. DISCUSSION

The concept of eigenenergy of the system fails in the time-dependent treatment. So we should first define a pseudo-energy of the system via the expectation value of the corresponding hamiltonian, which is explicitly time-dependent and is in fact related to the dynamic phase of the system. For convenience of the theoretical discussion, we assume the system is in the ground state (n=0). Thus it becomes, using Eqs.(6)–(9).

$$E(t) = <|H|> = \frac{D(0)}{2\sqrt{\pi}}\left[\frac{1}{D(t)^3} + \frac{G_1(t) + G_2(t)}{D(t)} - \sqrt{\pi}\dot{\alpha}(\gamma_1^2(t) + \gamma_2^2(t)) + \sqrt{\pi}(\dot{\beta}_1\gamma_1(t) + \dot{\beta}_2\gamma_2(t))\right], \quad (14)$$

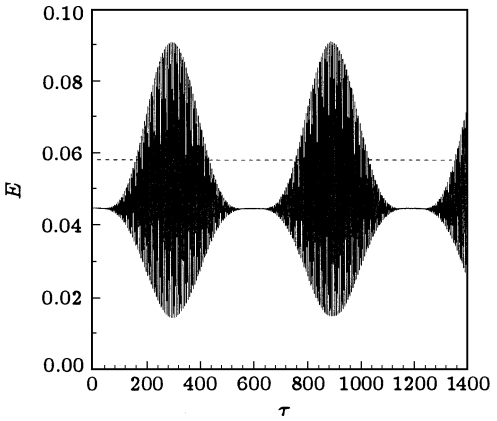


Fig.1. The variation of pseudo-energy of Eq.(3) with time, where $\tau = \nu t$, a=0.1, q=0.25, ν=15 MHz, A=0.01, and B=1.0. The dashed line is the result under secular approximation.

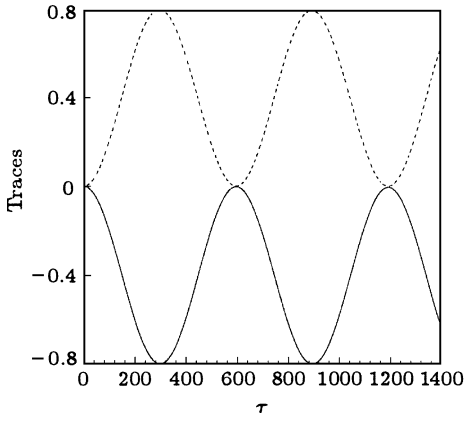


Fig.2. The variation of traces of the peaks of wave packets $\Psi_i^{*0}(\xi,t)\,\Psi_i^0(\xi,t)$ with time, where the values of parameters are the same as in Fig.1, and i=1 and 2 correspond to solid and dashed curves, respectively.

where, and in the following, $|>$ denotes $\frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_e\\ \Phi_g\end{pmatrix}$, and its variation in time is plotted in Fig.1, where a=0.1, q=0.25 and ν=15.0 MHz according to the order of magnitude of the parameters in the ion trap experiment. From this figure we could see that the pseudo-energy E has intensive fluctuations appearing periodically in the time evolution, which is physically understood as the exchange of energy between the trapped ion and the source including the trap and the laser. However, during a period of long time, E could be still averaged as a constant. Meanwhile, the numer-

ical results under the secular approximation[12] is alos plotted in Fig.1, where $\omega_0^2(t)$ in Eqs.(4) and (5) are replaced by $\frac{1}{8}(a+2q^2)$ in the calculation. We can find that at this time the secular approximation is valid due to being close to the average value of pseudo-energy. Figure 2 demonstrates the traces of peaks of $\Psi_i^{0*}(\xi,t)\,\Psi_i^0(\xi,t)(i=1,2)$ at different time, from which we could find that these two wave packets oscillate strictly symmetrically in opposite directions. Corresponding to Fig.1, the closing of the wave packets of $\Psi_1^0(\xi,t)$ and $\Psi_2^0(\xi,t)$ corresponds to the silence of the system, and going apart corresponds to the energy exchange. As $\Phi_e=(\Psi_1+\Psi_2)/2$ and $\Phi_g=(\Psi_1-\Psi_2)/2$, making use of Eqs.(6)–(9), we would obtain the wave packets of motional ground states entangled respectively with the excited and ground internal states of the ion,

$$\Phi_i^*\,\Phi_i = \frac{D(0)}{4\sqrt{\pi}D(t)}\Bigg\{\exp\left[-\left(\frac{\xi}{D(t)}+\gamma_1(t)\right)^2\right]+\exp\left[-\left(\frac{\xi}{D(t)}+\gamma_2(t)\right)^2\right] \pm 2\cos\theta\exp\left[-\frac{1}{2}\left(\frac{\xi}{D(t)}+\gamma_1(t)\right)^2-\frac{1}{2}\left(\frac{\xi}{D(t)}+\gamma_2(t)\right)^2\right]\Bigg\}, \tag{15}$$

with $i=e$ or g corresponding to the notation of $\pm$ respectively, and $\theta=(\beta_2-\beta_1)\xi-\int^t(G_1(t')-G_2(t'))\mathrm{d}t'$. Equation (15) plotted in Fig.3 demonstrates that the wave packets expand and shrink simultaneously in time, whereas we have more possibility to find the ion when it stays at the excited internal state. Obviously, this characteristic results from the introduction of the resonant laser.

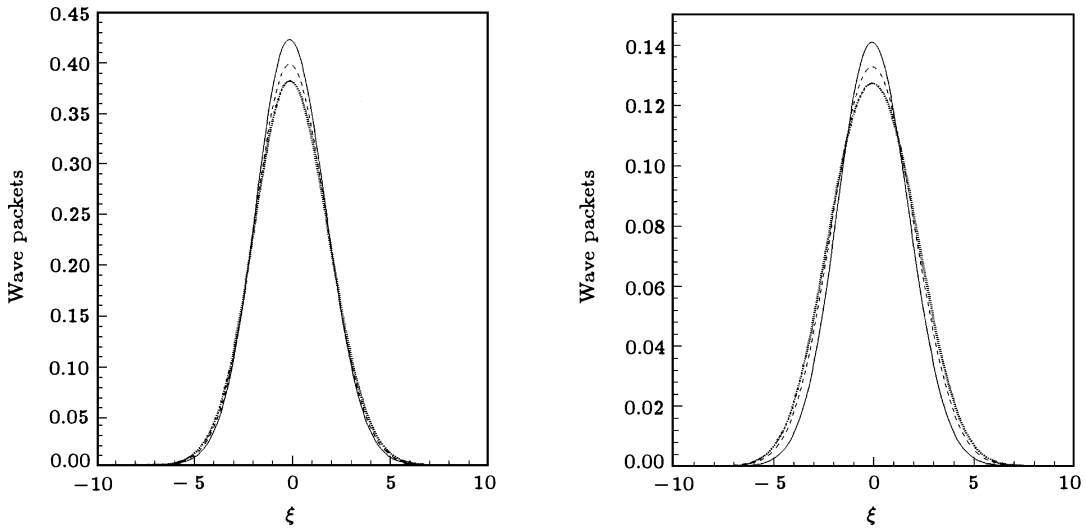


Fig.3. The time evolution of the wave packets of the motional states entangled with the excited internal state (left) and the ground one (right) of the ion, where the values of parameters are the same as in Fig.1, and the solid, dashed and dotted curves denote the cases at τ=0,200 and 300, respectively.

In former treatments of TDHO[9,10,11], the ion-trap system was modeled as a harmonic oscillator without consideration of the laser field, which describes the case that the laser is shut up after the ion has been cooled to the ultracold state. In such treatments, the solutions of the systems are similar to Eq.(6) with an auxiliary equation same as Eq.(7). For finding the contribution of the laser field, we should compare our solution with TDHO. The pseudo-energy of TDHO is plotted in Fig.4 with the same evaluations of the parameters as in Fig.1. We could see that, for the amplitude of the oscillation and the average value over time of the pseudo-energy, the values shown in Fig.4 are greater than those in Fig.1. From our viewpoint, the reason is that the interaction with the laser splits the ion's motion into two wave packets oscillating always symmetrically in opposite directions around the trap center, which makes the system more stable than that without the laser. In fact, if the statevector of the system is assumed to be (a) $\Psi=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\Phi$, or (b) $\Psi=\begin{pmatrix}1\\0\end{pmatrix}\otimes\Phi$, or (c)

$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Phi$, then Φ could be solved directly from the Schrödinger equation of Eq.(3). In such cases, the motional states of the ion are no longer constituted by two parts oscillating in opposite directions, and the motion of (a) would be a driven COM one, and (b) and (c) are of regular COM behavior. Therefore, it could be concluded that the ion-trap system with the entanglement of the motional and internal states of the ion is more stable than that with non-entangled states.

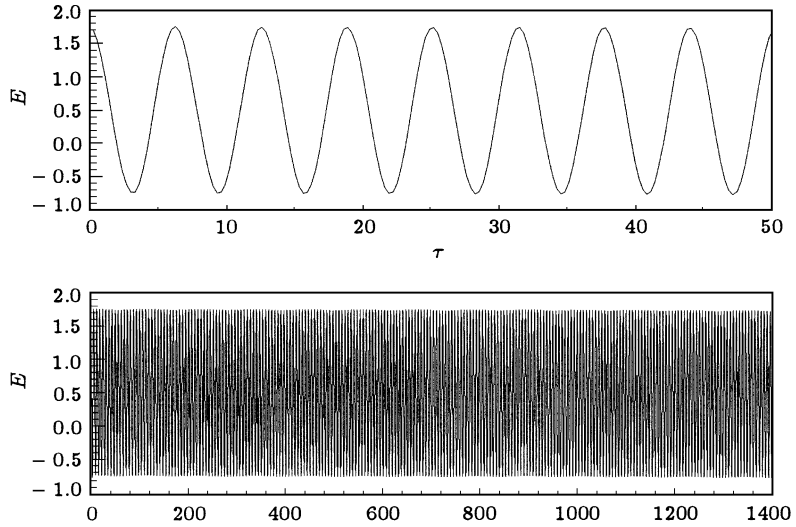


Fig.4. The time evolution of the pseudo-energy solved from the harmonic oscillator model, where the values of parameters are the same as in Fig.1.

Squeezing properties of the quadrature components in the TDHO model have been studied extensively for the ion-trap system.[13] Figure 5 demonstrates the fluctuations of the operators of $\hat{\xi}$ and $\hat{p}$ varying with time in such a case, that $\sqrt{\Delta p^2}(= \sqrt{< p^2 > - < p >^2}$ with $< \ldots >$ being expectation value with respect to wavefunction of ground state in the calculation) is squeezed under the quantum limit $\sqrt{\Delta p^2} = \frac{1}{2}$ at the expense of the increase of $\sqrt{\Delta \xi^2}$. If we consider the action of the laser, the situation would be completely different. Figure 6 tells that no squeezing would take place with the consideration of the laser, and the variations of $\sqrt{\Delta \xi^2}$ and $\sqrt{\Delta p^2}$ in time, with rapid fluctuations produced by the micromotion attached, are modulated by the laser field. Further calculation shows that this result is still valid even if the values of a and q are increased. This means that the squeezing of fluctuations of the quadrature components of an ion-trap system is destroyed by the resonant laser field.

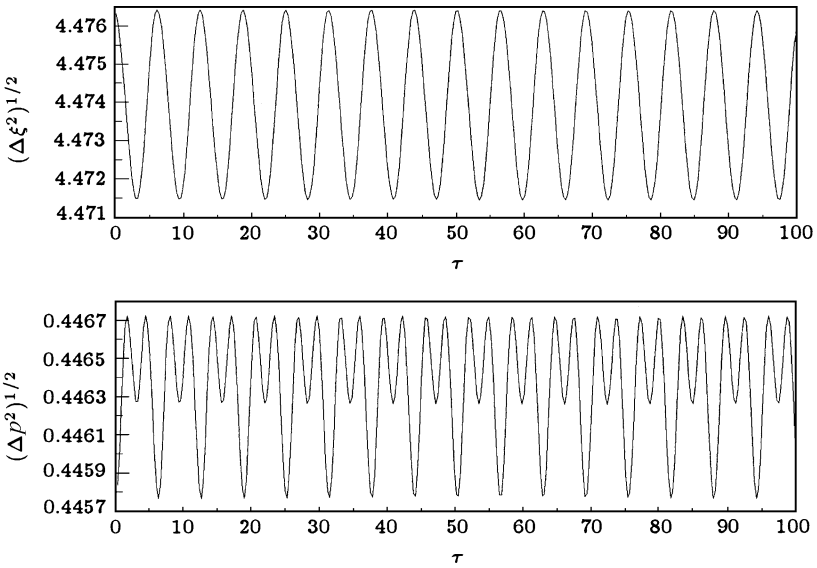


Fig.5. The variation of $\sqrt{\Delta \xi^2}$ and $\sqrt{\Delta p^2}$ with time solved from the harmonic oscillator model, where the values of parameters are the same as in Fig.1.

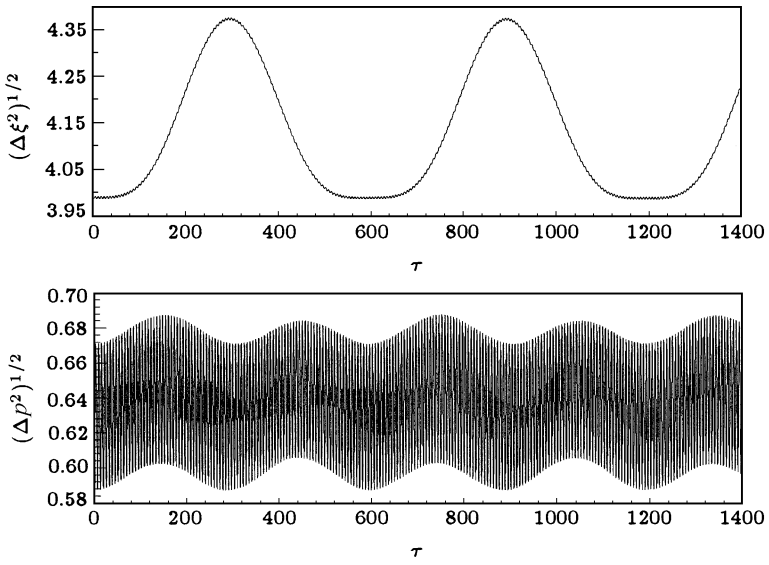


Fig.6. The variation of $\sqrt{\Delta\xi^2}$ and $\sqrt{\Delta p^2}$ with time solved from our model, where the values of parameters are the same as in Fig.1.

We should note that, in the calculation of Eqs.(12) and (13), we took the initial values of $\gamma_i(0)$=0 and $\dot{\gamma}_i(0)$=0(i=1,2). As $D(t)\gamma_i(t)$ represents the deviation of the peaks of wave packets $\Psi_i^{0*}(\xi,t)\,\Psi_i^0(\xi,t)$($i$=1,2) from the trap center, $\dot{\gamma}_i(t)$ is the variable related to the velocity of the wave packets. At the initial time, the ion should be confined stably at the trap center, and no split is taken place. Therefore, $\gamma_i(0)$=0 and $\dot{\gamma}_i(0)$=0 are the requirement of physical condition of the system. Moreover, the increase of the values of a and q will lead to the increase of the pseudo-energy of the system, and the enlargement of the difference between our solution and the treatment of secular approximation (Fig.7). It is understandable that the secular approximation is valid only in the case of $a < q \ll 1$. Comparing with Ref.[11] that the stability region of the trapped ion was studied by means of FSE method, we see that the equations in the present work for FSE method are identical to those in Ref.[11]. Therefore, the stability region remains unchanged regardless of the introduction of the resonant laser. From this point,we think that the region of validity for the secular approximation should be also identical to that without consideration of the resonant laser, that is, $a < q < 0.4$,as proved in Ref.[12].

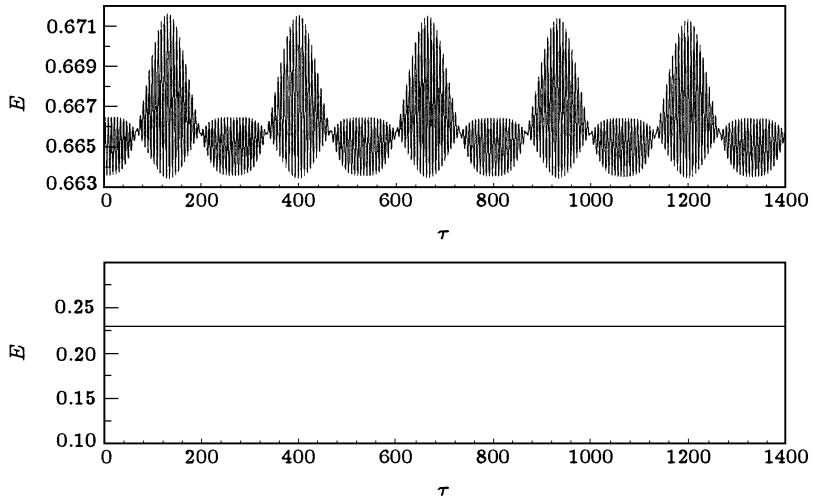


Fig.7. (a) The time evolution of the pseudo-energy of Eq.(3), where the values of the parameters are the same as those in Fig.(1) except a=0.5 and q=0.75. (b) The results solved from secular approximation.

We also noted that, a semiclassical treatment was made in Ref.[14] to discuss the chaos for a single Paul-trapped ion experiencing the standing wave of a resonant laser by means of similar hamiltonian to Eq.(2). For convenience of their investigation, the authors assumed $<\sigma_x>=<\cos\theta>$=constant, and found chaos of the ion's motion for a wide range of parameters and initial conditions in the case of a sufficiently strong laser field. Moreover, some quantum festures associated with the classical chaos have also been found for a single ion interacting with the large-detuning laser field.[15] It seems that the high-order terms related to the ion's position, extended from the interaction, lead to these intensive nonlinear effects. However, as our analytical treatment fails for the strong excitation regime, we have no way to compare our purely quantum results with those results.

IV. SUMMARY

In summary, with our approach of decoupling the internal and motional states of the trapped ion, instead of the former treatment of mapping the motional dynamics into the internal one of the ion, the resonant interaction of an ultracold ion with a standing wave laser in a Paul trap has been studied. As we could present the analytical forms of the wave functions of the system, we focused on studying the dynamics of the wave packets of the ion's motion in the laser wave, which enables us to investigate conveniently the mechanical properties of the ion-trap system and discover the contribution of the resonant laser wave to the stability of the trapped ion. Without the need of the rotating-wave approximation used in the endoscopy approach, the time-dependence effect of the trapping potential and the action of the laser were treated simultaneously in our solutions, whose results are different from those in the secular approximation treatment and TDHO model. We demonstrated the periodic fluctuation in the time evolution of the pseudo-energy of the system, and found that the motion of the ion in the entanglement of motional and internal states consists of two parts oscillating always in opposite directions around the trap center. The new results in this work can be briefly concluded as follows: (i) the trapped ion radiated by a resonant laser is more stable when it is in the entanglement of motional and internal states; (ii) the system loses the squeezing properties of the quadrature components appearing in the TDHO model when the resonant laser is introduced. Note that, in this work, most of the values of a and q are set to be smaller than 0.4. If increasing these values, one can find that the variation due to the micromotion for the pseudo-energy, Δx^2 and Δp^2, etc., will be enhanced, whereas the above two results will be still correct. However, the above conclusions are only valid for the case of trapped ion-laser interaction in resonance because our analytical approach could not generalized to treat the detuning case unless the case of large-detuning interaction and the ion staying at the ground internal state.[15] Our approach would also be invalid for the case beyond LDL or strong excitation regime, which includes the high-order terms of $\eta\xi$ and must resort to numerical simulation. As the parameter a is in the denominator of the expression of FSE, the situation of a=0 could not be discussed in this work although it usually appears in the actual Paul trap experiment.

APPENDIX

The generalized time-dependent harmonic oscillator has been solved previously.[16] Here we use the time-space transformation method to solve the following Schrödinger equation in the unit of $\hbar$=1

$$i\frac{\partial}{\partial t}\Psi(x,t) = -\frac{1}{2M(t)}\frac{\partial^2}{\partial x^2}\Psi(x,t) + [g_2(t)x^2 + g_1(t)x + g_0(t)]\Psi(x,t), \tag{A.1}$$

where $\Psi(x,t)$ is the function we want to obtain, $M(t)$, $g_2(t)$, $g_1(t)$ and $g_0(t)$ are time-dependent variables.

Performing a unitary transformation $\Psi(x,t) = \phi(x,t)\exp[i(\alpha x^2+\beta x)]$ where α, β are time-dependent variables determined later, Eq.(A1) can be rewritten as

$$i\frac{\partial}{\partial t}\phi(x,t) = -\frac{1}{2M(t)}\frac{\partial^2}{\partial x^2}\phi(x,t) - \frac{i}{M(t)}(2\alpha x+\beta)\frac{\partial}{\partial x}\phi(x,t) + \left[g_2(t) + \frac{2\alpha^2}{M(t)} + \dot{\alpha}\right]x^2\phi(x,t) + \left[g_1(t) + \frac{2\alpha\beta}{M(t)} + \dot{\beta}\right]x\phi(x,t) + \left[g_0(t) + \frac{1}{2M(t)}\beta^2 - i\frac{\alpha}{M(t)}\right]\phi(x,t), \tag{A.2}$$

where the dot on the variable denotes the differentiation with respect to time. Setting

$$y = x/D(t) + \gamma(t), \qquad s = \int^t \frac{\mathrm{d}\sigma}{M(\sigma)D(\sigma)^2}, \tag{A.3}$$

where $\gamma(t)$ and $D(t)$ will be determined later. Equation (A2) is changed to

$$\frac{i}{M(t)D^2(t)}\frac{\partial f(y,s)}{\partial s}+i[-D^{-2}(t)\dot{D}(t)x+\dot{\gamma}(t)]\frac{\partial f(y,s)}{\partial y}=-\frac{1}{2M(t)D^2(t)}\frac{\partial^2 f(y,s)}{\partial y^2}$$
$$-\frac{i}{M(t)D(t)}(2\alpha x+\beta)\frac{\partial f(y,s)}{\partial y}+\left[g_2(t)+\dot{\alpha}+\frac{2\alpha^2}{M(t)}\right][y-\gamma(t)]^2D^2(t)f(y,s)$$
$$+\left[g_1(t)]+\frac{2\alpha\beta}{M(t)}+\dot{\beta}\right][y-\gamma(t)]D(t)f(y,s)+\left[g_0(t)+\frac{\beta^2}{2M(t)}-\frac{i\alpha}{M(t)}\right]f(y,s), \tag{A.4}$$

in which $\phi(x,t)=f(y,s)$ is used. To delete the term of $\frac{\partial f(y,s)}{\partial y}$, we set $\alpha=\frac{M(t)\dot{D}(t)}{2D(t)}$, $\beta=-M(t)D(t)\dot{\gamma}(t)$. Thus

$$\frac{i}{D(t)^2M(t)}\frac{\partial}{\partial s}f(y,s)=-\frac{1}{2D(t)^2M(t)}\frac{\partial^2}{\partial y^2}f(y,s)+\left[g_2(t)+\frac{2\alpha^2}{M(t)}+\dot{\alpha}\right]D(t)^2y^2f(y,s)$$
$$+\left[-2\gamma(t)\left(g_2(t)+\frac{2\alpha^2}{M(t)}+\dot{\alpha}\right)D^2(t)+\left(g_1(t)+\frac{2\alpha\beta}{M(t)}+\dot{\beta}\right)D(t)\right]yf(y,s)$$
$$+G(t)f(y,s)-i\frac{\alpha}{M(t)}f(y,s), \tag{A.5}$$

where $G(t)=\left[g_2(t)+\frac{2\alpha^2}{M(t)}+\dot{\alpha}\right]D(t)^2\gamma(t)^2-\left[g_1(t)+\frac{2\alpha\beta}{M(t)}+\dot{\beta}\right]\gamma(t)D(t)+g_0(t)+\frac{\beta^2}{2M(t)}$. To obtain the standard Schrödinger equation of time-independent harmonic oscillator, we have to assume

$$f(y,s)=f'(y,s)\exp\left[-i\int^t G(t')\mathrm{d}t'\right]$$
$$\exp\left[-\int^t\alpha(t')\mathrm{d}t'/M(t')\right], \tag{A.6}$$

$$\left[g_2(t)+\frac{2\alpha^2}{M(t)}+\dot{\alpha}\right]D(t)^4=1/2M(t), \tag{A.7}$$

$$g_1(t)+\frac{2\alpha\beta}{M(t)}+\dot{\beta}=2\gamma(t)\left[g_2(t)+\dot{\alpha}+\frac{2\alpha^2}{M(t)}\right]D(t), \tag{A.8}$$

Eq.(A5) becomes

$$i\frac{\partial}{\partial s}f'(y,s)=-\frac{1}{2}\frac{\partial^2}{\partial y^2}f'(y,s)+\frac{1}{2}y^2f'(y,s), \tag{A.9}$$

with the familiar solution

$$f'_n(y,s)=N_n\exp[-y^2/2]H_n(y)\exp[-i(n+1/2)s], \tag{A.10}$$

where $N_n=(2^nn!\sqrt{\pi})^{-1/2}$, H_n is the Hermitian polynomial. Reversing the above deduction, we obtain the statevector of Eq.(A1)

$$\Psi_n(x,t)=N_n\sqrt{\frac{D(0)}{D(t)}}\exp[-(x/D(t)+\gamma(t))^2/2]H_n(x/D(t)+\gamma(t))$$
$$\exp\left[-i(n+1/2)\int^t\frac{\mathrm{d}\sigma}{M(\sigma)D(\sigma)^2}\right]\exp\left[-i\int^t G(t')\mathrm{d}t'\right]\exp[i(\alpha x^2+\beta x)], \tag{A.11}$$

in which $D(t)$ is determined by Eq.(A7), and $\gamma(t)$ could be solved from Eq.(A8).

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