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SOME NEW AND EXACT EXPLICIT SOLUTIONS OF THE HIGHER ORDER BROER-KAUP EQUATIONS*

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A simple solitary wave solution and infinitely many rational function solutions, especially the closed form of the solution for the initial value problem of the higher order Broer-Kaup (BK) systems in (1+1) and (2+1) dimensions, are obtained by using the extended homogeneous balance method.

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I. INTRODUCTION

In this paper we will consider the higher-order Broer-Kaup (BK) systems in (1+1)-dimensions,

$$H_t = -4(H_{xx} + H^3 + 6HG - 3HH_x)_x, \quad (1)$$
$$G_t = -4(G_{xx} + 3G_xH + 3H^2G + 3G^2)_x, \quad (2)$$

and (2+1)-dimensions,

$$H_t = -4(H_{xx} + H^3 - 3HH_x + 3H\partial_z^{-1}G_x + 3\partial_z^{-1}(GH)_x)_x, \quad (3)$$
$$G_t = -4(G_{xx} + 3G_xH + 3H^2G + 3G\partial_z^{-1}G_x)_x, \quad (4)$$

where $\partial_z^{-1} = \int \mathrm{d}z$. We will construct some new and exact solutions of the above systems by using the extended homogeneous balance method.[1−3,5] In these solutions a simple solitary wave solution and infinitely many rational function solutions, especially, the closed form of the solution for the initial value problem of these systems are included.

These systems may be reduced from Kadomtsev-Petviashvili equation by using symmetry constraints.[4,5]

II. EQUATIONS THAT THE QUASI-SOLUTIONS SATISFY

A. Differential equation which the quasi-solution of the higher-order BK system in (1+1) dimensions satisfies

According to the extended homogeneous balance method, let

$$H(x,t) = f'(\varphi)\varphi_x + P(x,t), \quad (5)$$
$$G(x,t) = g^{(2)}(\varphi)\varphi_x^2 + g'(\varphi)\varphi_{xx} + Q(x,t), \quad (6)$$

where $P(x,t)$ and $Q(x,t)$ are known solutions of the system expressed by Eqs.(1) and (2). That is, $P(x,t)$ and $Q(x,t)$ satisfy

$$P_t + 4(P_{xx} + P^3 + 6PQ - 3PP_x)_x = 0, \quad (7)$$
$$Q_t + 4(Q_{xx} + 3Q_xP + 3P^2Q + 3Q^2)_x = 0. \quad (8)$$

Substituting Eqs.(5) and (6) into Eqs.(1) and (2), we obtain

$$\begin{aligned}
&H_t + 4(H_{xx} + H^3 + 6HG - 3HH_x)_x = (4f^{(4)} + 12f'^2f'' + 24f''g'' + 24f'g^{(3)} - 12f^{(3)}f' - 12f''^2)\varphi_x^4 \\
&+ (24f^{(3)}\varphi_x^2\varphi_{xx} + 12f'^3\varphi_x^3\varphi_{xx} + 72f'g''\varphi_x^2\varphi_{xx} + 24f''g'\varphi_x^2\varphi_{xx} + 24f'g''\varphi_x^2\varphi_{xx} - 60f''f'\varphi_x^2\varphi_{xx} \\
&+ 24f'f''P\varphi_x^3 + 24g^{(3)}P\varphi_x^3 - 12f^{(3)}P\varphi_x^3) \\
&+ (f''\varphi_x\varphi_t + 12f''\varphi_{xx}^2 + 16f''\varphi_x\varphi_{xxx} + 24f'^2\varphi_x\varphi_{xx}P + 12f'^2\varphi_x^2P_x + 24g''\varphi_x^2P + 72g''\varphi_x\varphi_{xx}P \\
&+ 24f'g'\varphi_{xx}^2 + 24f'g'\varphi_x\varphi_{xxx} + 14f''\varphi_x^2Q - 12f'^2\varphi_x\varphi_{xxx} - 12f'^2\varphi_{xx}^2 - 24f''\varphi_x^2P_x - 36f''\varphi_x\varphi_{xx}P) \\
&+ (\varphi_{xt}f' + 4\varphi_{xxxx}f' + 24P\varphi_{xxx}g' - 12P\varphi_{xxx}f' + 12P^2\varphi_{xx}f' + 24\varphi_xPP_xf' + 24\varphi_{xx}Qf' + 24\varphi_xQ_xf' \\
&- 24\varphi_{xx}P_xf' - 12\varphi_xP_{xx}f' + 24\varphi_{xx}P_xg') \\
&+ (P_t + 4P_{xxx} + 12P^2P_x + 24P_xQ + 24PQ_x - 12P_x^2 - 12PP_{xx}) \quad (9)
\end{aligned}$$

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$$
\begin{aligned}
&G+4(G_{xx}+3G_xH+3H^2G+3G^2)_x=(4g^{(5)}+12g^{(4)}f'+12g^{(3)}f''+24g''f''f'\\
&+12g^{(3)}f'^2+24g^{(3)}g'')\varphi_x^5+[(40g^{(4)}+108g^{(3)}g'+36g''f''+60f'^2g''+24f'f''g'+72g''^2)\varphi_x^3\varphi_{xx}\\
&+(12g^{(4)}+24f''g''+24f'g^{(3)})P\varphi_x^4]+[g^{(3)}\varphi_x^2\varphi_t+(60g^{(3)}+72g''f'+24f'^2g'+72g'g'')\varphi_x\varphi_{xx}^2\\
&+(40g^{(3)}+48g''f'+12g'f''+12f'^2g'+24g''g')\varphi_x^2\varphi_{xxx}+(72g^{(3)}+96f'g''+24f''g')\varphi_x^2\varphi_{xx}P\\
&+(12g^{(3)}+24f'g'')\varphi_x^3P_x+12g^{(3)}\varphi_x^3P^2+(24f'f''+24g^{(3)})\varphi_x^3Q]+[g''\varphi_{xx}\varphi_t+2g''\varphi_x\varphi_{xt}\\
&+(40g''+12g'f'+24g'^2)\varphi_{xx}\varphi_{xxx}+(20g''+12f'g')\varphi_x\varphi_{xxxx}+(36g''+24f'g')\varphi_{xx}^2P\\
&+(48g''+24f'g')\varphi_x\varphi_{xxx}P+(36g''+24f'g')\varphi_x\varphi_{xx}P_x+(12f''+12f'^2+24g'')\varphi_x^2Q_x\\
&+24g''\varphi_x^2PP_x+36g''\varphi_x\varphi_{xx}P^2+(24f'^2+72g'')\varphi_x\varphi_{xx}Q+24f''\varphi_x^2PQ]\\
&+[g'\varphi_{xxt}(12\varphi_xQ_{xx}+12\varphi_{xx}Q_x+24\varphi_xPQ_x+24\varphi_{xx}PQ+24\varphi_xQP_x)f'\\
&+(4\varphi_{xxxxx}+12\varphi_{xxxx}P+12\varphi_{xxx}P_x+24\varphi_{xx}PP_x+12\varphi_{xxx}P^2+24\varphi_{xxx}Q+24\varphi_{xx}Q_x)g']\\
&+(Q_t+4Q_{xxx}+12PQ_{xx}+12PQ_x+12Q_xP^2+24QPP_x+24QQ_x) \qquad (10)
\end{aligned}
$$

Setting the coefficients of φ_x^4 in Eq.(9) and φ_x^5 in Eq.(10) to zero yields an ordinary differential system for $f(\varphi)$ and $g(\varphi)$,

$$4f^{(4)}+12f'^2f''+24f''g''+24f'g^{(3)}-12f^{(3)}f'-12f''^2=0, \qquad (11)$$

$$4g^{(5)}+12g^{(4)}f'+12g^{(3)}f''+24g''f''f'+12g^{(3)}f'^2+24g^{(3)}g''=0. \qquad (12)$$

There exist special solutions for Eqs.(11) and (12):

$$f(\varphi)=g(\varphi)=\ln(\varphi). \qquad (13)$$

Thereby

$$f'^4=-\frac{1}{6}f^{(4)},f''^2=-\frac{1}{6}f^{(4)},f'f^{(3)}=-\frac{1}{3}f^{(4)},f'^2f''=\frac{1}{6}f^{(4)},f'^3=\frac{1}{2}f^{(3)},f'f''=-\frac{1}{2}f^{(3)},f'^2=-f^{(2)}. \qquad (14)$$

Using Eqs.(13) and (14), expressions (9) and (10) can be simplified as

$$
\begin{aligned}
&H_t+4(H_{xx}+H^3+6HG-3HH_x)_x=(\varphi_x\varphi_t+4\varphi_x\varphi_{xxx}+12\varphi_x\varphi_{xx}P-12\varphi_x^2P_x+12\varphi_x^2P^2+24\varphi_x^2Q)f^{(2)}\\
&+(\varphi_{xt}+4\varphi_{xxxx}+12\varphi_{xxx}P+12\varphi_{xx}P^2+24\varphi_xPP_x+24\varphi_{xx}Q+24\varphi_xQ_x-12\varphi_xP_{xx})f', \qquad (15)\\
&G_t+4(G_{xx}+3G_xH+3H^2G+3G^2)_x=(\varphi_x^2\varphi_t+4\varphi_x^2\varphi_{xxx}+12\varphi_x^2\varphi_{xx}P+12\varphi_x^3P^2+12\varphi_x^3Q)f^{(3)}\\
&+(\varphi_{xx}\varphi_t+2\varphi_x\varphi_{xt}+4\varphi_{xx}\varphi_{xxx}+8\varphi_x\varphi_{xxxx}+12\varphi_{xx}^2P+24\varphi_x\varphi_{xxx}P+12\varphi_x\varphi_{xx}P_x+24\varphi_x^2Q_x+24\varphi_x^2PP_x\\
&+36\varphi_x\varphi_{xx}P^2+24\varphi_x^2PQ+48\varphi_x\varphi_{xx}Q)f''\\
&+(\varphi_{xxt}+4\varphi_{xxxxx}+12\varphi_xQ_{xx}+12\varphi_{xxxx}P+12\varphi_{xxx}P_x+12\varphi_{xx}Q_x+24\varphi_{xx}PP_x+12\varphi_{xxx}P^2+24\varphi_xPQ_x\\
&+24\varphi_{xx}PQ+24\varphi_xP_xQ+24\varphi_{xxx}Q+24\varphi_{xx}Q_x)f'. \qquad (16)
\end{aligned}
$$

Letting the coefficients of the $f^{(3)},f'',f'$ in Eqs.(15), (16) to zero yields a set of partial differential equations for $\varphi(x,t)$,

$$\varphi_x\varphi_t+4\varphi_x\varphi_{xxx}+12\varphi_x\varphi_{xx}P-12\varphi_x^2P_x+12\varphi_x^2P^2+24\varphi_x^2Q=0, \qquad (17)$$

$$\varphi_{xt}+4\varphi_{xxxx}+12\varphi_{xxx}P+12\varphi_{xx}P^2+24\varphi_xPP_x+24\varphi_{xx}Q+24\varphi_xQ_x-12\varphi_xP_{xx}=0, \qquad (18)$$

$$\varphi_x^2\varphi_t+4\varphi_x^2\varphi_{xxx}+12\varphi_x^2\varphi_{xx}P+12\varphi_x^3P^2+12\varphi_x^3Q=0, \qquad (19)$$

$$
\begin{aligned}
&\varphi_{xx}\varphi_t+2\varphi_x\varphi_{xt}+4\varphi_{xx}\varphi_{xxx}+8\varphi_x\varphi_{xxxx}+12\varphi_{xx}^2P+24\varphi_x\varphi_{xxx}P+12\varphi_x\varphi_{xx}P_x\\
&+24\varphi_x^2Q_x+24\varphi_x^2PP_x+36\varphi_x\varphi_{xx}P^2+24\varphi_x^2PQ+48\varphi_x\varphi_{xx}Q=0, \qquad (20)\\
&\varphi_{xxt}+4\varphi_{xxxxx}+12\varphi_xQ_{xx}+12\varphi_{xxxx}P+12\varphi_{xxx}P_x+12\varphi_{xx}Q_x+24\varphi_{xx}PP_x+12\varphi_{xxx}P^2\\
&+24\varphi_xPQ_x+24\varphi_{xx}PQ+24\varphi_xP_xQ+24\varphi_{xxx}Q+24\varphi_{xx}Q_x=0. \qquad (21)
\end{aligned}
$$

Analyzing Eqs.(17)–(21), We find that when $\varphi_x \neq 0$, the independent equation in Eqs.(17)–(21) reads

$$\varphi_t + 12P^2\varphi_x + 12(P\varphi_x)_x + 4\varphi_{xxx} = 0. \tag{22}$$

And

$$P_x = Q. \tag{23}$$

Under the conditions of Eqs.(22) and (23), Eqs.(17)–(21) are satisfied automatically. Thereby, under the condition of $P_x = Q$, the quasi-solution $\varphi(x,t)$ is determined completely by Eq.(22).

B. Differential equation which the quasi-solution of the higher-order BK system in (2+1) dimensions satisfies

We suppose that the solution of Eqs.(3), (4) is of the form

$$H = f'\varphi_x + P(x,z,t), \tag{24}$$
$$G = g''\varphi_x\varphi_z + g'\varphi_{xz} + Q(x,z,t). \tag{25}$$

Through the discussion similar to the case of the higher-order BK system in (1+1) dimensions, we obtain

$$\varphi_t + 12P^2\varphi_x + 12(P\varphi_x)_x + 4\varphi_{xxx} = 0, \tag{26}$$
$$P_z = Q, \tag{27}$$

where $\varphi_{(x,z,t)}$ is the quasi-solution of the higher-order BK system in (2+1) dimensions, Eqs.(3), (4); $P(x,z,t), Q(x,z,t)$ are the known solutions of Eqs.(3) and (4), that is,

$$\begin{aligned} P_t =& -4(P_{xx} + P^3 - 3PP_x + 3P\partial_z^{-1}Q_x \\ &+ 3\partial_z^{-1}(QP)_x)_x, \end{aligned} \tag{28}$$
$$Q_t = -4(Q_{xx} + 3Q_xP + 3P^2Q + 3Q\partial_z^{-1}Q_x)_x. \tag{29}$$

And the new solution of Eqs.(3), (4) may be expressed as

$$H(x,z,t) = P(x,z,t) + \frac{\partial}{\partial x}\ln(\varphi), \tag{30}$$
$$G(x,z,t) = \frac{\partial}{\partial z}H(x,z,t). \tag{31}$$

III. NEW AND EXACT EXPLICIT SOLUTIONS OF THE HIGHER ORDER BK SYSTEM IN (1+1) DIMENSIONS

A. Solitary wave solution and the closed form of the solution for the initial value problem

It can be verified that there exists a special solution for Eqs.(7) and (8), that is

$$P_x = Q = 0, P = C_0 = \text{const}. \tag{32}$$

In this case, the solution of Eq.(22) with respect to solitary wave solution of Eqs.(1) and (2) reads

$$\varphi(x,t) = 1 + \exp(\alpha x + \beta t + \gamma). \tag{33}$$

Substituting Eq.(33) and $P_x = Q = 0, P = C_0 =$ const into Eqs.(5) and (6), and using $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain a solitary wave solution of the system (Eqs.(1) and (2)) as follows

$$\begin{aligned} H(x,t) &= \frac{1}{2}\alpha\tanh\left[\frac{1}{2}(\alpha x + \beta t + \gamma)\right] + \frac{1}{2}\alpha + C_0 \\ &= \frac{1}{2}\alpha + C_0 + \frac{\partial}{\partial x}\ln\varphi_1(x,t), \end{aligned} \tag{34}$$
$$\begin{aligned} G(x,t) &= \frac{1}{4}\alpha^2\sec h^2\left[\frac{1}{2}(\alpha x + \beta t + \gamma)\right] \\ &= \frac{\partial}{\partial x}H(x,t), \end{aligned} \tag{35}$$

where

$$\varphi_1(x,t) = \cosh\left[\frac{1}{2}(\alpha x + \beta t + \gamma)\right], \tag{36}$$
$$\beta = -12C_0^2\alpha - 12C_0\alpha^2 - 4\alpha^3, \tag{37}$$

and C_0, α and γ are arbitrary constants.

It is important that in the case of $P_x = Q = 0, P = C_0 =$ const, the solution of the initial value problem for the higher order BK system in (1+1) dimensions may be written as a closed form,

$$H(x,t) = C_0 + \frac{\partial}{\partial x}\ln[\varphi(x,t)], \tag{38}$$
$$G(x,t) = \frac{\partial}{\partial x}H(x,t), \tag{39}$$

where

$$\begin{aligned} \varphi(x,t) = \frac{\varphi(0,0)}{2\pi(12t)^{1/3}}\int_{-\infty}^{\infty}\exp\Bigg[\int_0^{\zeta}H(x,0)\mathrm{d}x - C_0\zeta \\ + C_0(x-\zeta) - 20C_0^3t\Bigg]\mathrm{Ai}\left(\frac{24C_1^2t - (x-\zeta)}{(12t)^{1/3}}\right)\mathrm{d}\zeta, \end{aligned} \tag{40}$$

and $\mathrm{Ai}(x)$ is the Airy function.

In fact, making Fourier transformation for Eq.(22) with respect to variable x, we obtain

$$\begin{aligned} &\Psi_t(k,t) + 12C_0^2\mathrm{i}k\,\Psi(k,t) - 12C_0k^2\,\Psi(k,t) \\ &- 4\mathrm{i}k^3\,\Psi(k,t) = 0, \end{aligned} \tag{41}$$

where

$$\Psi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x,t)\exp(-\mathrm{i}kx)\mathrm{d}x. \tag{42}$$

Integrating the ordinary differential equation (41) with respect to variable t, we obtain

$$\Psi(k,t) = \Psi(k,0)\exp[(4\mathrm{i}k^3 + 12C_0k^2 - 12C_0^2\mathrm{i}k)t], \tag{43}$$

where

$$\Psi(k,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\zeta,0)\exp(-\mathrm{i}kx)\mathrm{d}x. \quad (44)$$

Making the inverse Fourier transformation of $\Psi(k,t)$ with respect to x, we obtain

$$\varphi(x,t)=\frac{1}{2\pi(12t)^{1/3}}\int_{-\infty}^{\infty}\varphi(\zeta,0)\exp[C_0(x-\zeta)-20C_0^3t]\mathrm{Ai}\left(\frac{24C_1^2t-(x-\zeta)}{(12t)^{1/3}}\right)\mathrm{d}\zeta, \quad (45)$$

Starting from Eq.(5) and using $P(x,t)=C_0$, we obtain

$$\varphi(\zeta,0)=\varphi(0,0)\exp\left[\int_0^{\zeta}H(x,0)\mathrm{d}x-C_0\zeta\right]. \quad (46)$$

Substituting $\varphi(\zeta,0)$ into Eq.(45), we complete the proof of Eq.(40). Because $\varphi(0,0)$ does not appear in the solution (Eqs.(38) and (39)), the initial value problem of Eqs.(1) and (2) is determined completely besides containing an arbitrary constant C_0.

B. Rational function solutions

It can be verified that there is another special solution of Eqs.(7) and (8) as follows

$$P_x=Q=-\frac{C}{(x+x_0)^2},\ P=\frac{C}{x+x_0},\ C=\frac{3\pm1}{2}, \quad (47)$$

where x_0 is an arbitrary constant.

In order to solve Eq.(22), supposing

$$\varphi(x,t)=\sum_{n=-\infty}^{n=\infty}a_n(t)(x+x_0)^n, \quad (48)$$

and substituting $\varphi(x,t)$ and $P(x,t)$ in Eq.(47) into Eq.(22), we obtain

$$\sum_{n=-\infty}^{n=\infty}\left[\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}+12C^2a_{n+3}(t)(n+3)+12Ca_{n+3}(t)(n+3)(n+1)+4a_{n+3}(t)(n+3)(n+2)(n+1)\right](x+x_0)^n=0. \quad (49)$$

Setting the coefficient of $(x+x_0)^n$ for any n to zero yields a set of ordinary differential equations for $a_n(t)$

$$\frac{\mathrm{d}a_n(t)}{\mathrm{d}t}+12C^2a_{n+3}(t)(n+3)+12Ca_{n+3}(t)(n+3)(n+1)+4a_{n+3}(t)(n+3)(n+2)(n+1)=0. \quad (50)$$

In order to solve the set of ordinary differential equations, we must cut the series solution $\varphi(x,t)$ in Eq.(48) off at some n artificially. For example, if we let $a_n\equiv0$ (for $n\geq N, N$ is an arbitrary constant), we may solve the system as follows

$$\frac{\mathrm{d}a_N(t)}{\mathrm{d}t}=0, \quad (51)$$

$$\frac{\mathrm{d}a_{N-1}(t)}{\mathrm{d}t}=0, \quad (52)$$

$$\frac{\mathrm{d}a_{N-2}(t)}{\mathrm{d}t}=0, \quad (53)$$

$$\frac{\mathrm{d}a_{N-3}(t)}{\mathrm{d}t}+[12C^2N+12CN(N-2)+4N(N-1)(N-2)]a_N(t)=0, \quad (54)$$

$$\frac{\mathrm{d}a_{N-4}(t)}{\mathrm{d}t}+[12C^2(N-1)+12C(N-1)(N-3)+4(N-1)(N-2)(N-3)]a_{N-1}(t)=0, \quad (55)$$

$$\frac{\mathrm{d}a_{N-5}(t)}{\mathrm{d}t}+[12C^2(N-2)+12C(N-2)(N-4)+4(N-2)(N-3)(N-4)]a_{N-2}(t)=0, \quad (56)$$

.....

Solving of the above equations is direct and simple, the results read

$$a_N(t)=C_N, \quad (57)$$

$$a_{N-1}(t)=C_{N-1}, \quad (58)$$

$$a_{N-2}(t)=C_{N-2}, \quad (59)$$

$$a_{N-3}(t)=-[12C^2N+12CN(N-2)+4N(N-1)(N-2)]C_Nt+C_{N-3}, \quad (60)$$

$$a_{N-4}(t)=-[12C^2(N-1)+12C(N-1)(N-3)+4(N-1)(N-2)(N-3)]C_{N-1}t+C_{N-4}, \quad (61)$$

$$a_{N-5}(t)=-[12C^2(N-2)+12C(N-2)(N-4)+4(N-2)(N-3)(N-4)]C_{N-2}t+C_{N-5}, \quad (62)$$

.....

Substituting $a_N(t)$,$a_{N-1}(t)$, $a_{N-2}(t)$, $a_{N-3}(t)$, $a_{N-4}(t)$..... in Eqs.(57)–(62) and $a_n(t)$ (for $n\geq N$) into Eq.(48), we obtain

$$\varphi_N(x,t) = C_n(x+x_0)^N + C_{N-1}(x+x_0)^{N-1} + C_{N-2}(x+x_0)^{N-2} + \{-[12C^2N + 12CN(N-2) + 4N(N-1)(N-2)]C_N t + C_{N-3}\}(x+x_0)^{N-3} + \ldots \quad (63)$$

Substituting $\varphi_N(x,t)$ and $P(x,t), Q(x,t)$ in Eq.(47) into Eqs.(5) and (6) and using $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain rational solutions $H_N(x,t)$ and $G_N(x,t)$.

Because the terms reserved in the serious solution (Eq.(49)) are arbitrary, we may obtain arbitrary many solutions similar to those in Eq.(50).

IV. EXACT SOLUTIONS OF THE HIGHER-ORDER BK SYSTEM IN (2+1)-DIMENSIONS

It can be verified that there exists a special solution for Eqs.(28) and (29),

$$P(x,z,t) = P_0(z), \quad (64)$$

where $P_0(z)$ is an arbitrary function of z, so Eq.(26) may be written as

$$\varphi_t + 4\varphi_{xxx} + 12P_0(z)\varphi_{xx} + 12(P_0(z))^2\varphi_x = 0. \quad (65)$$

Solving of Eq.(65) is simple and direct. The result reads

$$\varphi(x,z,t) = 1 + \exp[\alpha(z)x + \beta(z)t + \gamma(z)], \quad (66)$$

where

$$\beta(z) = -4\alpha^3(z) - 12P(z)\alpha^2(z) - 12P^2(z)\alpha(z), \quad (67)$$

and $P(z), \alpha(z), \gamma(z)$ are arbitrary functions of z.

Through the discussion similar to the case of the higher-order BK system in (1+1)-dimensions, we obtain the closed form of the solution for the initial value problem of the higher-order BK system in (2+1)-dimensions.

$$H(x,z,t) = P_0(z) + \frac{\partial}{\partial x}\ln[\varphi(x,z,t)], \quad (68)$$

$$G(x,z,t) = \frac{\partial}{\partial z}H(x,z,t), \quad (69)$$

where

$$\varphi(x,z,t) = \frac{1}{2\pi(12t)^{1/3}}\int_{-\infty}^{\infty}(\zeta,0,0)\exp[(x-\zeta)P_0(z) - 20tP_0(z)^3]\mathrm{Ai}\left(\frac{24P_0(z)^2t - (x-\zeta)}{(12t)^{1/3}}\right)\mathrm{d}\zeta, \quad (70)$$

and $\mathrm{Ai}(x)$ is an Airy function.

Similarly, for the special solution of higher-order BK system in (2+1)-dimensions

$$P = \frac{C}{x + P_0(z)},\ P_z = Q,\ C = \frac{3 \pm 1}{2}. \quad (71)$$

We obtain arbitrary many solutions of Eq.(26) as follows

$$\varphi_N(x,z,t) = C_n[x+P_0(z)]^N + C_{N-1}[x+P_0(z)]^{N-1} + C_{N-2}[x+P_0(z)]^{N-2} + \{-[12C^2N + 12CN(N-2) + 4N(N-1)(N-2)]C_N t + C_{N-3}\}[x+P_0(z)]^{N-3} + \ldots, \quad (72)$$

where N is an arbitrary positive integer. Substituting this $\varphi_N(x,z,t)$ and $P(x,z,t), Q(x,z,t)$ in Eq.(71) into Eqs.(24) and (25) and using $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain the corresponding rational solutions $H_N(x,z,t), G_N(x,z,t)$.

V. CONCLUSION

A simple solitary wave solution and infinitely many rational function solutions, especially, the closed form of the solution for the initial value problem of the higher order BK system in (1+1) and (2+1)-dimensions, are obtained by using the extended homogeneous balance method.

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