Volume 9, Number 2
1009-1963/2000/09(02)/0131-07
CHINESE PHYSICS
February, 2000
© 2000 Chin. Phys. Soc.

QUANTUM GAP SOLITON IN PARAMETRIC PROCESS

Li Xiang(李 翔) and Guo Guang-can(郭光灿)

Center of Nonlinear Science, University of Science and Technology of China, Hefei 230026, *China*

(Received 22 July 1999)

The Hamiltonian of the process of cascaded second harmonic generation is found from Maxwell equations. In the double-gap model and under rotating-wave and effective-mass approximations, it is quantized and the generalized quantum nonlinear Shrödinger equation (GQNSE) is obtained. Tri-photon and quadri-photon bound states are found based on general solutions of GQNSE solved via Bethe's Ansatz method. Quantum parametric gap soliton (QPGS) solution is constructed consequently, and the existence of the double-gap QPGS is predicted for the first time.

PACC: 4250

I. INTRODUCTION

The integration of nonlinearity and periodicity, the two fundamental properties in physics, has brought to optics, as well as many other fields, plenty of interesting topics.[1−4] Gap soliton is a typical product of such an integration.[5] Since 1987, its first discovery in optics,[6] gap soliton has aroused wide interest in particle physics,[7] atom physics,[8,9] condensed matter physics,[10−12] electronic engineering,[13] and consequently, its concept has quickly spread into many branches of physics. The ubiquity of co-existence and cooperation between nonlinearity and periodicity as well as the abundant and unique properties of gap soliton will attract more people to further their research and application in this field.

Since the end of 80's, the study of soliton has entered the domain of quantum physics.[14,15] In 1995, Cheng et al.[16−18] proposed the theory of quantum gap soliton (QGS) in one-dimensional (ID) periodic dielectric based on Kerr effect for the first time. In this paper, we try to extend it into the domain of parametric dielectric, and find the quantum parametric gap soliton (QPGS) solution in the process of cascaded second harmonic generation.

We know in classic electrodynamics that the parametric counterparts of many Kerr effects do exist.[19−23] Recent discovery of spatial gap soliton due to cascaded second order nonlinearity[24−26] confirms our feeling that, though they are different physics processes, second order and Kerr nonlinearity can be described by similar mathematical structures under certain conditions. For example, under the condition of large phase mismatch, the process of cascaded second harmonic generation is completely controlled by the coupled modes equations which govern the process of Kerr gap soliton generation, except that $\chi^{(3)}_{\rm eff} \sim [\chi^{(2)}]^2$.[24] On the other hand, as stated above, studies in one-dimensional nonlinear periodic structure show that the quantum version of gap soliton does exist in the form of optical multiple exciton (OME). Therefore, we have reasons to believe that in the same structure with second order nonlinearity, the parametric counterpart of QGS can also exist and its dynamics should be similar to that of Kerr QGS except that the nonlinear coefficient $\chi^{(3)}_{\rm eff} \sim [\chi^{(2)}]^2$. In this paper, we will realize the above ideas and construct the QGS in the process of cascaded second harmonic generation.

In section Ⅱ, we start from the method proposed by Duan and Guo,[27] and find the Hamiltonian that controls the process of cascaded second order harmonics generation. In the double-gap model and under the rotating-wave and effective-mass approximations, we quantize the classical Hamiltonian and obtain the generalized quantum nonlinear Schrödinger equation (GQNSE). Subsequently, we use Bethe's ansatz method[28] to get the general solution of GQNSE in section Ⅲ, find the tri- and quadri-photon bound states corresponding to OME in section V, and based on them, construct the QGS solution directly according to Lai and Haus's methods[14,15,29] in section V. Finally in section Ⅵ, we briefly discuss the result we get from previous sections.

Ⅱ. CLASSICAL HAMILTONIAN AND ITS QUANTIZATION

A. Classical Hamiltonian

Now we consider the process of cascaded second harmonic generation. The field here is composed of

the fundamental and second harmonics

$$\boldsymbol{E}(t)=\sum_{i=1}^{2}\boldsymbol{E}_i\mathrm{e}^{-\mathrm{i}\omega_i t}+\mathrm{H.c.}, \tag{1}$$

where $\omega_2=2\omega_1$. For

$$\boldsymbol{D}_1=\epsilon_0\left(n_1^2\boldsymbol{E}_1+\chi_1^{(2)}:\boldsymbol{E}_1^+\boldsymbol{E}_2\right), \tag{2}$$

$$\boldsymbol{D}_2=\epsilon_0\left(n_2^2\boldsymbol{E}_2+\frac{1}{2}\chi_2^{(2)}:\boldsymbol{E}_1\boldsymbol{E}_1\right), \tag{3}$$

we get

$$\begin{aligned}\boldsymbol{E}_1=&\epsilon_0^{-1}n_1^{-2}\boldsymbol{D}_1-\epsilon_0^{-2}\Delta_1:\boldsymbol{D}_1^+\boldsymbol{D}_2\\&+\epsilon_0^{-3}n_2^2\Delta_1:\boldsymbol{D}_1^+(\Delta_2:\boldsymbol{D}_1\boldsymbol{D}_1)\\&+\epsilon_0^{-3}n_1^2\Delta_1:(\Delta_1^*:\boldsymbol{D}_1\boldsymbol{D}_2^+)\boldsymbol{D}_2, \qquad (4)\\ \boldsymbol{E}_2=&\epsilon_0^{-1}n_2^{-2}\boldsymbol{D}_2-\frac{1}{2}\epsilon_0^{-2}\Delta_2:\boldsymbol{D}_1\boldsymbol{D}_1\\&+\frac{1}{2}\epsilon_0^{-3}n_1^2\Delta_2:\boldsymbol{D}_1(\Delta_1:\boldsymbol{D}_1^+\boldsymbol{D}_2)\\&+\frac{1}{2}\epsilon_0^{-3}n_1^2\Delta_2:(\Delta_1:\boldsymbol{D}_1^+\boldsymbol{D}_2)\boldsymbol{D}_1. \qquad (5)\end{aligned}$$

So according to Maxwell equations and under the assumption of large phase mismatch, the Hamiltonian of the process of cascaded second harmonic generation is

$$\begin{aligned}H=&\int\mathrm{d}^3\boldsymbol{r}\ \mathcal{H}(\boldsymbol{r},t)=\int\boldsymbol{E}\mathrm{d}\boldsymbol{D}\\=&\int\mathrm{d}^3\boldsymbol{r}\Big\{\frac{1}{2}\sum_{i=1}^{2}\Big[\frac{1}{\epsilon_0\epsilon_i}(\boldsymbol{D}_i\mathrm{e}^{-\mathrm{i}\omega_i t}+\mathrm{H.c.})^2\\&+\frac{1}{\mu_0}(\boldsymbol{B}_i\mathrm{e}^{-\mathrm{i}\omega_i t}+\mathrm{H.c.})^2\Big]\\&+\Lambda_1^{(3)}::\boldsymbol{D}_1^+\boldsymbol{D}_1^+\boldsymbol{D}_1\boldsymbol{D}_1\\&+\Lambda_2^{(3)}::\boldsymbol{D}_2^+\boldsymbol{D}_1\boldsymbol{D}_1^+\boldsymbol{D}_2\Big\}, \qquad (6)\end{aligned}$$

where $\Lambda_1^{(3)}=\epsilon_0^{-3}n_1^2\Delta_2:\Delta_3$, $\Lambda_2^{(3)}=\epsilon_0^{-3}n_2^2\Delta_3:\Delta_1$,

$$\Delta_1=\frac{\chi^{(2)}(\omega_1=\omega_2-\omega_1)}{\chi_1^{(1)2}(\omega_1)\,\chi_2^{(1)}(\omega_2)}, \tag{7}$$

$$\Delta_2=\frac{\chi^{(2)}(\omega_2=2\omega_1)}{\chi_1^{(1)2}(\omega_1)\,\chi_2^{(1)}(\omega_2)}. \tag{8}$$

Finally in this subsection, we consider the Gaussian incident light

$$\boldsymbol{D}(\boldsymbol{r},t)=\sqrt{2/\pi\rho^2}\mathrm{e}^{-r^2/\rho^2}D(x,t)\hat{\boldsymbol{r}}, \tag{9}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\sqrt{2/\pi\rho^2}\mathrm{e}^{-r^2/\rho^2}B(x,t)\hat{\boldsymbol{\theta}}, \tag{10}$$

and the classical Hamiltonian for the process of cascaded second harmonic generation can be reduced into a 1D one

$$\begin{aligned}H=&\int\mathrm{d}x\Big[\sum_{i=1}^{2}\Big(\frac{1}{2\epsilon_0\epsilon_i}D_i^+D_i+\frac{1}{2\mu_0}B_i^+B_i\Big)\\&+\frac{1}{\pi\rho^2\epsilon_0^3\epsilon_1^3\epsilon_2^1}(\chi_1^{(3)}D_1^+D_1^+D_1D_1+\chi_2^{(3)}D_2^+D_1^+D_1D_2)\Big],\end{aligned} \tag{11}$$

where $\chi_1^{(3)}=\frac{1}{\epsilon_2}\chi_1^{(2)}:\chi_2^{(2)}$, $\chi_2^{(3)}=\frac{1}{\epsilon_1}\chi_2^{(2)}:\chi_1^{(2)}$.

B. Quantization

It is well known that in 1D periodic optical dielectric, the spectrum of frequency is divided into gaps and bands. In each band, a set of eigen-waves form an orthonormal complete set. Therefore, any wave propagating in the band can be expressed as a linear combination of them, i.e.,

$$D_\alpha(x,t)=\sum_k\mathcal{A}_{\alpha k}(t)\mathcal{D}_{\alpha k}(x), \tag{12}$$

$$B_\alpha(x,t)=\sum_k\mathcal{A}_{\alpha k}(t)\mathcal{B}_{\alpha k}(x), \tag{13}$$

and

$$\int\mathrm{d}x\epsilon^{-1}\mathcal{D}^*_{\alpha'k'}(x)\mathcal{D}_{\alpha k}(x)=\hbar\omega_{\alpha k}\epsilon_0\delta_{\alpha\alpha'}\delta_{kk'}, \tag{14}$$

$$\int\mathrm{d}x\mathcal{B}^*_{\alpha'k'}(x)\mathcal{B}_{\alpha k}(x)=\hbar\omega_{\alpha k}\mu_0\delta_{\alpha\alpha'}\delta_{kk'}, \tag{15}$$

where k and $\omega_{\alpha k}$ satisfy dispersion relationship

$$\omega_{\alpha k}=\omega_{\alpha k}(k_\alpha), \tag{16}$$

in which α is the label of bands, $\mathcal{D}_{\alpha k}(x),\mathcal{B}_{\alpha k}(x)$ are normalized Bloch functions, and $\mathcal{A}_{\alpha k}(t)$ are corresponding amplitudes.

In this paper, we assume that the incident light and its second harmonic lie in different gaps of the periodic structure. So, we have

$$D_i(x,t)=\sum_\alpha\sum_k\mathcal{A}_{i\alpha k}(t)\mathcal{D}_{i\alpha k}(x), \tag{17}$$

$$B_i(x,t)=\sum_\alpha\sum_k\mathcal{A}_{i\alpha k}(t)\mathcal{B}_{i\alpha k}(x), \tag{18}$$

where $i=1,2$ is the label of gaps corresponding to the first and second harmonics, $\alpha=+,-$ represents the valence and conducting bands of each gap, respectively. Considering the assumption of large phase mismatch which implies that the intensity of the second harmonic is much weaker than that of the fundamental harmonic, we make a transformation here. Let $\mu_1=\sqrt{I_2/I_1}\ll 1,\mu_2=1$, we define that

$$D_{i\alpha k}(x)=\mu_i^{-1}\mathcal{D}_{i\alpha k}(x), \tag{19}$$

$$B_{i\alpha k}(x)=\mu_i^{-1}\mathcal{B}_{i\alpha k}(x), \tag{20}$$

$$a_{i\alpha k}(t)=\mu_i\mathcal{A}_{i\alpha k}(t). \tag{21}$$

Then

$$D_i(x,t)=\sum_{\alpha}\sum_{k}a_{i\alpha k}(t)D_{i\alpha k}(x), \tag{22}$$

$$B_i(x,t)=\sum_{\alpha}\sum_{k}a_{i\alpha k}(t)B_{i\alpha k}(x). \tag{23}$$

As $\mathcal{D}_{1\alpha k}(x)$ and $\mathcal{D}_{2\alpha k}(x)$ are of the same order, $D_{1\alpha k}(x)\gg D_{2\alpha k}(x)$. On the other hand, this definition minimizes $A_{1\alpha k}$ by μ times i.e., $a_{1\alpha k}$ corresponds to a quantity (operator) of a virtual particle which consists of μ^{-2} photons.

By substituting Eqs.(22) and (23) into the classical Hamiltonian (11), considering a and a^+ as operators that satisfy the Bose commutation relationship

$$[a_{i\alpha k},a^+_{j\beta k'}]=\delta_{ij}\delta_{\alpha\beta}\delta_{kk'}, \tag{24}$$

and using rotating-wave approximation, we find the original Hamiltonian takes the form of

$$H=H_0+H_1+H_2, \tag{25}$$

$$H_0=\sum_{i=1}^{2}\sum_{\alpha=_-^+}\sum_{k}\hbar\omega_{i\alpha k}a^+_{i\alpha k}a_{i\alpha k}, \tag{26}$$

$$\begin{aligned}H_1=&\frac{1}{L}\sum_{\alpha=_-^+}\sum_{k,k',q}I_{\alpha q}(k,k')a^+_{1\alpha,k'+q}a^+_{1\alpha,-k'}a_{1\alpha,-k}a_{1\alpha,k+q}\\&+\frac{1}{L}\sum_{k,k',q}I_{iq}(k,k')a^+_{1-,k'+q}a^+_{1+,-k'}a_{1+,-k}a_{1-,k+q},\end{aligned} \tag{27}$$

$$H_2=\frac{1}{L}\sum_{\alpha,\beta=_-^+}\sum_{k,k',q}V_{\alpha\beta q}(k,k')a^+_{1\alpha,k'+q}a^+_{2\beta,-k'}a_{2\beta,-k}a_{1\alpha,k+q}, \tag{28}$$

where

$$I_{\alpha,iq}(k,k')=\frac{4}{(3\delta_{\alpha,\alpha'}+1)}\frac{lN^2}{\pi\rho^2\epsilon_0^3}\int_0^l\frac{\chi_1^{(3)}}{\epsilon_1^3\epsilon_2^1}D^*_{1\alpha',k'+q}D^*_{1\alpha,-k'}D_{1\alpha,-k}D_{1\alpha',k+q}\mathrm{d}x, \tag{29}$$

$$V_{\alpha\beta q}(k,k')=\frac{lN^2}{\pi\rho^2\epsilon_0^3}\int_0^l\frac{\chi_2^{(3)}}{\epsilon_1^3\epsilon_2^1}D^*_{2\alpha,k'+q}D^*_{1\beta,-k'}D_{1\beta,-k}D_{2\alpha,k+q}\mathrm{d}x. \tag{30}$$

Here considering only the k values in the vicinity of the origin, and resetting the coordinate by letting the origin locate at the minimum or maximum of the dispersion curve, we make the effective-mass approximation,

$$\omega_{i\alpha k}=\frac{\hbar^2k_{i\alpha}^2}{2m_{i\alpha}}, \tag{31}$$

where

$$m_{i\alpha}=\left(\frac{2}{\hbar}\frac{\partial\omega_{i\alpha k}}{\partial k_{i\alpha}}\right)^{-1}\Big|_{k_{i\alpha}=0}. \tag{32}$$

Then we perform the Fourier transformation of a,a^+ from the wave vector space to coordinate space:

$$\phi_{i\alpha}(x)=\frac{1}{\sqrt{L}}\int\frac{\mathrm{d}k}{2\pi}a_{i\alpha,k}\mathrm{e}^{\mathrm{i}kx}. \tag{33}$$

Noting

$$a_{i\alpha,k}=\frac{1}{\sqrt{L}}\int\mathrm{d}x\phi_{i\alpha}(x)\mathrm{e}^{-\mathrm{i}kx} \tag{34}$$

and the Bose commutation relationship of a,a^+, it is easy to see that

$$[\phi_{i\alpha}(x),\phi_{j\beta}(y)]=\delta_{ij}\delta_{\alpha\beta}\delta(x-y). \tag{35}$$

Then we can finally get the quantized Hamiltonian expressed by field operators

$$H=H_{\mathrm{i}}+H_{\mathrm{v}}, \tag{36}$$

$$H_{\mathrm{i}}=\sum_{i=1}^{2}\sum_{\alpha=_-^+}\hbar\omega_{i\alpha}N_{i\alpha}, \tag{37}$$

$$H_{\mathrm{v}}=H_0+H_{\mathrm{I}}+H_{\mathrm{V}}, \tag{38}$$

$$H_0=\sum_{i=1}^{2}\sum_{\alpha=_-^+}\int\mathrm{d}x\frac{\hbar^2}{2m_{i\alpha}}\partial_x\phi^+_{i\alpha}\partial_x\phi_{i\alpha}, \tag{39}$$

$$H_{\mathrm{I}}=\int\mathrm{d}x\left(\sum_{\alpha=_-^+}I_\alpha\phi^+_{1\alpha}\phi^+_{1\alpha}\phi_{1\alpha}\phi_{1\alpha}+I_i\phi^+_{1-}\phi^+_{1+}\phi_{1+}\phi_{1-}\right), \tag{40}$$

$$H_{\mathrm{V}}=\sum_{\alpha,\beta=_-^+}\int\mathrm{d}xV_{\alpha\beta}\phi^+_{2\alpha}\phi_{2\alpha}\phi^+_{1\beta}\phi_{1\beta}, \tag{41}$$

where H_{i} is the invariant Hamiltonian which is invariant when the photon number of each band is given, and H_{v} is the variable Hamiltonian which consists of H_0, the Hamiltonian of kinetic energy, H_{I}, the interaction Hamiltonian of the bands neighboring the first

gap, and $H_{\rm V}$, the interaction Hamiltonian between bands of the first and second gaps. Here the interaction potentials are defined as

$$I_\alpha = I_{\alpha 0}, \tag{42}$$

$$I_i = I_{i0}, \tag{43}$$

$$V_{\alpha\beta} = V_{\alpha\beta 0}. \tag{44}$$

Considering the intensity of the interaction is proportional to the inverse of the distance between different bands, we can assume without losing the generality that $|V_{-+}| \ll |V_{++}|, |V_{--}| \ll |V_{+-}|$.

Ⅲ. BETHE'S ANSATZ SOLUTION

Firstly, we consider the basic quantum nonlinear Shrödinger equation (QNSE)

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = H_0\,\Psi, \tag{45}$$

where

$$H_0 = \int \mathrm{d}x \left(\frac{\hbar^2}{2m}\partial\phi^+\partial\phi + V\phi^+\phi^+\phi\phi\right). \tag{46}$$

Let

$$\Psi(t) = \Psi \mathrm{e}^{-\mathrm{i}Et/\hbar}, \tag{47}$$

and substitute it into Eq.(45), we obtain the stationary QNSE

$$H\,\Psi = E\,\Psi. \tag{48}$$

This equation has N-body Bethe's ansatz solution[28]

$$|\,\Psi_N(k)\rangle = \int \prod_{i=1}^{N} [\phi^+(x_i)\mathrm{e}^{\mathrm{i}k_i x_i}\mathrm{d}x_i] \times \prod_{i<j}^{N}\left[1 - \frac{\mathrm{i}2V}{K_i - K_j}\mathrm{sgn}(x_i - x_j)\right]|0\rangle, \tag{49}$$

where

$$K_i = \frac{\hbar^2 k_i}{m}, \tag{50}$$

$$E = \sum_{i=1}^{N}\frac{\hbar^2 k_i^2}{2m}. \tag{51}$$

In the case of GQNSE

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = H_{\rm v}\,\Psi, \tag{52}$$

we follow the method proposed by Cheng, and get the eigen-function of GQNSE as

$$\begin{aligned}|\,\Psi_{N_{1_-^+},N_{2_-^+}}(k)\rangle = \int & \prod_{\alpha=_-^+}\prod_{i,j=1,i<j}^{N_{1\alpha}}\left(1 - \frac{\mathrm{i}2I_\alpha}{K_{1\alpha i} - K_{1\alpha j}}\mathrm{sgn}(x_{1\alpha i} - x_{1\alpha j})\right) \\ & \times \prod_{i=1}^{N_{1-}}\prod_{j=1}^{N_{1+}}\left(1 - \frac{\mathrm{i}I_i}{K_{1-i} - K_{1+j}}\mathrm{sgn}(x_{1-i} - x_{1+j})\right) \\ & \times \prod_{\alpha,\beta=_-^+}\prod_{i=1}^{N_{2\alpha}}\prod_{j=1}^{N_{1\beta}}\left(1 - \frac{\mathrm{i}V_{\alpha\beta}}{K_{2\alpha i} - K_{1\beta j}}\mathrm{sgn}(x_{2\alpha i} - x_{1\beta j})\right) \\ & \times \prod_{i=1}^{2}\prod_{\alpha=_-^+}\prod_{j=1}^{N_{i\alpha}}[\phi^+_{i\alpha}(x_{i\alpha j})\mathrm{e}^{\mathrm{i}k_{i\alpha j}x_{i\alpha j}}\mathrm{d}x_{i\alpha j}]|0\rangle,\end{aligned} \tag{53}$$

where

$$K_{i\alpha j} = \frac{\hbar^2 k_{i\alpha j}}{m_{i\alpha}}, \tag{54}$$

and

$$E = \sum_{i=1}^{2}\sum_{\alpha=_-^+}\sum_{j=1}^{N_{i\alpha}}\frac{\hbar^2 k_{i\alpha j}^2}{2m_{i\alpha}}. \tag{55}$$

Ⅳ. BOUND STATES

A. Tri-photon Solution

In previous section, we obtain the general solution of GQNSE which is satisfactory mathematically, but too complicated to allow physical interpretations. We therefore go to simplify it by discussing its physical significance, and find its bound states.

Under the assumption of large phase mismatch, i.e., $D_2 \ll D_1$, which implies that $V \ll I$ and $H_{\rm V} \ll H_{\rm I}$, we try to find bound states in the solutions of $H_{\rm v}$ here. In the first step, we only consider the bound states of $H'_{\rm v} = H_0 + H_{\rm I}$. Its eigenfunction is bound state only when

$$I_i^* = -\mathrm{sgn}(m_{1-} + m_{1+})I_i \tag{56}$$

is negative i.e., I_i^* is attractive.[17] When $I_i < 0$, both I_α and $V_{\alpha\beta}$ are less than zero, if the bound states of $H'_{\rm v}$ do exist, we must have $(m_{1-} + m_{1+}) < 0$. Consider the effective potential between bands in different gaps

$$V_{++}^* = -\mathrm{sgn}(m_{2+} + m_{1+})V_{++}, \tag{57}$$

$$V^*_{+-} = -\mathrm{sgn}(m_{2+} + m_{1-})V_{+-}, \tag{58}$$

$$V^*_{-+} = -\mathrm{sgn}(m_{2-} + m_{1+})V_{-+}, \tag{59}$$

$$V^*_{--} = -\mathrm{sgn}(m_{2-} + m_{1-})V_{--}. \tag{60}$$

Noting that $m_{i+} < 0$, $m_{i-} > 0$ and $|m_{2\alpha}| > |m_{1\alpha}|$, we find $V^*_{++}, V^*_{+-} < 0, V^*_{--} > 0$. For $|V^*_{-+}| \ll |V^*_{--}|$, whether V^*_{-+} is positive or negative, only photons in the second valence band could be bound to the photons in the first gap. Therefore, we disconsider those terms which are irrelevant to H_{I} and V^*_{+-}, and simplify the general solution into the form of

$$\begin{aligned}|\Psi_{N_{1^+_-},N_{2+}}(k)\rangle = &\int \prod_{\alpha=-}^{+} \prod_{i,j=1,i<j}^{N_{1\alpha}} \left(1 - \frac{\mathrm{i}2I_\alpha}{K_{1\alpha i} - K_{1\alpha j}}\mathrm{sgn}(x_{1\alpha i} - x_{1\alpha j})\right) \\ &\times \prod_{i=1}^{N_{1+}} \prod_{j=1}^{N_{1-}} \left(1 - \frac{\mathrm{i}I_i}{K_{1+i} - K_{1-j}}\mathrm{sgn}(x_{1+i} - x_{1-j})\right) \\ &\times \prod_{i=1}^{N_{2+}} \prod_{j=1}^{N_{1-}} \left(1 - \frac{\mathrm{i}V_{+-}}{K_{2+i} - K_{1-j}}\mathrm{sgn}(x_{2+i} - x_{1-j})\right) \prod_{\alpha=-}^{+} \prod_{j=1}^{N_{1\alpha}} [\phi^+_{1\alpha}(x_{1\alpha j})\mathrm{e}^{\mathrm{i}k_{1\alpha j}x_{1\alpha j}}\mathrm{d}x_{1\alpha j}] \\ &\times \prod_{i=1}^{N_{2+}} [\phi^+_{2+}(x_{2+i})\mathrm{e}^{\mathrm{i}k_{2+i}x_{2+i}}\mathrm{d}x_{2+i}]|0\rangle. \end{aligned} \tag{61}$$

Besides real values, k could be complex. Let

$$N_{1\alpha} = N_{2+} = N, \tag{62}$$

$$K_{1+j} = \frac{\hbar^2}{M}K + \mathrm{i}(V + I_i) - \mathrm{i}I_-(N - 2j + 1), \tag{63}$$

$$K_{1-j} = \frac{\hbar^2}{M}K + \mathrm{i}V - \mathrm{i}I_-(N - 2j + 1), \tag{64}$$

$$K_{2+j} = \frac{\hbar^2}{M}K + \mathrm{i}\,(V + V_{+-}) - \mathrm{i}I_-(N - 2j + 1), \tag{65}$$

where

$$M = \sum_{\alpha=-}^{+} m_{1\alpha} + m_{2+}, \tag{66}$$

$$K = \frac{1}{N}\sum_{j=1}^{N} k_j = \frac{1}{N}\sum_{j=1}^{N}\left[\sum_{\alpha=-}^{+}\frac{m_{1\alpha}}{\hbar^2}K_{1\alpha j} + \frac{m_{2+}}{\hbar^2}K_{2+j}\right], \tag{67}$$

$$V = -\frac{1}{M}(m_{1+}I_i + m_{2+}V_{+-}), \tag{68}$$

we get the tri-photon bound-state solution

$$\begin{aligned}|\Psi_{3N}(K)\rangle = &A_{3N}\int \exp\left\{\mathrm{i}\frac{K}{M}\sum_{j=1}^{N}\left[\sum_{\alpha=-}^{+} m_{1\alpha}x_{1\alpha j} + m_{2+}x_{2+j}\right] - \frac{1}{\hbar^2}\sum_{j=1}^{N}[|m_{1+}(V + I_i)||x_{1+j} - x_{1-j}|\right. \\ &+ |m_{2+}(V + V_{+-})||x_{2+j} - x_{1-j}|] - \frac{1}{\hbar^2}|I_-|\sum_{i<j}^{N}[|m_{1+}||x_{1+i} - x_{1+j}| + |m_{1-}||x_{1-i} - x_{1-j}| \\ &\left. + |m_{2+}||x_{2+i} - x_{2+j}|]\right\} \prod_{\alpha=-}^{+}\prod_{j=1}^{N}[\phi^+_{1\alpha}(x_{1\alpha j})\mathrm{e}^{\mathrm{i}k_{1\alpha j}x_{1\alpha j}}\mathrm{d}x_{1\alpha j}]\prod_{i=1}^{N}[\phi^+_{2+}(x_{2+i})\mathrm{d}x_{2+i}]|0\rangle. \end{aligned} \tag{69}$$

B. Quadri-photon bound states

When $I_i > 0$, I_α and $V_{\alpha\beta} > 0$. If the bound states of H'_{v} exist, we must have $(m_{1-} + m_{1+}) > 0$. As in the previous subsection, we investigate the effective potential between band sides in different gaps (see Eqs.(57)–(60)). Considering $m_{i+} < 0$, $m_{i-} > 0$ and $|m_{2\alpha}| > |m_{1\alpha}|$, we find $V^*_{--} < 0, V^*_{-+}, V^*_{++} > 0$. For $|V^*_{-+}| \ll |V^*_{--}|$, photons in the second conducting band and the first gap are always bound together. When $m_{2+} + m_{1-} < 0$, i.e., $V^*_{+-} < 0$, because $|V^*_{++}| \ll |V^*_{+-}|$, photons in the second valence band and the first gap are repulsive to each other, we get the tri-photon solution similar to that we obtain in previous subsection, expect that the photons that are coupled to the first gap are from the second conduction band instead of the second valence band. But when $m_{2+} + m_{1-} > 0$, i.e., $V^*_{+-} > 0$, the photons in the second valence band are coupled to the first gap too. So, we dismiss the terms in H_{V} which are irrelevant to V^*_{+-}, V^*_{--}, and simplify the general solution into the form of

$$|\Psi_{N_{1_-^+},N_{2_-^+}}(k)\rangle=\int\prod_{\alpha=_-^+}\prod_{i,j=1,i<j}^{N_{1\alpha}}\left(1-\frac{\mathrm{i}2I_\alpha}{K_{1\alpha i}-K_{1\alpha j}}\mathrm{sgn}(x_{1\alpha i}-x_{1\alpha j})\right)$$
$$\times\prod_{i=1}^{N_{1+}}\prod_{j=1}^{N_{1-}}\left(1-\frac{\mathrm{i}I_i}{K_{1+i}-K_{1-j}}\mathrm{sgn}(x_{1+i}-x_{1-j})\right)\prod_{i=1}^{N_{2+}}\prod_{j=1}^{N_{1-}}\left(1-\frac{\mathrm{i}V_{+-}}{K_{2+i}-K_{1-j}}\mathrm{sgn}(x_{2+i}-x_{1-j})\right)$$
$$\times\prod_{i=1}^{N_{2-}}\prod_{j=1}^{N_{1-}}\left(1-\frac{\mathrm{i}V_{--}}{K_{2-i}-K_{1-j}}\mathrm{sgn}(x_{2-i}-x_{1-j})\right)\prod_{i=1}^{2}\prod_{\alpha=_-^+}\prod_{j=1}^{N_{i\alpha}}[\phi_{i\alpha}^+(x_{i\alpha j})\mathrm{e}^{\mathrm{i}k_{i\alpha j}x_{i\alpha j}}\mathrm{d}x_{i\alpha j}]|0\rangle. \quad (70)$$

Similarly, let

$$K_{1+j}=\frac{\hbar^2}{M}K+\mathrm{i}(V+I_i)+\mathrm{i}I_+(N-2j+1), \quad (71)$$

$$K_{1-j}=\frac{\hbar^2}{M}K+\mathrm{i}V+\mathrm{i}I_+(N-2j+1), \quad (72)$$

$$K_{2+j}=\frac{\hbar^2}{M}K+\mathrm{i}(V+V_{+-})+\mathrm{i}I_+(N-2j+1), \quad (73)$$

$$K_{2-j}=\frac{\hbar^2}{M}K+\mathrm{i}(V+V_{--})+\mathrm{i}I_+(N-2j+1), \quad (74)$$

where

$$M=\sum_{i=1}^{2}\sum_{\alpha=_-^+}m_{i\alpha}, \quad (75)$$

$$K=\frac{1}{N}\sum_{j=1}^{N}k_j=\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{2}\sum_{\alpha=_-^+}\frac{m_{i\alpha}}{\hbar^2}K_{i\alpha j}, \quad (76)$$

$$V=-\frac{1}{M}(m_{1+}I_i+m_{2+}V_{+-}+m_{2-}V_{--}), \quad (77)$$

we obtain the quardi-photon solution

$$|\Psi_{4N}(K)\rangle=A_{4N}\int\exp\left\{\mathrm{i}\frac{K}{M}\sum_{j=1}^{N}\sum_{i=1}^{2}\sum_{\alpha=_-^+}m_{i\alpha}x_{i\alpha j}-\frac{1}{\hbar^2}\sum_{j=1}^{N}|m_{1+}(V+I_i)||x_{1+j}-x_{1-j}|\right.$$
$$+|m_{2+}(V+V_{+-})||x_{2+j}-x_{1-j}|+|m_{2-}(V+V_{--})|$$
$$\times|x_{2-j}-x_{1-j}|-\frac{1}{\hbar^2}|I_+|\sum_{i<j}^{N}[|m_{1+}||x_{1+i}-x_{1+j}|$$
$$+|m_{1-}||x_{1-i}-x_{1-j}|+|m_{2+}||x_{2+i}-x_{2+j}|$$
$$\left.+|m_{2-}||x_{2-i}-x_{2-j}|]\right\}\prod_{i=1}^{2}\prod_{\alpha=_-^+}\prod_{j=1}^{N}[\phi_{i\alpha}^+(x_{i\alpha j})\mathrm{d}x_{i\alpha j}]|0\rangle. \quad (78)$$

What we want to stress here is that the so-called tri-photon and quardi-photon solutions are not genuinely composed of three or four photons. According to the definition in section Ⅱ, we know that a_1, a_1^+ are the annihilation and creation operators of virtual particles which represent a set of μ^{-2} photons. So, the real situation is that massive photons in the first valence and conducting band (instead of two photons) are bound together with one or two photons in the second gap and transmit through the nonlinear periodic dielectric, which is in consistence with the assumption of large phase mismatch.

V. QUANTUM GAP SOLITON SOLUTION

In accordance with the widely accepted method of constructing quantum solitons[14,15,29], we establish here the QPGS states by direct superposition of the bound states we obtained from previous sections

$$|\Phi_{\mathrm{p}}(x,t)\rangle=\sum_{N=1}^{\infty}c_{\mathrm{p}N}|\Phi_{\mathrm{p}N}(x,t)\rangle, \quad (79)$$

where

$$\sum_{N=1}^{\infty}|c_{\mathrm{p}N}|^2=1, \quad (80)$$

$$\Phi_{\mathrm{p}N}(x,t)=C_{\mathrm{p}N}\int_{-\infty}^{\infty}g_{\mathrm{p}N}(K)\exp[\mathrm{i}N(Kx-p\hbar\Omega_{\mathrm{p}N}t)]|\Psi_{\mathrm{p}N}(K)\rangle\mathrm{d}K. \quad (81)$$

In the above equation,

$$g_{\mathrm{p}N}(K)=\frac{1}{(\Delta K)^{1/2}\pi^{1/4}}\exp\left[-\frac{1}{2}\frac{K^2}{(\Delta K)^2}-\mathrm{i}NKX_0\right], \quad (82)$$

which satisfies

$$\int|g_{\mathrm{p}N}(K)|^2\mathrm{d}K=1, \quad (83)$$

$$\Delta K\ll 4K_0. \quad (84)$$

Noting that

$$|C_{\mathrm{p}N}|^2\langle\Psi_{\mathrm{p}N}(K')|\Psi_{\mathrm{p}N}(K)\rangle=\delta_{NN'}\delta(K-K'), \quad (85)$$

it is obvious that

$$\langle\Phi_{\mathrm{p}}(t)|\Phi_{\mathrm{p}}(t)\rangle=1. \quad (86)$$

Here, X_0 is the position of QGS at the zero time, $p = 3, 4$ represents tri- and quadri-photon OMEs respectively. K_0 is the positon of the origin in the wave vector space. ΔK represents the line width of $g_{\mathrm{p}N}(K)$. $\Omega_{\mathrm{p}N}$ is the effective frequency of the photons of the bound states. The photon number distribution could be obtained by experimentally measuring the spectrum of the output intensity.

$$I_{\mathrm{p}}(x,t) = \sum_{N=1}^{\infty} I_{\mathrm{p}}(x,t), \tag{87}$$

$$I_{\mathrm{p}}(x,t) = |c_{\mathrm{p}N}|^2 \langle \Phi_{\mathrm{p}N}(t)|E_{\mathrm{p}}^{+}(x)E_{\mathrm{p}}(x)|\Phi_{\mathrm{p}N}(t)\rangle, \tag{88}$$

where $E_3(x) = \prod_{i=1}^{2} \prod_{\alpha,\beta=\frac{+}{-}} \phi_{i\alpha}(x)$,

$$E_4(x) = \prod_{\alpha,\beta=\frac{+}{-}} \phi_{1\alpha}(x) + \phi_{2+}(x). \tag{89}$$

$I_{\mathrm{p}}(x,t)$ has the sec spatial shape of common solitons.

VI. CONCLUSION

The principles of QGS were established 4 years ago, but the existence of parametric QGS is still a question, even though its classical counterpart has been found for years. Compared with Cheng's result, we prove in the context of the second order nonlinear effect the existence of QGS, and extend the concept and the form of OME by finding the tri- and quardiphoton bound states generated by interaction between the fundamental and second harmonics. The establishment of the theory in this paper convinces us of the close intrinsic connection between the second and third order nonlinearity.

Our work shows that in the process of cascaded second harmonic generation, if the phase mismatch is large enough, behavior of photons in second-order nonlinear periodic structure will be similar to that in Kerr periodic dielectric–(both will form OMEs), and transmit through the structure with little impedance, which is forbidden in purely linear case. This proves once again the non-existence of the absolute border between parametric and Kerr processes.

However, noting that the effective Kerr coefficient of the cascaded second harmonic generation is proportional to the square of its second-order nonlinear coefficient which is much smaller than the nonlinear coefficient of common Kerr process, we find that our result loses the advantage of second-order nonlinear process, therefore is only of theoretical significance. For this reason, further research in general parametric quantum gap solitons is welcome.

ACKNOWLEDGMENTS

The authors are grateful to Professor Cheng Ze and Dr. Duan Lu-ming for helpful discussions.

REFERENCES

[1] A. Hasegawa, F. Tappert, *Appl. Phys. Lett.*, **23**(1973), 142.
[2] L. F. Mollenauer, R. H. Stolen and J. P. Gordon, *Phys. Rev. Lett.*, **45**(1980), 1095.
[3] J. Callaway, Quantum Theory of the Solid State (Academic, New York, 1974), p.409.
[4] Photonic Bandgaps and Localization, edited by C. Soukoulis (Plemun, New York, 1993).
[5] C. M. de Sterke and J. E. Sipe, *Progress in Optics* XXXIII, edited by E. Wolf (North-Holland, Amsterdam, 1994), Chap. III.
[6] W. Chen, D. L. Mills, *Phys. Rev. Lett.*, **58**(1987), 160.
[7] A. S. Belanov *et al., Izv. Akad. Nauk. Fiz.*, **60**(1996), 39.
[8] O. Zobay *et al., Phys. Rev.*, **A59**(1999), 643.
[9] V. I. Rupasov, M. Singh, *Phys. Rev. Lett.*, **77**(1996), 338.
[10] G. X. Huang *et al., Phys. Rev.*, **B55**(1997), 336.
[11] G. X. Huang *et al., Int. J. Bifurcat. Chaos*, **6**(1996), 1775.
[12] K. J. Kasunic, E. M. Wright, *J. Opt. Soc. Am.*, **B16**(1999), 96.
[13] B. Z. L. Ambassa, T. C. Kofane, *Physica*, **D106**(1997), 207.
[14] Y. Lai and H. A. Haus, *Phys. Rev.*, **A40**(1989), 844.
[15] Y. Lai and H. A. Haus, *Phys. Rev.*, **A40**(1989), 854.
[16] Z. Cheng, G. Kurizki, *Phys. Rev. Lett.*, **75**(1995), 3430.
[17] Z. Cheng and G. Kurizki, *Phys. Rev.*, **A54**(1996), 3576.
[18] Z. Cheng, *Commun. Theor. Phys.*, **30**(1998), 517.
[19] R. De Salvo, D. J. Hagen, M. Sheik-Bahae, G. I. Stegemen, E. W. Van Stryland, *Opt. Lett.*, **A17**(1992), 28.
[20] G. I. Stegemen *et al., Opt. Lett.*, **18**(1993), 13.
[21] P. Pliszka, P. P. Banerjee, *J. Mod. Opt.*, **40**(1993), 1909.
[22] R. Danlielius *et al., Opt. Lett.*, **18**(1993), 574.
[23] S. Nitti *et al., Opt. Commun.*, **106**(1994), 263.
[24] Y. S. Kivshar, *Phys. Rev.*, **E51**(1995), 1613.
[25] M. J. Werner, P. D. Drummond, *Opt. Lett.*, **19**(1994), 613.
[26] A. V. Buryak, Y. S. Kivshar, *Opt. Lett.*, **13**(1994), 1612.
[27] L. -M. Duan and G. C. Guo, *Phys. Rev.*, **A56**(1997), 925.
[28] H. B. Thacker, *Rev. Modern Phys.*, **53**(1981), 253.
[29] L. -M. Duan and G. C. Guo, *Phys. Rev.*, **A52**(1995), 874.