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EXPLICIT REPRESENTATION OF PLANAR WAVE SPEED AND DISPERSIVE RELATION IN GENERAL EXCITABLE MEDIA

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We transform the problem of seeking the planar wave speed in general excitable media into a cubically algbraic equation, thus we can find the explicit representation of the planar wave speed (in one-dimensional case, the traveling wave speed) and dispersive relation. Especially for a few typical excitable media, we give the formula for calculating the corresponding wave speed.

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I. INTRODUCTION

So far there have been many reports[1−6] investigating the dependence of wave speed on system parameters and the dispersive relation in some excitable media (EM) through numerical researches. Because the authors of those papers did not know the explicit representation of the wave speed and also the explicit dispersive relation, what they did was only to carry out numerical studies of the corresponding partial differential equation by adopting different numerical methods and taking into account the assumed necessary factors. Therefore, the results obtained were approximate ones and showed some differences between themselves.[7,8]

In this paper, we give the explicit representation of the planar wave speed and the dispersive relation in general EM. In particular, we give the formulas of the corresponding wave speeds of a few typical EM, such as various FitzHungh-Nagmo models[9], Tyson model[10] and Ginzburg-Landau model.[11] However, in order to obtain these formulas, we make an assumption that a slow variable does not change considerably so that we may regard it a constant within the thin boundary layer. According to the nature of EM, this assumption is reasonable.

Obtaining the explicit representation of wave speed is based on the fact: under the translational invariance, we can transform the problem of seeking the planar wave speed in some typical EM into solving the algbraic equation in the following form:

$$a_n u^{(n)} + a_{n-1} u^{(n-1)} + \cdots + a_1 u^{(1)} + \sum_{i=-k}^{m} b_i u^i = 0, \quad (1)$$

where m is a positive integer, $m \neq 1, \dfrac{n}{m-1}$ is negative(generally negative integer), a_i and b_i are constants which possibly depend on wave speed μ for at least one a_i, and $u^{(i)}$ are derivatives with respect to the argument: $z = \boldsymbol{n} \cdot \boldsymbol{x} - \mu t$ ($\boldsymbol{n}$ is a unit constant vector, $\boldsymbol{x}$ is a coordinate vector). We can prove that Eq.(1) is of Painlevé property[12,13], thus we do Painlevé analysis on it. Under the condition $b_m \neq 0$, in order to get pole exponent, we suppose $u \sim (z - z_0)^\alpha$, then $a_n u^{(n)} \sim a_n \alpha(\alpha - 1)(\alpha - n + 1)(z - z_0)^{\alpha - n}$, which must balance with the term $b_m u^m$, (this means they have the same exponent and the equal coefficients with opposite signs). Hence we have $\alpha = -\dfrac{n}{m-1}$, and we can suppose $u(z) = \sum\limits_{i=\alpha}^{+\infty} c_i (z - z_0)^i$. Substituting it into the Eq.(1), balancing the coefficients of various powers of $z - z_0$, according to Painlevé charateristic, there must exist a resonance point. So we can obtain the alge braic equation for the wave speed μ.

In Sec.Ⅱ, we give the explicit representation of the planar wave speed of two general types of EM.; Sec.Ⅲ gives formulas of wave speeds of a few typical EM.; Sec.Ⅳ discusses the dispersive relation; Sec.V gives the conclusion that after knowing the wave speed, we can also find the series solution of the planar wave.

Ⅱ. EXPLICIT REPRESENTATION OF WAVE SPEED

Jus as mentioned in the introduction, the previous works are confined only to numerical analyses except for finding certain explicit representations of the planar wave speeds for some special models. In this section two kinds of EM models are included (because of the nature of EM, these two EM are typical and do not lose generality) and give some cubically algebraic equations which their planar wave speeds satisfy. In

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the following section we shall apply the result obtained in this section to the usual EM's models and show that the results in Refs.[7,14,15] can be easily derived from our algebraic equations. However, comparisons between our results and those in Refs.[7,14,15] will be left over until the next section. Moreover, different from Keener's method[14] which is suited only to his models, the method we use here is a general one and can be used to deal with similar problems in other nonlinear partial differential equations.

The general EM can be described as follows:

$$\begin{cases} \epsilon\dfrac{\partial u}{\partial t} = \epsilon^2 D\Delta u + f(u,v), \\ \dfrac{\partial v}{\partial t} = \epsilon D_1\Delta v + g(u,v), \end{cases} \tag{2}$$

where D and D_1 are diffusional coefficients, ϵ is a small parameter ($0 < \epsilon \ll 1$), so that the time scale of u is much faster than that of v; Δ is Laplacian symbol which may be one, two or three-dimensions; $f(u,v) = 0$ generally has three branches, denoted by u_+, u_0, u_-: $u = u_-$ for $v \geq v_{\min}$; $u = u_0$ for $v_{\min} \leq v \leq v_{\max}$; $u = u_+$ for $v \leq v_{\max}$. The null-cline $g(u,v) = 0$ is a monotonic function. The local kinetics, the dynamics in the absence of spatial derivative, has a stable but excitable fixed point at the intersection of the null-clines $f(u,v) = 0$ and $g(u,v) = 0$. The appending boundary conditions are $\lim\limits_{z\to+\infty} u(z) = u_-(v_0)$, $\lim\limits_{z\to-\infty} u(z) = u_+(v_0)$. Let $z = \dfrac{(\boldsymbol{n}\cdot\boldsymbol{x} - \mu t)}{\epsilon}$, then Eq.(2) becomes

$$Du'' + \mu u' + f(u,v) = 0, \tag{3}$$

$$-\mu v' = D_0 v'' + \epsilon g(u,v). \tag{3a}$$

Because $0 < \epsilon \ll 1$ and v is bounded, we can obtain $v = v_0 + O(\varepsilon)$ from Eq.(3a). Substituting it into Eq.(3) and omitting the small quantity in ϵ, we have

$$Du'' + \mu u' + f(u,v_0) = 0. \tag{4}$$

Since the figure of the null-clines $f(u,v_0)$ is of N-type or upside-down S-type, for convinience, Here we study the function f of only two typical forms as follows:

$$f(u,v_0) = \frac{a}{u} + b + cu + du^2 + eu^3, \tag{5}$$

in wich case 1: $a = 0, e \neq 0$; case 2: $a \neq 0, e = 0$.

A. Case 1. $a = 0, e \neq 0$

We begin by determining what kind of pole a solution of Eq.(4) can have. If $u(z) \sim (z - z_0)^{-\alpha}$, then the Du'' and eu^3 must balance with each other. Hence

$$u'' \sim (-\alpha)(-\alpha-1)(z-z_0)^{-\alpha-2} \sim (z-z_0)^{-3\alpha} \sim u^3.$$

This gives $\alpha = 1$ for a nontravial solution. Thus we look for a solution of the form

$$u(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{+\infty} a_k z^k, \tag{6}$$

where we have made use of the translational invariance of Eq.(4) to drop the parameter z_0. With this, we have

$$\begin{aligned} u' =& -\frac{a_{-1}}{z^2} + \sum_{k=1}^{+\infty} k a_k z^{k-1}, \\ u^2 =& \frac{a_{-1}^2}{z^2} + \frac{2a_{-1}a_0}{z} + (2a_{-1}a_1 + a_0^2) \\ & + 2(a_{-1}a_2 + a_0 a_1)z + \cdots, \\ u^3 =& \frac{a_{-1}^3}{z^3} + \frac{3a_{-1}^2 a_0}{z^2} + \frac{3(a_{-1}^2 a_1 + a_{-1}a_0^2)}{z} \\ & + (3a_{-1}^2 a_2 + 6a_{-1}a_0 a_1 + a_0^3) \\ & + 3(a_{-1}^2 a_3 + a_{-1}a_1^2 + a_0^2 a_1 + 2a_0 a_{-1} a_2)z + \cdots \end{aligned}$$

Substituting them into Eq.(4) and balancing the coefficients of the various powers of z, we obtain a group of equations as follows: $2Da_{-1} + ea_{-1}^3 = 0$ which gives $a_{-1}^2 = -\dfrac{2D}{e}$; $-\mu a_{-1} + \mathrm{d}a_{-1}^2 + 3ea_{-1}^2 a_0 = 0$, which gives $a_0 = \dfrac{1}{3e}\left(\dfrac{\mu}{a_{-1}} - d\right)$;

$$ca_{-1} + 2da_{-1}a_0 + 3e(a_{-1}^2 a_1 + a_{-1}a_0^2) = 0, \tag{7}$$

which gives $a_1 = -\dfrac{c + 2da_0 + 3ea_0^2}{3ea_{-1}}$.

$$\begin{aligned} & 2Da_2 + \mu a_1 + b + ca_0 + d(2a_{-1}a_1 + a_0^2) \\ & + e(3a_{-1}^2 a_2 + 6a_{-1}a_0 a_1 a_0^3) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} & 6Da_3 + 2\mu a_2 + ca_1 + 2d(a_{-1}a_2 + a_0 a_1) \\ & + 3e(a_{-1}^2 a_3 + a_{-1}a_1^2) + a_0^2 a_1 + 2a_0 a_{-1} a_2) = 0. \end{aligned} \tag{9}$$

In Eq.(9), since the coefficient of a_3 is zero, through the coefficients of a_0 and a_{-1} obtained above and Eq.(7), we have $a_2 = 0$. Let $x = -\dfrac{\mu}{ea_{-1}}$. From (8), we obtain

$$8ex^3 - 6\left(\frac{d^2}{e} + 3c\right)x + \frac{2d^3}{e^2} - \frac{9dc}{e} + 27b = 0. \tag{10}$$

In the next section we shall see that Keener's result[14] is easily derived from Eq.(10). For the rest of coefficients of Eq.(6), we have the following recursions

$$\begin{aligned} a_n =& \frac{-1}{n(n-1)D + 3ea_{-1}^2}\Big\{[2da_{-1} + (n-1)\mu]a_{n-1} \\ & + ca_{n-2} + 2d\sum_{i+j=n-2} \epsilon'_{ij} a_i a_j + 3ea_{-1}\sum_{i+j=n-1} \epsilon'_{ij} a_i a_j \\ & + 3e\sum_{i+j+k=n-2} \epsilon'_{ijk} a_i a_j a_k \Big\}, \end{aligned}$$

where $n>3, a_3$ is an arbitrary constant.

$$\epsilon'_{ijk}=\begin{cases}0, & \text{if } i<0 \text{ or } j<0 \text{ or } k<0;\\ \dfrac{1}{3}, & \text{if } i=j=k;\\ 1, & \text{otherwise.}\end{cases}$$

$$\epsilon'_{ij}=\begin{cases}0, & \text{if } i<0 \text{ or } j<0;\\ 1, & \text{if } i\neq j;\\ \dfrac{1}{2}, & \text{if } i=j.\end{cases}$$

B. Case 2. $a\neq 0, e=0$

Similar to the discussion in case 1, we have the pole exponent $\alpha=-2$. Thus we may suppose:

$$u(z)=\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+\sum_{k=0}^{+\infty}a_k z^k. \tag{11}$$

With this, we have:

$$\begin{aligned}u'=&-\frac{2a_{-2}}{z^3}-\frac{a_{-1}}{z^2}+\sum_{k=1}^{+\infty}ka_k z^{k-1},\\ u''=&\frac{6a_{-2}}{z^4}+\frac{2a_{-1}}{z^3}+\sum_{k=2}^{+\infty}k(k-1)a_k z^{k-2} \text{ and}\\ u^2=&\frac{a_{-2}^2}{z^4}+\frac{2a_{-2}a_{-1}}{z^3}+\frac{a_{-1}^2+2a_{-2}a_1}{z^2}\\ &+\frac{2(a_{-2}a_1+a_{-1}a_0)}{z}+(2a_{-2}a_2+a_0^2+2a_{-1}a_1)\\ &+2(a_{-2}a_3+a_0a_1+a_{-1}a_2)z+(2a_{-2}a_4\\ &+2a_{-1}a_3+2a_0a_2+a_1^2)z^2+\cdots\end{aligned}$$

Besides, we also suppose

$$\frac{1}{u(z)}=\sum_{k=2}^{+\infty}b_k z^k. \tag{12}$$

By balancing the powers of z of Eq.(12), we have

$$b_2=\frac{1}{6}, b_n=-\frac{1}{6}\sum_{i+j=n-2}a_i b_j\epsilon_{ij}, \tag{13}$$

where $n\geq 3$,

$$\epsilon_{ij}=\begin{cases}0, & \text{if } 0<j<2 \text{ or } i<-1;\\ 1, & \text{otherwise.}\end{cases}$$

Furthermore, by substituting Eq.(11) into Eq.(4) and balancing the powers of z, respectively, we have

$$\begin{aligned}a_{-2}=&-\frac{6D}{d},\ a_{-1}=\frac{6\mu}{5d},\\ a_0=&\frac{1}{2d}\left(\frac{\mu^2}{25D}-c\right),\ a_1=-\frac{\mu^3}{250dD^3},\\ a_2=&\frac{7\mu^4}{5000dD^3}-\frac{c^2-4bd}{40dD},\\ a_3=&\frac{79\mu^5}{75000dD^4}+\frac{11\mu}{150D}\left(\frac{-c^2}{4d}+b\right).\end{aligned}$$

In particular, when comparing the coefficient of z^2, we obtain

$$\begin{aligned}&12Da_4+3\mu a_3+ab_2+ca_2+d(2a_{-2}a_4\\ &+2a_{-1}a_3+2a_0a_2+a_1^2)=0,\end{aligned}$$

which gives

$$-\frac{18}{3125dD^3}\mu^6+\frac{2}{5D}\mu^2\left(-\frac{c^2}{4d}+b\right)-\frac{ad}{6}=0, \tag{14}$$

from which we can easily obtain the expression of the planar wave speed in Tyson's model (see part 3 in the next section). For the rest of coefficients of Eq.(11), we have the following recursions:

$$\begin{aligned}a_n=&-\frac{1}{n(n-1)D+2da_{-2}}\Big\{[2da_{-1}+(n-1)\mu]a_{n-1}\\ &+ca_{n-2}+ab_{n-2}+2d\sum_{i+j=n-2}\epsilon'_{ij}a_ia_j\Big\},\end{aligned}$$

where $n>4, a_4$ is an arbitrary constant.

$$\epsilon'_{ij}=\begin{cases}0, & \text{if } i<0 \text{ or } j<0;\\ 1, & \text{if } i\neq j;\\ \dfrac{1}{2}, & \text{if } i=j.\end{cases}$$

For the algebraic equation (10) or (14) of the wave speed, we can use Cardan formula to find its explicit solution.

Ⅲ. THE FORMULA OF CALCULATION ON THE WAVE SPEED OF SOME TYPICAL EM

In this section, by applying the equations of wave speeds obtained above to some usual EM, we obtain their concrete solutions.

1. *Keener*[14] *studied the following model*:

$$\begin{cases}\epsilon\dfrac{\partial u}{\partial t}=\epsilon^2D\Delta u+f(u,v),\\ \dfrac{\partial v}{\partial t}=g(u,v),\end{cases} \tag{15}$$

where $f(u,v)=-A(u-u_0)(u-u_1)(u-u_2), u_0<u_1<u_2$. Let $x=-\dfrac{\mu}{\sqrt{2AD}}$. According to Eq.(10), we have

$$8x^3-6(p^2-3q)x+2p^3-9pq+27r=0, \tag{16}$$

where $p=u_0+u_1+u_2, q=u_0u_1+u_0u_2+u_1u_2, r=u_0u_1u_2$. By solving Eq.(16), we obtain three roots:

$$\begin{aligned}x_1=&\frac{2u_0-u_1-u_2}{2},\ x_2=\frac{2u_2-u_1-u_0}{2},\\ x_3=&-\frac{u_0-2u_1+u_2}{2}.\end{aligned}$$

Considering the boundary conditions, we have the formula of wave speed: $\mu = \sqrt{\frac{AD}{2}}(u_0 - 2u_1 + u_2)$ (in Ref.[14], Keener gave the form: $\mu = \frac{\sqrt{AD}}{2}(u_0 - 2u_1 + u_2)$, which is incorrect)

2. *FitzHung-Nagmo model*

FitzHung-Nagmo model has different forms, here we choose one of them[15] which has the form

$$\begin{cases} \epsilon\dfrac{\partial u}{\partial t} = \epsilon^2 D\Delta u + f(u,v), \\ \dfrac{\partial v}{\partial t} = g(u,v), \end{cases} \tag{17}$$

where $f(u,v) = u(1-u)(u-\alpha) - v, g(u,v) = u - v$. From Eq.(10), we have

$$x^3 - 3(\alpha^2 - \alpha + 1)x - [2(\alpha+1)^3 - 9\alpha(\alpha+1) + 27v_0] = 0, \tag{18}$$

where $x = -\dfrac{2\mu}{\sqrt{2D}}$. We omit the concrete formula of μ, because of its complication. Especially, when $v_0 = 0$, Eq.(18) has three roots

$$x_1 = 2 - \alpha,\ x_2 = 2\alpha - 1,\ x_3 = -\alpha - 1.$$

According to the boundary conditions, if $0 < \alpha < 1$, then the wave speed $\mu = \dfrac{1-2\alpha}{2}\sqrt{2D}$; if $\alpha > 1$, then $\mu = \dfrac{\alpha - 1}{2}\sqrt{2D}$; if $\alpha < 0$, then $\mu = \dfrac{1+\alpha}{2}\sqrt{2D}$.

3. *Tyson model*[16]

$$\begin{cases} \epsilon\dfrac{\partial u}{\partial t} = \epsilon^2 D\Delta u + f(u,v), \\ \dfrac{\partial v}{\partial t} = \epsilon D_1\Delta u + g(u,v), \end{cases} \tag{19}$$

where $f(u,v) = u(1-u) - fv\dfrac{u-h}{u+h}, g(u,v) = u - v$ and $0 < h \ll 1, 0 < f < +\infty$. According to Eq.(14), we have

$$-\frac{18}{3125D^3}\mu^6 + \frac{1 - 4fv_0}{10D}\mu^2 + \frac{fhv_0}{3} = 0. \tag{20}$$

If $h = 0$, then we have $\mu = \dfrac{5\sqrt{D}}{\sqrt{6}}(1 - 4fv_0)^{1/4}$.

4. *Ginzburg-Landau medol*[7]

Considering the equal diffusion and $\alpha + \beta = 1$, we have

$$\frac{\partial\alpha}{\partial t} = \Delta\alpha - \alpha(1-\alpha)^2 - k\alpha(1-\alpha). \tag{21}$$

In Ref.[7], the author discussed the problem of wave speed under the two cases: $0 \le k \le 1/2$ and $k > 1/2$, and through the numerical experiment, observed that for $0 \le k \le 1/(2\mu) = 1/\sqrt{2} + k\sqrt{2}$ and for $k > 1/2$, $\mu = 2\sqrt{k}$. Here according to the above idea, we have

$$\alpha'' + \mu\alpha' - \alpha^3 + (2+k)\alpha^2 - (k+1)\alpha = 0. \tag{22}$$

From Eq.(10), we can derive

$$x^3 - 3(k^2 + k + 1)x - (2k^3 + 3k^2 - 3k - 2) = 0, \tag{23}$$

where $x = -\sqrt{2}\mu$. Equation (23) has three roots:

$$x_1 = 1 - k,\ x_2 = 2k + 1,\ x_3 = -2 - k. \tag{24}$$

With this and according to the boundary conditions, we have for $k \ge 1$, $\mu = \sqrt{2} + k/\sqrt{2}$; for $0 < k \le 1$, $\mu = \sqrt{2}k + 1/\sqrt{2}$; for $k < 0$, $\mu = (1-k)/\sqrt{2}$. Thus it can be seen that the results in Ref.[7] are only partially correct.

IV. DISPERSIVE RELATION

After having known the wave speed, we can easily obtain the dispersive relation. According to Ref.[14], the period of the wave has the following expression:

$$T(v_0) = \int_{v_0}^{v_1}\left(\frac{1}{g(u_+(v),v)} - \frac{1}{g(u_-(v),v)}\right)\mathrm{d}v, \tag{25}$$

where $u_+(v)$ and $u_-(v)$ are the roots of $f(u,v) = 0$, and v_1 is determinited by $\mu(v_0) = -\mu(v_1)$, $v_0 \ne v_1$ (where if v_0 is regarded as a location of wave front, then v_1 is one of wave back). Because $u_+(v)$ and $u_-(v)$ are known, the relation between the period and wave speed can be explicitly obtained. For Tyson model[16], we have derived its concrete representation of the dispersive relation (here we will omit the corresponding complicated representation).

V. CONCLUSION

It is well known that the key step in determining the dispersive relation of propagating waves in EM is to know the expression of wave speed. In this paper we give a typical method for finding it. The similar method can be also used to seek planar wave solutions and their wave speeds in other nonlinear partial differential equations. Besides, according to the above procedures, we can also obtain the series representations of the planar wave solutions.

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