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USING OPTIMAL FEEDBACK CONTROL FOR CHAOS TARGETING

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Since the conventional open-loop optimal targeting of chaos is very sensitive to noise, a close-loop optimal targeting method is proposed to improve the targeting performance under noise. The present optimal targeting model takes into consideration both precision and speed of the targeting procedure. The parameters, rather than the output, of the targeting controller, are directly optimized to obtain optimal chaos targeting. Analysis regarding the mechanism is given from physics aspect and numerical experiment on the Hénon map is carried out to compare the targeting performance under noise between the close-loop and the open-loop methods.

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I. INTRODUCTION

Controlling chaos has attracted increasing interest recently(see Refs.[1,2] and the references therein) and many methods have been proposed since Ott, Grebogi and Yorke introduced the famous OGY method[3] in 1990. Among these methods, many, including the OGY method,[3] are local linear controllers that are effective only in the linear neighborhood of the target unstable periodic orbit(UPO),[4] that is, they are not started until the system visits the linear neighborhood of the desired UPO. Due to the ergodicity of chaos, chaotic trajectory will visit the linear neighborhood in a finite but usually rather long time. Thus it is necessary to impose control on the system to direct the trajectory into the working area of the local stabilizer in an acceptable short time, and this procedure is called chaos directing or chaos targeting, which can be considered as the preprocess of chaos control.

The idea of chaos targeting was first introduced by Shinbrot[5] who, with small parameter perturbation, directed the chaotic trajectory into a specified region in a short time. After that, adaptive direct methods were studied,[6,7] and Paskota applied the open-loop optimal control method to chaos targeting and analyzed the geometry of chaos targeting.[8,9,10] For chaos targeting based on open-loop optimal control model, the control sequence or the perturbation sequence is computed numerically and imposed on the system for the specific starting point and target point. As is well known, open-loop is highly sensitive to noise especially in chaotic systems. For the non-chaotic systems, the open-loop method would not lead to serious problem, and as a matter of fact, optimal control is usually open-loop except in those systems whose optimal control is analytically available. However, due to the sensitivity to initial conditions of chaotic dynamic system, the open-loop targeting methods will be seriously subject to noise and the trajectory will deviate from the optimal one in the presence of noise. To address the problem Paskota applied a close loop online correction when imposing the prescribed open-loop optimal perturbation.[9,10] In the present paper, we present a targeting method based on close-loop optimal control model to enhance the performance of optimal chaos target in the presence of noise. Rather than determining the perturbation sequence directly, the close-loop optimal targeting is devised to design the parameters for the feedback controllers, such as the feedback coefficients, so that the output of the optimal controller can be adapted to the noisy environment. From the viewpoint of nonlinear system feedback control, the method is to design the feedback parameters using optimal control method. Unlike Paskota's method[9,10] which need two phases, i.e, one to compute the optimal open-loop controller, the other to design a close-loop corrector which is local to the optimal trajectory, our method is implemented in only one stage by directly computing the optimal close-loop controller.

This paper is organized as follows. First, an introduction to the feedback targeting of chaos is presented followed by the optimal feedback control model used to determine the feedback coefficients. Then a comparison is made between the performance of open-loop optimal targeting and the proposed close-loop optimal targeting based on a great deal of numerical experiments on the frequently studied Hénon mapping system. The experiments are carried out by numerically solving the optimal control problem in the absence

of noise and applying the resulting optimal control laws to direct the chaotic trajectory under the additive Gaussian noise as well as additive uniform noise. Analysis regarding the mechanism is also given.

Ⅱ. TARGETING OF CHAOTIC TRAJECTORY

Given the m-dimensional discrete-time dynamic system defined by

$$\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n, p), n = 0, 1, 2, \cdots, \tag{1}$$

where $\boldsymbol{x}_n \in R^m$ is the state variable, $\boldsymbol{f} : R^m \to R^m$ is the nonlinear mapping function, $p \in R$ is an adjustable system parameter. Assume that system (1) is chaotic for $p = p^*$, with a corresponding unstable fixed point $\boldsymbol{x}_F$=$x_F(p^*)$. The chaos targeting using parameter perturbation is to perturb the parameter p with a sequence of perturbation δp such that the system will reach the ε neighborhood of the target point $\boldsymbol{x}_g$ in a given short time after starting from the initial point $\boldsymbol{x}_0$. Mostly the target point $\boldsymbol{x}_g$ is, but not limited to, the fixed point $\boldsymbol{x}_F$ of the system, and without loss of generality, we assume $\boldsymbol{x}_g = \boldsymbol{x}_F$ in the present paper. The controlled system can be modeled as

$$\begin{cases} p_{n+1} = p^* + \delta p_n, \\ \boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n, p_{n+1}), \end{cases} \quad n = 0, 1, 2, \cdots \tag{2}$$

The parameter perturbation mentioned above can also be in the following form:

$$p_{n+1} = p_n + \delta p_n. \tag{3}$$

The perturbation can be determined as

$$\delta p_n = \boldsymbol{k}_n^T(\boldsymbol{f}(\boldsymbol{x}_n, p^*) - \boldsymbol{x}_g), \tag{4}$$

where $\boldsymbol{k}_n^T \in R^m$ is the matrix of feedback coefficients, which can be either time-invariant or time-variant. In the present paper we will use the time-variant feedback matrix $\boldsymbol{k}_n$ that is capable of providing an adaptive controller. However, the computation of designing the coefficients can be heavy due to the large number of undetermined coefficients. For simplicity, the matrix $\boldsymbol{k}_n$ is reduced by predefining some of its components as zero, that is,

$$k_n = (k_1 \ k_2 \ \cdots \ k_s \ 0 \ 0 \ \cdots \ 0)^T, \ 1 \le s < m. \tag{5}$$

Since the controller for chaos targeting has to work in the whole phase space rather than in the small neighborhood of the target UPO, it is inappropriate to approximately linearize Eq.(1) near the target orbit to design the feedback matrix, which will result in a local controller effective only in the linear neighborhood of the target. To design a global feedback controller, we will use the optimal control method.

Ⅲ. OPTIMAL FEEDBACK TARGETING

There are three main constraints for chaos targeting: 1). Scale of the target region, i.e., the radius ε of the target linear neighborhood; 2). the allowed directing time $T_{\max}$; 3). the upper bound of the perturbation $U_{\max}$. These constraints actually correspond to three different performance indexes (accuracy, speed and energy) of the directing procedure respectively. The indexes are conflicting with each other and we have to make a trade off among them by assigning each index with a specific weight. Thus, based on the resulting compositive index, we setup a general feedback optimal control model which differs from the conventional open-loop optimal control model in that the present model solves the feedback coefficients while the open-loop model computes the control sequences directly. The feedback optimal control model comes as follows.

Given the radius ε of the target linear neighborhood, the maximal directing time $T_{\max}$ and the upper bound of the perturbation $U_{\max}$, the problem is to decide the sequence of feedback coefficients $\boldsymbol{K} = \{\boldsymbol{k}_1 \ \boldsymbol{k}_2 \ \cdots \ \boldsymbol{k}_{T_{\max}}\}$, such that the controlled system of Eq.(2) reaches the linear neighborhood of the target orbit with the minimal objective value defined by the following formal description,

$$\begin{aligned} \min J(\boldsymbol{K}) &= \alpha\|\boldsymbol{x}_{T_e} - \boldsymbol{x}_g\| + \sum_{n=0}^{T_e}\{\beta + \gamma(\delta p_n)^2\} \\ &= \alpha\|\boldsymbol{x}_{T_e} - \boldsymbol{x}_g\| + \beta T_e + \gamma\sum_{n=0}^{T_e}(\delta p_n)^2 \\ s.t. \ \|\boldsymbol{x}_{T_e} &- \boldsymbol{x}_g\| < \varepsilon, \quad T_e < T_{\max}, \quad |\delta p_n| < U_{\max} \end{aligned} \tag{6}$$

Where $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$ are the weights for the corresponding performance indexes (accuracy, speed, and energy) respectively; T_e is the ending time of the control procedure, i.e., the time for the directed orbit to reach the target region, for instance, T_e is the number of iterations of mapping function (2) in a mapping system; $\boldsymbol{K}$, the sequence of feedback coefficients to be decided, has the order of $m \times T_{\max}$. However, since some components of $\boldsymbol{K}$ has been predefined as zero, and the control procedure will end as soon as the trajectory enters the target region, the actual order of $\boldsymbol{K}$ becomes $s \times T_e$, which will significantly simplify the

computation. And $\boldsymbol{x}_{T_e}$ stands for the system state at the time the trajectory reaches the desired region, while $\boldsymbol{x}_g$ is the target state, usually the goal UPO. $\|\cdot\|$ is the Euclidean norm. Alternatively we can derive from the above model three other models respectively corresponding to the three indexes: the minimal deviation control($\alpha = 1$, $\beta = \gamma = 0$), the shortest time control ($\alpha = \gamma = 0, \beta = 1.0$) and the least energy control ($\alpha = \beta = 0, \gamma = 1$). For chaos targeting, the amount of energy cost is less important than accuracy and speed, so γ is set to zero while $\alpha > 0$ and $\beta > 0$ in this paper. The value of deviation error $\|\boldsymbol{x}_{T_e} - \boldsymbol{x}_g\|$ is usually quite small in comparison with that of T_e, thus the weight α have to be much larger than the weight β if α is nonzero so that deviation error in the objective function will not be numerically neglected during the optimization procedure. It is recommended that $\alpha/\beta \geq 20$ unless α is zero.

The optimal target model is a quadratic optimal programming problem which can be solved analytically according to the Bellman principle in case of a linear controlled system with an a priori mathematical model. But for nonlinear system, it is well known that analytical solution is rarely available. The present paper will solve the problem numerically with optimization algorithm. In the current numerical optimization problem, as a result of the parameter sensitivity of chaos, the surface of the objective function can be very complex, even with fractal structure, and this makes it difficult for an effective search for the global optimal solution, thus a conventional derivative-based algorithm is likely to trap into local minimum. Genetic algorithm (GA)[11] will be applied in this paper to search for a global optimal solution. As a randomized optimization algorithm based on the simulation of Darwinian natural selection, gene crossover and mutation, the genetic algorithm is capable of obtaining the global optimum at the probability of 1.0 and has been widely applied in the field of artificial intelligence (AI), automatic control, and engineering design. The floating point GA, which has the advantage of accuracy and speed over binary GA, will be used in the present paper.

IV. NUMERICAL RESULTS AND DISCUSSION

Numerical experiments were carried out on Hénon map to compare the performance of feedback optimal targeting with that of open-loop optimal targeting. We solve the feedback optimal control problem and the open-loop optimal control problem for chaos targeting in the absence of noise, and then apply the resulting control law, close-loop and open-loop respectively, to direct the chaotic trajectory in the presence of additive Gaussian noise, in order to observe their sensitivity to the noisy environment.

The two-dimensional Hénon map[4] is defined by:

$$x_{n+1} = \phi(x_n, y_n, p) = p - x_n^2 + q y_n, \quad y_{n+1} = x_n, \tag{7}$$

where $p = 1.4$, $q = 0.3$ are the system parameters. The above system has an unstable fixed point

$$\boldsymbol{F} = \Big((q - 1 + \sqrt{(1+q)^2 + 4p}\,)/2,$$
$$(q - 1 + \sqrt{(1+q)^2 + 4p}\,)/2\Big) = (0.883896, 0.883896).$$

So the chaos targeting is to direct the map from some initial point $\boldsymbol{S}$ into the region $\boldsymbol{G} = \{x \mid \|x - \boldsymbol{F}\| < \varepsilon\}$ in an iteration number of $T_{\max}$. The starting point $\boldsymbol{S}$ can be any point on the attractor with the exception of $\boldsymbol{F}$. Any point near the attractor can also serve as the starting point.

With the feedback perturbation to parameter p, Eq.(4) becomes

$$\delta p_n = k_n(\phi(x_n, y_n, p) - x_g), \tag{8}$$

where all operands, i.e. k_n, x_n, y_n and so on, are scalar and $x_g = 0.883896$. Apparently, in Eq.(8) only the x component of Hénon map is feeded back and $s = 1$ for Eq.(5).

A large number of starting points on the attractor are randomly selected, and both feedback control and open-loop control are numerically obtained for them according to the corresponding optimal control model without consideration of noise effect. Then the resulting control laws, either close-loop or open-loop, are applied in the presence of additive noise to see whether they can perform successful targeting in noisy environment. The results are illustrated in Table 1 with the upper bound of perturbation $U_{\max} = 0.05$ for all items. Table 2, where $U_{\max} = 0.02$, is similar to Table 1 except that all control laws correspond to the same initial point $\boldsymbol{S}$ randomly taken from the Hénon attractor, as shown in Fig.1. The applied Gaussian noise is of zero mean and its standard deviation, denoted by σ in the paper, is used as the noise level.

Table 1. Percentage of the successful control laws under additive Gaussian noise

Experimental conditions		$\varepsilon = 0.001$	$\varepsilon = 0.01$	$\varepsilon = 0.02$
1000 points,	open-loop	$0/43 = 0\%$	$2/448 = 0.4464\%$	$11/935 = 1.1765\%$
$\sigma = 5\%U_{\max}$	close-loop	$280/859 = 32.60\%$	$406/930 = 43.66\%$	$445/933 = 47.70\%$
2000 points,	open-loop	$1/238 = 0.4202\%$	$7/1756 = 0.3986\%$	$16/1979 = 0.8085\%$
$\sigma = 10\%U_{\max}$	close-loop	$310/1885 = 16.44\%$	$482/1934 = 24.92\%$	$530/1935 = 27.39\%$

Table 2. Percentage of the successful control laws under additive Gaussian noise

Experimental conditions		$\varepsilon = 0.001$	$\varepsilon = 0.01$	$\varepsilon = 0.02$
1000 trials,	open-loop	$0/159 = 0\%$	$4/879 = 0.4551\%$	$11/997 = 1.103\%$
$\sigma = 5\%U_{\max}$	close-loop	$130/990 = 13.13\%$	$251/1000 = 25.1\%$	$290/1000 = 29.0\%$
1000 trials,	open-loop	$0/154 = 0\%$	$4/862 = 0.464\%$	$8/993 = 0.8056\%$
$\sigma = 10\%U_{\max}$	close-loop	$97/991 = 9.79\%$	$155/1000 = 15.5\%$	$184/1000 = 18.4\%$

In the two tables, the denominators are the numbers of consequent numerical optimal control laws that are able to successfully direct the trajectory from the corresponding initial point into the goal region. Obviously, they imply the efficiency of the optimization algorithm, and, from the viewpoint of problem evaluation, they suggest the degree of numerical difficulty of the optimal control problems. The numerators are the numbers of consequent optimal control laws that are able to perform successful targeting even in the presence of noise. These numbers show the noise sensitivity of the close-loop or open-loop optimal control laws. So the percentage of success targeting under noise environment are determined by the ratio of the numerators and the denominators. From the data shown in the tables, it can be seen that the optimal targeting model with feedback is more robust against noise than the model without feedback, and that the former is more easy to solve than the latter, in other words, the objective surface structure of the feedback optimal target model is more smooth than that of the open-loop one.

Now we study the targeting performance of a specific optimal control law based on a certain initial point. The initial point concerned, $\boldsymbol{S} = (1.270309, -0.7644366)$, is the point $\boldsymbol{S}$ shown in Fig.1. Without directing, it will take 18219 iterations for Hénon map to reach the 0.001 neighborhood of the target point $\boldsymbol{F}$. Among those control laws numerically obtained for $\boldsymbol{S}$, the quickest controllers for open-loop model and close-loop respectively, some of which are

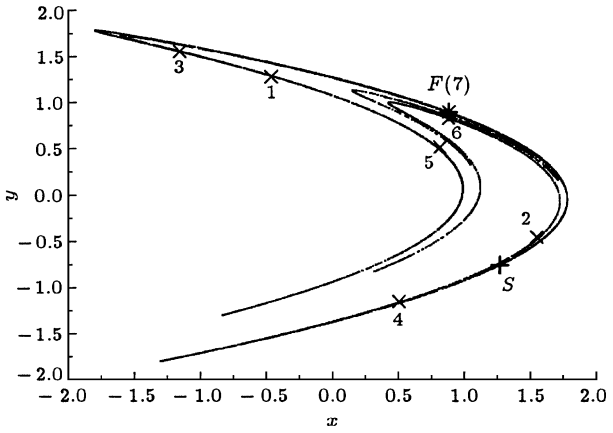


Fig.1. Attractor of Hénon map ($\boldsymbol{F}$: the fixed point, $\boldsymbol{S}$: the starting point, 1–7: the directed trajectory).

illustrated in Table 3, require 7 steps to accomplish the targeting mission. Figure 1 plots the trajectory directed by the close-loop optimal controller starting from point $\boldsymbol{S}$ to point $\boldsymbol{F}$ through points 1, 2, ..., and 7. The controllers in Table 3 were applied to Hénon map under additive Gaussian noise and uniform noise, respectively, for 50000 times. The percentage of the times when the directed trajectory reaches the desired region under noise of different level is taken and illustrated in Figs.2(a) and 2(b). Both the two kind of noises are of zero mean, and the standard deviation σ of the former is used as its noise level. The uniform noise is uniformly distributed in the interval $[-\theta, \theta]$, where $\theta > 0$ is the upper bound and represents the noise level as shown in the figure.

Table 3. Optimal Controller ($\varepsilon = 0.001$, steps $= T_e = 7$, starting point $\boldsymbol{S} = (1.270309, -0.7644366)$)

Type of Controller	Perturbation sequence(open-loop) or Feedback coefficients sequence(close-loop)
open-loop	$-0.01763335, -0.02, -0.02, -0.02, 0.02, 4.357647 \times 10^{-4}, 0.02$
close-loop	$1.035819, -0.02647953, 0.7503544, 0.683851, -0.7392056, -1.026027, -1.003532$

From the above numerical results, we can see that the close loop controller works better than the open loop one, this is consistent with the knowlege of control engineering. Now we will investigate the physical mechanism by means of Jacobian analysis. For simplicity and without loss of generality, our analysis will ignore the saturation of control perturbation. If approximately linearized along with the desired optimal trajectory, the system (2) has the following Jacobian matrix with respect to $\boldsymbol{x}_n$:

$$\boldsymbol{J}_x = \boldsymbol{f}_x + \boldsymbol{f}_p \boldsymbol{k}_n^T \boldsymbol{f}_x = (\boldsymbol{I} + \boldsymbol{f}_p \boldsymbol{k}_n^T)\boldsymbol{f}_x, \qquad (9)$$

where $\boldsymbol{f}_x$ and $\boldsymbol{f}_p$ stand for $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ and $\partial \boldsymbol{f}/\partial p$, respectively. Let

$$\boldsymbol{J}_{tr} = \boldsymbol{J}_{x_{T_e}} \boldsymbol{J}_{x_{T_e-1}} \cdots \boldsymbol{J}_{x_2} \boldsymbol{J}_{x_1}. \qquad (10)$$

We call $\boldsymbol{J}_{tr}$ the transition matrix from point $\boldsymbol{x}_1$ to point $\boldsymbol{x}_{T_e}$, and use $\rho(\boldsymbol{J}_{tr})$ to denote its spectral radius, the maximal absolute value of all eigenvalues of the matrix. The sensitivity of the system to noise can be measured in terms of spectral radius of the above transition matrix. The larger the spectral radius, the more sensitive the system. Now we will check the eigenvalues specified by Eq.(9) for the optimal trajectory directed by the close loop (i.e., $k_n \neq 0$) controller presented in Table 3. For the case of no feedback (i.e., $k_n \equiv 0$), $\rho(\boldsymbol{J}_{tr})$ is about 43.91, while for $k_n \neq 0$, the value is only 0.1156. This explains the difference between their sensitivity to noise. To obtain smaller $\rho(\boldsymbol{J}_{tr})$, we may need to slightly modify our optimization model(6) by adding corresponding constrains. Actually for $-1.5 < k_n < 1.5$ in system (7), smaller spectral radius can be easily obtained, thus no additional constrain was considered.

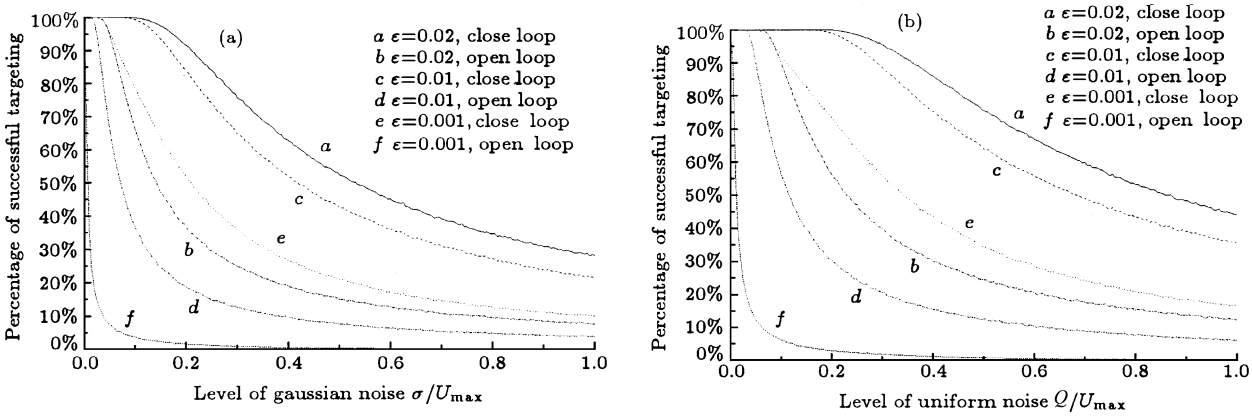


Fig.2. Percentage of times of successful targeting vs Noise Level (a) For Gaussian noise of zero mean and standard deviation σ (b) For noise uniformly distributed in the interval $[-\theta, \theta]$.

V. CONCLUSION

As is well known, the open-loop optimal control method is highly sensitive to noise; and meanwhile for nonlinear feedback controller, it is difficult to design an appropriate feedback matrix for global controllers. To solve the problems, the present paper proposed a close-loop optimal chaos targeting method which implement the nonlinear optimal control by optimizing the sequence of feedback matrices rather than that of the perturbation. As a close-loop targeting method, it is more robust against noise than the conventional open-loop targeting method. The mechanism relies on the fact that, in an intuitive sense of control engineering, the output of close-loop is capable of adapting itself to environmental noise, or, from a physics aspect, the spectral radius of the transition matrix for the close loop is smaller than that without feedback. Numerical experiments have confirmed the performance, and proved that the close-loop optimal control problem is numerically easier to solve than the open-loop one.

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