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INTEGRABILITY OF THE SPIN-1 XXX CHAIN WITH BOUNDARY IMPURITIES AND THEIR BETHE ANSATZ

Hou Bo-yu(侯伯宇), Shi Kang-jie(石康杰), Yue Rui-hong(岳瑞宏),
and Zhao Shao-you(赵少游)

Institute of Modern Physics, Northwest University, Xi'an 710069, *China*

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By using the fusion method, we give the spin-1 XXX chain associated with sl_2 with two arbitrary spin boundary impurities. Then the exact diagonalized Hamiltonian of this model was obtained by the means of Bethe ansatz method.

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I. INTRODUCTION

Recently, increasing attention has been focused on the Kondo problem. It is well known that the spin dynamics of the Kondo problem is equivalent to that of the spin chain system with magnetic impurities.[1] Although the magnetic impurity plays an important role in the system, it usually destroys the integrability of the system. So how to maintain the integrability is an interesting problem when one deals with the system. Till now, fruitful achievement has been obtained in different areas.[2−5]

Many papers devote themselves to the impurity problem of the Heisenberg chain since it is an important integrable model. Andrei and Johannesson[6] first considered the integrable impurities problem of the Heisenberg chain with periodic boundary condition. Then Lee and Schlottmann[7,8] generalized their result to arbitrary spin impurity. But They have to present some nonphysical terms in Hamiltonian to maintain the integrability, though these terms may be irrelevant.[9] Noticing the fact that the open boundary theory is very useful to formulate both the thermodynamics and the transport properties of the quantum chain with impurities, Gaudin[10] considered the nonlinear Schrödinger model and the spin-1/2 Heisenberg chain with simple open boundary, then Schulz and Alcaraz *et al.*[11,12] generalized it to Hubbard and other models.

The quantum inverse scattering method and the Bethe ansatz technique have been powerful tools to study the integrable impurity problems in 1-dimensional physical system, such as the t-j model[13−15] and the vertex model[9,16] with impurities.

In recent research, the impurities with arbitrary spin coupled to the spin-1/2 XXX and XXZ chains have been discussed in Refs.[9,16]. In this paper, we consider the Zamolodchikov-Fateev 19-vertex model[17], which is associated with spin-1 representation of sl_2.[18] This model is the simplest extension of the six-vertex model. Using the fusion method, Yue[19] has obtained the precise elliptic solution of the Yang-Baxter equation of the spin-1 Heisenberg chain. In this paper we only use the rational limit of the solution.

In Section Ⅱ, we recall the 19-vertex model. In Section Ⅲ, we discuss the integrability of Heisenberg chain with arbitrary spin impurities. Then we construct the transfer matrix and its eigenvalues by using the algebraic Bethe ansatz method in section Ⅳ. A brief discussion about our results is given in the last section.

Ⅱ. THE 19-VERTEX MODEL

The R-matrix of ordinary spin-1 XXX Heisenberg chain can be derived by using the fusion method.[17,19] It takes the form

$$R(u)=\rho(u)\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & a_5 & 0 & a_6 & 0 & 0 \\ 0 & a_3 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & 0 & a_7 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_6 & 0 & a_5 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 \end{pmatrix}, \tag{1}$$

where

$$a_1=(u+1)(u+2),\ a_2=u(u+1),$$
$$a_3=2(u+1),\ a_4=u(u-1),$$
$$a_5=2u,\ a_6=2,\ a_7=a_2+a_6.$$

It acts on the space $V\otimes V$. If we name the first space auxiliary space and the second quantum space, we can also write the R-matrix as L operator form

$$L_n(u)=u^2+u-2+2(\boldsymbol{S}_\tau\cdot\boldsymbol{S}_n)^2+2(u+1)\boldsymbol{S}_\tau\cdot\boldsymbol{S}_n, \quad (2)$$

where

$$S_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad S_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},$$
$$S_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},$$

$n=1,2,\cdots,N$ and τ presents the auxiliary space. Then we define

$$\begin{aligned}T(u)&=L_N(u)L_{N-1}(u)\cdots L_2(u)L_1(u),\\ \widetilde{T}(u)&=T^{-1}(-u)/\text{cont.}\\&=L_1(u)L_2(u)\cdots L_{N-1}(u)L_N(u).\end{aligned}\quad (3)$$

It satisfies the Yang-Baxter relation (YBR)

$$R_{12}(u-v)\overset{1}{T}(u)\overset{2}{T}(v)=\overset{2}{T}(v)\overset{1}{T}(u)R_{12}(u-v), \quad (4)$$

where R_{12} acts on the auxiliary space $v_1\otimes v_2$, and the operator T acts on quantum spaces as $v_1\otimes v_2\otimes\cdots\otimes v_N$. In this paper, we denote $\overset{1}{A}=A\otimes 1,\overset{2}{A}=1\otimes A$

In order to construct the open boundary condition consistent with the integrability, we introduce the reflection equation:

$$\begin{aligned}&R_{12}(u-v)\overset{1}{K}(u)R_{21}(u+v)\overset{2}{K}(v)\\=&\overset{2}{K}(v)R_{12}(u+v)\overset{1}{K}(u)R_{21}(u-v),\end{aligned}\quad (5)$$

where K is the reflecting matrix, which contributes a nontrivial boundary term in the Hamiltonian. One can prove that the double-row monodromy matrix defined as $U(u)=T(u)K(u)T^{-1}(-u)$ also satisfies the reflection equation

$$\begin{aligned}&R_{12}(u-v)\overset{1}{U}(u)R_{21}(u+v)\overset{2}{U}(v)\\=&\overset{2}{U}(v)R_{12}(u+v)\overset{1}{U}(u)R_{21}(u-v).\end{aligned}\quad (6)$$

The dual K-matrix $K_+(u)$ can be defined by the automorphism[20]

$$\phi: K(u)\to K_+(u)=K^t(-u-1). \quad (7)$$

It satisfies the dual reflection equation

$$\begin{aligned}&R_{12}(-u+v)\overset{1}{K}_+(u)R_{12}(-u-v-2)\overset{2}{K}_+(v)\\=&\overset{2}{K}_+(v)R_{12}(-u-v-2)\overset{1}{K}_+(u)R_{12}(-u+v).\end{aligned}\quad (8)$$

Then the transfer matrix can be defined by

$$t(u)=\text{tr}K_+(u)U(u),\quad (i=1\ \text{ or }\ 2). \quad (9)$$

One can check that the transfer matrix defined above satisfies the commutative relation

$$[t(u),t(v)]=0. \quad (10)$$

Ⅲ. THE FUSION OF THE BOUNDARY *L* OPERATOR

In this section we discuss the fusion procedure of the boundary L operator. The R-matrix of the 6-vertex model associated with the spin-1/2 Heisenberg chain can be written as

$$R(u)\equiv L(u)=u+\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}), \quad (11)$$

where $\boldsymbol{\sigma}$ is the Pauli matrix. The boundary L operator can be constructed by[9]

$$L(u)=u+\frac{1}{2}+\boldsymbol{\sigma}\cdot\boldsymbol{d}, \quad (12)$$

where $\boldsymbol{d}$ is a spin operator of $SU(2)$. The permutation operator and the projection operators are defined by

$$P_{12}=R_{12}(0), \quad (13)$$
$$P_{12}^-=-R_{12}(-1)/2. \quad (14)$$
$$P_{12}^+=1-P_{12}^- \quad (15)$$

respectively. They satisfy the following properties

$$P_{12}^2=1,\ (P_{12}^\pm)^2=P_{12}^\pm,\ P_{12}^+P_{12}^-=P_{12}^-P_{12}^+=0. \quad (16)$$

Now we use fusion procedure to obtain the high-dimensional L operator. Taking $v=u+1$ in Eq.(4), the YBR gives

$$R_{12}(-1)L_{1d}(u)L_{2d}(u+1)=L_{2d}(u+1)L_{1d}(u)R_{12}(-1), \quad (17)$$

where d represent the boundary terms. Multiplying above equation by $P_{12}^\pm$ from the left and right respectively, we get

$$P_{12}^+L_{2d}(u+1)L_{1d}(u)P_{12}^-=0, \quad (18)$$
$$P_{12}^-L_{1d}(u)L_{2d}(u+1)P_{12}^+=0. \quad (19)$$

Define

$$\overset{<12>}{L_d}(u)=P_{12}^+\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)P_{12}^+, \quad (20)$$

$$\overset{<12>}{L'_d}(u) = P_{12}^{+}\overset{2}{L_d}(u)\overset{1}{L_d}(u-1)P_{12}^{+}, \quad (21)$$

which satisfy the YBR respectively,

$$R_{<12><34>}(u-v)\overset{<12>}{L_d}(u)\overset{<34>}{L_d}(v) = \overset{<34>}{L_d}(v)\overset{<12>}{L_d}(u)R_{<12><34>}(u-v), \quad (22)$$

$$R'_{<12><34>}(u-v)\overset{<12>}{L'_d}(u)\overset{<34>}{L'_d}(v) = \overset{<34>}{L'_d}(v)\overset{<12>}{L'_d}(u)R_{<12><34>}(u-v), \quad (23)$$

where $R_{<12><34>}(u-v)$ is the fused R-matrix[19]

$$R'_{<12><34>}(u) = P_{12}^{+}P_{34}^{+}R_{14}(u-1)R_{13}(u) P_{34}^{+}P_{34}^{+}R_{24}(u)R_{23}(u+1)P_{34}^{+}P_{12}^{+}$$

acting on $V_{<12>}\otimes V_{<34>}$. Here we give the proof for $R(u)$:

$$\begin{aligned}
\text{LHS} =& R_{<12><34>}(u-v)P_{12}^{+}\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)P_{12}^{+}P_{34}^{+}\overset{3}{L_d}(v)\overset{4}{L_d}(v+1)P_{34}^{+}\\
=& P_{12}^{+}P_{34}^{+}R_{14}(u-v-1)R_{13}(u-v)P_{34}^{+}P_{34}^{+}R_{24}(u-v)R_{23}(u-v+1)\\
&\times P_{34}^{+}P_{12}^{+}\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)\overset{3}{L_d}(v)\overset{4}{L_d}(v+1)P_{12}^{+}P_{34}^{+}\\
=& R_{14}(u-v-1)R_{13}(u-v)R_{24}(u-v)R_{23}(u-v+1)\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)\\
&\times \overset{3}{L_d}(v)\overset{4}{L_d}(v+1)P_{12}^{+}P_{34}^{+}\\
=& R_{14}(u-v-1)R_{24}(u-v)R_{13}(u-v)\overset{1}{L_d}(u)L_{3d}(v)\overset{1}{L_d}(u+1)\\
&\times R_{23}(u-v+1)\overset{4}{L_d}(v+1)P_{12}^{+}P_{34}^{+}\\
=& \overset{3}{L_d}(v)\overset{4}{L_d}(v+1)\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)R_{14}(u-v-1)R_{13}(u-v)\\
&\times R_{24}(u-v)R_{23}(u-v+1)P_{34}^{+}P_{12}^{+}\\
=& P_{34}^{+}\overset{3}{L_d}(v)\overset{4}{L_d}(v+1)P_{34}^{+}P_{12}^{+}\overset{1}{L_d}(u)\overset{2}{L_d}(u+1)P_{12}^{+}P_{12}^{+}P_{34}^{+}R_{14}(u-v-1)\\
&\times R_{13}(u-v)P_{34}^{+}P_{34}^{+}R_{24}(u-v)R_{23}(u-v+1)P_{34}^{+}P_{12}^{+}\\
=& \overset{<34>}{L_d}(v)\overset{<12>}{L_d}(u)R_{<12><34>}(u-v)\\
=& \text{RHS}.
\end{aligned} \quad (24)$$

The proof for the other formula is similar. Substituting relation (12) into Eq.(20) and taking the transformation

$$L_d(u) :\rightarrow (1/\sqrt{2})L_d(u)\begin{pmatrix}1\\1/\sqrt{2}\\1\end{pmatrix}, \quad (25)$$

we have

$$\begin{aligned}
L_d(u) &= \begin{pmatrix}(u+d_z)(u+1+d_z) & d^-\sqrt{2}(u+d_z) & (d^-)^2\\ \sqrt{2}(u+d_z)d^+ & u(u+1)+d^2-2d_z^2 & \sqrt{2}(u-d_z)d^-\\ (d^+)^2 & d^+\sqrt{2}(u-d_z) & (u-d_z)(u+1-d_z)\end{pmatrix}\\
&= u(u+1)-d^2+2(u+1)\boldsymbol{s}\cdot\boldsymbol{d}+2(\boldsymbol{s}\cdot\boldsymbol{d})^2,
\end{aligned} \quad (26)$$

where we have used the notations $d^2 = \frac{1}{2}d^-d^+ + \frac{1}{2}d^+d^- + d_z^2$ and $s^2 = 2$. It satisfies the unitary relation

$$\rho_d(u) = (d^2-u^2-u)(d^2-u^2+u). \quad (27)$$

Now we couple the spin-1 Heisenberg chain with two boundary arbitrary spin impurities. Define

$$T'(u) = L_b(u+c_b)L_N(u)L_{N-1}(u)\cdots L_2(u)L_1(u),$$

$$\widetilde{T}'(u) = L_1(u)L_2(u)\cdots L_{N-1}(u)L_N(u)L_b(u-c_b), \quad (28)$$

where c_b is an arbitrary constant. Let

$$
\begin{aligned}
&K_+(u)=1,\\
&K(u)=L_a(u+c_a)L_a(u-c_a),\\
&(c_a \text{ is also a constant}).
\end{aligned}
\tag{29}
$$

It is easy to show that $K(u)$ and $K_+(u)$ satisfy the reflection equation (5) and the dual reflection equation (8) respectively. Thus the Hamiltonian of this model can be written as

$$
H=\frac{\mathrm{d}}{\mathrm{d}u}t(u)|_{u=0}=\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{tr}_\tau K_+(u)U(u)|_{u=0}. \tag{30}
$$

After some calculation up to a constant, we obtain the precise result of the Hamiltonian

$$
\begin{aligned}
H=&\frac{1}{4}\sum_{j=1}^{N-1}[\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}-(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})^2]\\
&+\frac{(c_a^2-1+d^2)\boldsymbol{S}_1\cdot\boldsymbol{d}_a-(\boldsymbol{S}_1\cdot\boldsymbol{d}_a)^2}{(d^2-c_a^2-c_a)(d^2-c_a^2+c_a)}\\
&+\frac{(c_b^2-1+d^2)\boldsymbol{S}_N\cdot\boldsymbol{d}_b-(\boldsymbol{S}_N\cdot\boldsymbol{d}_b)^2}{(d^2-c_b^2-c_b)(d^2-c_b^2+c_b)}.
\end{aligned}
\tag{31}
$$

IV. ALGEBRAIC BETHE ANSATZ FOR THE IMPURITY MODEL

For convenience, we rewrite the monodromy matrix as follows:

$$
U(u)=T''(u)\widetilde{T}''(u), \tag{32}
$$

where

$$
\begin{aligned}
T''(u)=&L_b(u+c_b)L_N(u)L_{N_{-1}}(u)\\
&\cdots L_1(u)L_a(u+c_a),\\
\widetilde{T}''(u)=&L_a(u-c_a)L_1(u)L_2(u)\\
&\cdots L_N(u)L_b(u-c_b).
\end{aligned}
$$

As usual, we write $T''(u)$ and $\widetilde{T}''(u)$ as

$$
\begin{aligned}
T''(u)&=\begin{pmatrix} A(u) & B_1(u) & B_2(u)\\ C_1(u) & D_1(u) & B_3(u)\\ C_2(u) & C_3(u) & D_2(u)\end{pmatrix},\\
\widetilde{T}''(u)&=\begin{pmatrix} \overline{A}(u) & \overline{B}_1(u) & \overline{B}_2(u)\\ \overline{C}_1(u) & \overline{D}_1(u) & \overline{B}_3(u)\\ \overline{C}_2(u) & \overline{C}_3(u) & \overline{D}_2(u)\end{pmatrix}
\end{aligned}
\tag{33}
$$

The vacuum state is defined by

$$
d_z|0\rangle=m|0\rangle,\quad s^+|0\rangle=d^+|0\rangle=0. \tag{34}
$$

Recalling the definition of the monodromy matrix, we have

$$
\begin{aligned}
A(u)|0\rangle=&\overline{A}(u)|0\rangle=a_1^N(u)a_1(u+c_b)a_1(u-c_a)|0\rangle,\\
D_1(u)|0\rangle=&\overline{D}_1(u)|0\rangle=a_2^N(u)a_2(u+c_b)a_2(u-c_a)|0\rangle,\\
D_2(u)|0\rangle=&\overline{D}_2(u)|0\rangle=a_4^N(u)a_4(u+c_b)a_4(u-c_a)|0\rangle,\\
C_i(u)|0\rangle=&\overline{C}_i(u)|0\rangle=0,(i=1,2,3)\\
B_i(u)|0\rangle=&\overline{B}_i(u)|0\rangle\neq 0.
\end{aligned}
\tag{35}
$$

It is easy to obtain the results of $\widetilde{T}'(u)$ acting on the vacuum state which have similar form to the above. Applying the double-row monodromy matrix

$$
U(u)=T''(u)\widetilde{T}''(u)=\begin{pmatrix}\mathcal{A}(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u)\\ \mathcal{C}_1(u) & \mathcal{D}_1(u) & \mathcal{B}_3(u)\\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{D}_2(u)\end{pmatrix}. \tag{36}
$$

on the vacuum state, one can directly check

$$
\begin{aligned}
&\mathcal{B}_i(u)|0\rangle\neq 0(i=1,2,3),\\
&\mathcal{C}_1(u)|0\rangle=\mathcal{C}_2(u)|0\rangle=0.
\end{aligned}
\tag{37}
$$

To obtain the value of the other elements acting on the vacuum state, we rewrite the Yang-Baxter equation (4) as

$$
\overset{2}{\widetilde{T}''}(v)R_{12}(u-v)\overset{1}{T''}(u)=\overset{1}{T''}(u)R_{12}(u-v)\overset{2}{\widetilde{T}''}(v). \tag{38}
$$

Letting $v=-u$, we can find

$$
\begin{aligned}
&\overline{B}_1(u)C_2(u)a_2(2u)+\overline{A}(u)C_1(u)a_5(2u)\\
&=C_2(u)\overline{B}_1(u)a_1(2u)+C_3(u)\overline{D}_1(u)\\
&+D_2(u)\overline{C}_3(u)a_5(2u).
\end{aligned}
\tag{39}
$$

Then we can see that

$$
\mathcal{C}_3(u)|0\rangle=C_2(u)\overline{B}_1(u)|0\rangle=0. \tag{40}
$$

We now study the diagonal elements of the monodromy matrices. From the multiplication rule of matrices, we have

$$
\begin{aligned}
\mathcal{A}(u)|0\rangle=&A(u)\overline{A}(u)|0\rangle,\\
\mathcal{D}_1(u)|0\rangle=&C_1(u)\overline{B}_1(u)|0\rangle+D_1(u)\overline{D}_1(u)|0\rangle,\\
\mathcal{D}_2(u)|0\rangle=&C_2(u)\overline{B}_2(u)|0\rangle+C_3(u)\overline{B}_3(u)|0\rangle\\
&+D_2(u)\overline{D}_2(u)|0\rangle.
\end{aligned}
\tag{41}
$$

Applying both sides of the relation (38) on the vacuum state $|0\rangle$, we find the following relations

$$C_1(u)\overline{B}_1(u)|0\rangle = \frac{1}{u+1}(\overline{A}(u)A(u) - \overline{D}_1(u)D_1(u))|0\rangle,$$
$$C_2(u)\overline{B}_2(u)|0\rangle = \frac{u\overline{A}(u)A(u) - (2u+1)\overline{D}_1(u)D_1(u) + (u+1)\overline{D}_2(u)D_2(u)}{u(u+1)(2u+1)}|0\rangle,$$
$$C_3(u)\overline{B}_3(u)|0\rangle = \frac{1}{u}(\overline{D}_1(u)D_1(u) - \overline{D}_2(u)D_2(u))|0\rangle. \quad (42)$$

To get only one wanted term in the commutative relation which will be studied later, we define

$$\widehat{\mathcal{D}}_1(u) = \mathcal{D}_1(u) - \frac{1}{u+1}\mathcal{A}(u),$$
$$\widehat{\mathcal{D}}_2(u) = \mathcal{D}_2(u) - \frac{1}{u}\widehat{\mathcal{D}}_1(u) - \frac{1}{(u+1)(2u+1)}\mathcal{A}(u). \quad (43)$$

Then the transfer matrix changes into

$$t(u) = \mathcal{A}(u) + \mathcal{D}_1(u) + \mathcal{D}_2(u)$$
$$= w_1^+\mathcal{A}(u) + w_2^+\widehat{\mathcal{D}}_1(u) + w_3^+\widehat{\mathcal{D}}_2(u)$$
$$= \frac{2u+3}{2u+1}\mathcal{A}(u) + \frac{u+1}{u}\widehat{\mathcal{D}}_1(u) + \widehat{\mathcal{D}}_2(u), \quad (44)$$

where we have introduced w_i^+ (i=1, 2, 3) for convenience.

Substituting Eq.(43) into Eq.(41), we arrive at

$$\mathcal{A}(u)|0\rangle = w_1|0\rangle,\ \widehat{\mathcal{D}}_1(u)|0\rangle = w_2|0\rangle,\ \widehat{\mathcal{D}}_2(u)|0\rangle = w_2|0\rangle, \quad (45)$$

where

$$w_1 = (u+1)^{2N}(u+2)^{2N}\prod_{k=a,b}\prod_{r=\pm1} \cdot (u+rc_k+m)(u+rc_k+1+m),$$
$$w_2 = \frac{u}{u+1}u^{2N}(u+1)^{2N}\prod_{k=a,b}\prod_{r=\pm1} \cdot [(u+rc_k)(u+rc_k+1)+d^2-2m^2]$$
$$w_3 = \frac{2u-1}{2u+1}u^{2N}(u-1)^{2N}\prod_{k=a,b}\prod_{r=\pm1} \cdot (u+rc_k-m)(u+rc_k+1-m).$$

In order to obtain the eigenvalue of the transfer matrix, we should find the commutative relation. From the reflection equation (6), we have

$$a_1^-a_2^+\mathcal{B}_1(u)\mathcal{A}(v) = a_1^+a_2^-\mathcal{A}(u)\mathcal{B}_1(v) + a_3^+a_2^-\mathcal{B}_1(u)\mathcal{D}_1(v) + a_2^+a_3^-\mathcal{B}_1(u)\mathcal{A}(v), \quad (46)$$

where the notations $a_i^\pm = a_i(u \pm v)$, $(i = 1, 2, \cdots, 7)$. Using the definition of $\widehat{\mathcal{D}}_1$, we find the commutative relation of the $\mathcal{A}(u)$ and $\mathcal{B}_1(v)$,

$$\mathcal{A}(u)\mathcal{B}_1(v)|0\rangle = \frac{(u+v)(u-v-2)}{(u-v)(u+v+2)}\mathcal{B}_1(v)\mathcal{A}(u)|0\rangle - \frac{2}{u+v+2}\mathcal{B}_1(u)\widehat{\mathcal{D}}_1(v)|0\rangle + \frac{2v}{(v+1)(u-v)}\mathcal{B}_1(u)\mathcal{A}(v)|0\rangle. \quad (47)$$

From the reflection Eq.(6), we find the following two equations

$$\mathcal{A}(u)\mathcal{B}_1(v)a_3^+a_3^- + \mathcal{D}_1(u)\mathcal{B}_1(v)a_2^+a_2^- + \mathcal{B}_1(u)\mathcal{D}_1(v)a_7^+a_3^- + \mathcal{B}_3(u)\mathcal{D}_1(v)a_5^+a_2^-$$
$$= \mathcal{B}_1(v)\mathcal{A}(u)a_5^+a_5^- + \mathcal{A}(v)\mathcal{B}_1(u)a_3^+a_7^- + \mathcal{B}_1(v)\mathcal{D}_1(u)a_7^+a_7^- + \mathcal{A}(v)\mathcal{B}_3(u)a_2^+a_5^- + \mathcal{B}_1(v)\mathcal{D}_2(u)a_5^+a_5^-, \quad (48)$$
$$\mathcal{B}_1(u)\mathcal{A}(v)a_5^+a_3^- + \mathcal{B}_3(u)\mathcal{A}(u)a_4^+a_2^-$$
$$= \mathcal{B}_1(v)\mathcal{A}(u)a_5^+a_6^- + \mathcal{A}(v)\mathcal{B}_1(u)a_3^+a_5^- + \mathcal{B}_1(v)\mathcal{D}_1(u)a_7^+a_5^- + \mathcal{A}(v)\mathcal{B}_3(u)a_2^+a_4^- + \mathcal{B}_1(v)\mathcal{D}_2(u)a_5^+a_4^-. \quad (49)$$

Solving the above equations and substituting the commutative relation of $\mathcal{A}$ and $\mathcal{B}_1$ into it, we obtain the commutative relation of $\widehat{\mathcal{D}}_1$ and $\mathcal{B}_1$

$$\widehat{\mathcal{D}}_1(u)\mathcal{B}_1(v)|0\rangle = \frac{(u+v)(u-v+1)(u-v-2)(u+v+3)}{(u-v)(u+v+1)(u-v-1)(u+v+2)}\mathcal{B}_1(v)\widehat{\mathcal{D}}_1(u)|0\rangle$$
$$- \frac{2[u(u+v)+(u+2v)]}{(u+1)(u-v)(u+v+1)}\mathcal{B}_1(u)\widehat{\mathcal{D}}_1(v)|0\rangle$$
$$+ \frac{2v[u(u-v)-(u+2v+4)]}{(u+1)(v+1)(u-v-1)(u+v+2)}\mathcal{B}_1(u)\mathcal{A}(v)|0\rangle$$
$$- \frac{2}{u+v+1}\mathcal{B}_3(u)\widehat{\mathcal{D}}_1(v)|0\rangle + \frac{2v}{(v+1)(u-v-1)}\mathcal{B}_3(u)\mathcal{A}(v)|0\rangle. \quad (50)$$

Using the similar procedure we can find the third commutative relation

$$\begin{aligned}\widehat{\mathcal{D}}_2(u)\mathcal{B}_1(v)|0\rangle =&\frac{(u-v+1)(u+v+3)}{(u-v-1)(u+v+1)}\mathcal{B}_1(v)\widehat{\mathcal{D}}_2(u)|0\rangle+\frac{4(u+1)}{u(2u+1)(u-v-1)}\mathcal{B}_1(u)\widehat{D}_1(v)|0\rangle\\&-\frac{4v(u+1)}{u(2u+1)(v+1)(u+v+1)}\mathcal{B}_1(u)\mathcal{A}(v)|0\rangle-\frac{2(u+1)}{u(u-v-1)}\mathcal{B}_3(u)\widehat{D}_1(v)|0\rangle\\&+\frac{2v(u+1)}{u(v+1)(u+v+1)}\mathcal{B}_3(u)\mathcal{A}(v)|0\rangle.\end{aligned}\tag{51}$$

Then we define the one-particle excited $|v\rangle$ over vacuum as

$$|v\rangle=\mathcal{B}_1(v)|0\rangle.\tag{52}$$

Applying the transfer matrix $t(u)$ on the one-particle state, we have

$$\begin{aligned}t(u)|v\rangle=&\left\{w_1w_1^+\frac{(u+v)(u-v-2)}{(u-v)(u+v+2)}\right.\\&+w_2w_2^+\frac{(u+v)(u-v+1)(u-v-2)(u+v+3)}{(u-v)(u+v+1)(u-v-1)(u+v+2)}\\&\left.+w_3w_3^+\frac{(u-v+1)(u+v+3)}{(u-v-1)(u+v+1)}\right\}|v\rangle,\end{aligned}\tag{53}$$

under the following constrain

$$\begin{aligned}&\frac{(v+2)^{2N}}{v^{2N}}\prod_{k=a,b}\prod_{r=\pm1}\\&\cdot\frac{(v+rc_k+m)(v+rc_k+1+m)}{(v+rc_k)(v+rc_k+1)+d^2-2m^2}=-1.\end{aligned}\tag{54}$$

From the reflection equation (4), we can construct the two-particle excited state as

$$\begin{aligned}|v_1,v_2\rangle=&\left\{\mathcal{B}_1(v_1)\mathcal{B}_1(v_2)+\frac{2}{v_1+v_2+1}\mathcal{B}_2(v_1)\mathcal{D}_1(v_2)\right.\\&\left.-\frac{2(v_1+v_2-1)}{(v_1-v_2-1)(v_1+v_2+1)}\mathcal{B}_2(v_1)\mathcal{A}(v_2)\right\}|0\rangle,\end{aligned}\tag{55}$$

which is symmetric in v_1, v_2 up to a whole factor. Applying the transfer matrix on the two-particle excited state, one can find a lot of "unwanted terms". From the demand that the eigenvalue is an entire function, which can be seen from the definition of the transfer matrix, one can obtain the corresponding Bethe ansatz equations. This is the so-called functional Bethe ansatz which was first proposed for the Ising model.[21] Here we cannot check directly that the "unwanted terms" cancel each other because the calculation turned out to be much more complicated than we had expected. Here we assume it vanishing. The coincidence with the results obtained by other approaches convinces us the result to be sound.

Tarasov *et al.* argued that the results deduced by the similar models can be generalized to n-particle excited state.[22−26] The eigenvalue of the transfer matrix for our model is as follows:

$$\begin{aligned}t(u|v_1,v_2,\cdots,v_n)=&w_1w_1^+\prod_{i=1}^n\frac{(u+v_i)(u-v_i-2)}{(u-v_i)(u+v_i+2)}\\&+w_2w_2^+\prod_{i=1}^n\frac{(u+v_i)(u-v_i+1)(u-v_i-2)(u+v_i+3)}{(u-v_i)(u+v_i+1)(u-v_i-1)(u+v_i+2)}\\&+w_3w_3^+\prod_{i=1}^n\frac{(u-v_i+1)(u+v_i+3)}{(u-v_i-1)(u+v_i+1)}\end{aligned}\tag{56}$$

where $v_1, v_2, \cdots, v_n$ should satisfy the following restriction

$$\begin{aligned}&\frac{(2v_j+1)(v_j+1)^{2N}}{(2v_j-1)(v_j-1)^{2N}}\prod_{k=a,b}\prod_{r=\pm1}\\&\cdot\frac{(v_j+rc_k+m-1)(v_j+rc_k+m)}{(v_j+rc_k-1)(v_j+rc_k)+d^2-2m^2}\\&=\prod_{\substack{i=1\\i\neq j}}^n\frac{(v_j-v_i+1)(v_j+v_i+1)}{(v_j+v_i-1)(v_j-v_i-1)}.\quad j=1,2,\cdots,n\end{aligned}\tag{57}$$

V. DISCUSSION

By using the fusion method, we obtained the integrable spin-1 XXX Heisenberg chain coupled with arbitrary spin impurities. With the help of algebraic Bethe ansatz method, we have diagonalized the Hamiltonian. The explicit expression of the eigenvalue and Bethe ansatz can be used to discuss the other properties of the system when impurities coupled to the chain, such as the thermodynamics, the transport property etc. The fusion procedure can also be used to construct more general integrable Heisenberg chains with arbitrary spin impurities. And we will discuss them in the next paper.

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