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GENERALIZED FRESNEL DIFFRACTION INTEGRAL AND ITS APPLICATIONS*

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Within the generalized definition of *ABCDGH* matrices, the conventional *ABCDGH* diffraction integral is extended for a misaligned complex paraxial system with a curved optical axis. On this basis the transformation law for Gaussian-Schell model beams through an *ABCDGH* systems is derived, the propagation behavior of the mutual coherence function through *ABCDGH* systems in the presence of random media is investigated as well.

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I. INTRODUCTION

As is well known, the Fresnel diffraction integral and the matrix-optics formalism are both useful for describing the beam propagation from one reference plane to another. In the past years, much effort has been made to combine the two analytical techniques in order to develop and improve the theory of beam propagation in paraxial optical systems. Baues[1] first related the diffraction theory to ray optics by showing that the diffraction integral can be written in terms of *ABCD* matrices. The extension of the above results to astigmatic systems was reported by Collins,[2] a compact form in terms of tensors was also presented later.[3] Next, several authors[4−6] independently proved that the *ABCD* diffraction integral is valid for complex-valued matrices. In addition, a 4×4 misalignment matrix and the corresponding diffraction integral have been developed to treat a paraxial system with misaligned optical elements.[7] Similarly, the misalignment diffraction integral is proper for complex-valued matrices.[8] Note that the 4×4 misalignment matrix is introduced by means of the coordinate transform, thus the misalignment diffraction integral cannot well determine the z-dependent phase (z denotes the propagation axis) and is limited to asymmetry owing to extrinsic misalignment. Recently, by using the separation of variables to solue the paraxial wave equation for a general complex lenslike medium including intrinsic asymmetry, Tovar and Casperson et al.[9] established a novel 3×3 *ABCDGH* matrix and obtained an exact expression for z-dependent phase. On the basis of Gaussian beamlets rather than Huygens' spherical wavelets, the Fresnel diffraction integral was extended so that it can be applied to the misaligned complex optical systems represented by 3×3 *ABCDGH* matrices.[10] In the present paper, we consider the effect of a curved optical axis by introducing an equivalent increment in complex propagation constant, and generalize the definition of the misalignment matrix elements G and H. From the results, we prove that the *ABCDGH* diffraction integral is also valid along a curved optical axis. Furthermore, two cases are investigated. First, the transformation law of Gaussian-Schell-model (GSM) beams through a complex *ABCDGH* systems is derived. Second, within the second-order Rytov approximation, an explicit expression for the mutual coherence function (MCF) propagating through complex *ABCDGH* systems in the presence of random media is obtained.

Note that in this analysis a basic assumption is that the profiles of gain (or loss) and refraction index have at most quadratic variables, which is at least a first-order approximation to any system having transverse variation in transmission.

II. A GENERALIZATION OF THE *ABCDGH* DIFFRACTION INTEGRAL

For simplicity, we restrict our discussion to one transverse dimension. Here we extend Tovar and Casperson's *ABCDGH* diffraction integral[10] for the general case of a curved optical axis, as illustrated in Fig.1. Let z denote the arc length along the axis of a curved medium and $C(z)$ the curvature. It is shown that within the paraxial approximation, a small curvature $C(z)$ is equivalent to an increment $k_0C(z)x$ in

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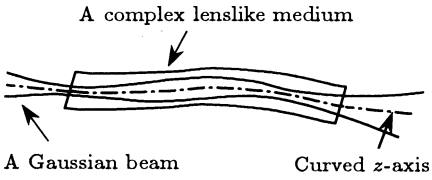


Fig.1. Illustration of a Gaussian beam through a complex lenslike medium with a curved z-axis.

propagation constant.[11] When k_0 is complex rather than real, the direct ray-optics interpretation for the additional increment in propagation constant, as in Arnaud's treatment,[11] seems no longer possible. Fortunately, Nazarathy and Shamir[5] showed that the Hamilton equation for first-order systems with gain or loss is in the same form as the conventional ray equation. In other words, the ray-optics solutions can be directly extended to complex-valued functions, known as complex rays. So, along a curved z-axis, the paraxial wave equation is approximated as

$$\frac{\partial^2 E(x,z)}{\partial x^2} - 2\mathrm{i}k_0\frac{\partial E(x,z)}{\partial z} + \left[k^2(x,z) - k_0^2(1-2C(z)x) - \mathrm{i}\frac{\mathrm{d}k_0}{\mathrm{d}z}\right]E(x,z) = 0, \quad (1)$$

where the complex propagation constant is of the form

$$k^2(x,z) = k_0^2(z) - k_0(z)(k_1(z)x + k_2(z)x^2). \quad (2)$$

For a Gaussian beam

$$E(x,z) = E_0\exp\left\{-\mathrm{i}\left(\frac{Q(z)}{2}x^2 + S(z)x + P(z)\right)\right\},$$

Eq.(1) may be separated into the set

$$Q^2 + k_0(z)\frac{\mathrm{d}Q}{\mathrm{d}z} + k_0(z)k_2(z) = 0, \quad (3a)$$

$$QS + k_0(z)\frac{\mathrm{d}S}{\mathrm{d}z} + \frac{k_0(z)k_1(z)}{2} - k_0^2(z)C(z) = 0, \quad (3b)$$

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{\mathrm{i}Q}{2k_0(z)} - \frac{S^2}{2k_0(z)} - \frac{\mathrm{i}}{2k_0(z)}\frac{\mathrm{d}k_0(z)}{\mathrm{d}z}. \quad (3c)$$

Equations (3a)–(3c) resemble the results of Ref.[10] except that there is an additional term $-k_0^2C(z)$ in Eq. (3b), which indicates the effect of a curved optical axis. Following the same procedures for analysis in Ref.[10], one can obtain a generalized beam matrix, in which $ABCD$ are the usual complex matrix elements for an alignment system as before, while G and H are written as

$$G(z) = \int_0^z\left(k_0(z')C(z') - \frac{k_1(z')}{2}\right)A(z')\mathrm{d}z', \quad (4a)$$

$$H(z) = \int_0^z\left(k_0(z')C(z') - \frac{k_1(z')}{2}\right)B(z')\mathrm{d}z', \quad (4b)$$

The results for lens waveguides and lenslike media with a curved axis[12] are merely particular cases of Eqs. (4a) and (4b). And the z-dependent phase $P(z)$ is in the same form as Ref.[10]. Based on the analysis above, it is indicated that the $ABCDGH$ diffraction integral is still valid along a curved optical axis but the elements G and H are defined by Eq.(4). That is,

$$E(x,z) = \sqrt{\frac{\mathrm{i}}{\lambda B}}\exp(-\mathrm{i}\varphi)\int \mathrm{d}\zeta E_{\mathrm{in}}(\zeta,z_0)\exp\left[-\mathrm{i}\frac{\pi}{\lambda B}(A\zeta^2 - 2\zeta x + Dx^2 + e\zeta + fx)\right], \quad (5)$$

where $e = \dfrac{\lambda}{\pi}(BG - AH)$, $\quad f = \dfrac{\lambda}{\pi}H$,

$$\varphi = -\frac{\lambda H}{4\pi}\left(\frac{BG - AH}{B}\right) + \frac{\lambda}{4\pi}\int_{z_0}^{z}\left(G\frac{\mathrm{d}H}{\mathrm{d}z'} - H\frac{\mathrm{d}G}{\mathrm{d}z'}\right)\mathrm{d}z' + k_0(z - z_0). \quad (6)$$

For a quasi-monochromatic field, the MCF is defined by

$$\Gamma(x_1,x_2;z) = <E(x_1,z)E^*(x_2,z)>, \quad (7)$$

where the angular brackets denote an ensemble average. Substitution of Eq. (5) into Eq. (7) yields

$$\Gamma(x_1,x_2;z) = \iint \mathrm{d}\zeta_1\mathrm{d}\zeta_2\mathrm{g}(x_1,\zeta_1) \times \mathrm{g}^*(x_2,\zeta_2)\Gamma_{\mathrm{in}}(\zeta_1,\zeta_2;z_0), \quad (8)$$

where $\Gamma_{\mathrm{in}}(\zeta_1,\zeta_2;z_0) = <E_{\mathrm{in}}(\zeta_1,z_0)E^*_{\mathrm{in}}(\zeta_2,z_0)>$, and g() is the kernel of the Fresnel diffraction integral written as

$$\mathrm{g}(x,\zeta) = \sqrt{\frac{\mathrm{i}}{\lambda B}}\exp(-\mathrm{i}\varphi)\exp\left[-\mathrm{i}\frac{\pi}{\lambda B}(A\zeta^2 - 2\zeta x + Dx^2 + e\zeta + fx)\right]. \quad (9)$$

Equation (8) governs the propagation rule of the MCF through any misaligned complex optical system represented by $ABCDGH$ matrices, which may be viewed as a generalization of the van Cittert-Zernike theorem. By means of the Wigner distribution function[13]

$$W(x,\theta) = \frac{1}{\lambda}\int \exp(-\mathrm{i}2\pi\theta s/\lambda)\Gamma(x + s/2, x - s/2)\mathrm{d}s, \quad (10)$$

the beam radius W and the far-field divergence Θ can be evaluated by

$$W^2 = 4\iint x^2 W(x,\theta)\mathrm{d}x\mathrm{d}\theta \Big/ \iint W(x,\theta)\mathrm{d}x\mathrm{d}\theta, \tag{11}$$

and

$$\Theta^2 = 4\iint \theta^2 W(x,\theta)\mathrm{d}x\mathrm{d}\theta \Big/ \iint W(x,\theta)\mathrm{d}x\mathrm{d}\theta. \tag{12}$$

Ⅲ. APPLICATIONS

A. Transformation of Gaussian schell-model beams through an arbitrary *ABCDGH* system

The Gaussian Schell-mode (GSM) beams, as representative of partially coherent beams, have been extensively investigated for many years.[14−16] Here we provide a more understandable derivation of the GSM beam transformation with the $ABCDGH$ diffraction integral as a mathematical tool, and get a more general result. A two-dimensional GSM source is characterized by the MCF of the form[14]

$$\Gamma_{\mathrm{in}}(\zeta_1,\zeta_2;z_0) = A_0 \exp\left(-\frac{\zeta_1^2+\zeta_2^2}{4\sigma_{\mathrm{i}0}^2} - \frac{(\zeta_1-\zeta_2)^2}{2\sigma_{\mathrm{c}0}^2}\right), \tag{13}$$

where A_0 is a constant, $\sigma_{\mathrm{i}0}^2$ and $\sigma_{\mathrm{c}0}^2$ are the variances of the intensity and the degree of coherence, respectively.

According to Ref.[14], the MCF may be represented by a mode expansion

$$\Gamma_{\mathrm{in}}(\zeta_1,\zeta_2;z_0) = \sum_n \alpha_n \phi_n(\zeta_1,z_0)\phi_n^*(\zeta_2,z_0). \tag{14}$$

Here, $\phi_n(\zeta)$ is the Hermite-Gaussian mode and

$$\alpha_n = \alpha_0 p^n, \tag{15}$$

where

$$\alpha_0 = \frac{\sqrt{8\pi}A_0\sigma_{\mathrm{i}0}}{M^2+1},\ p = \frac{M^2-1}{M^2+1},\ \text{and}\ M^2 = \sqrt{1+\frac{4\sigma_{\mathrm{i}0}^2}{\sigma_{\mathrm{c}0}^2}}. \tag{16}$$

It is remarked that M^2 is just the beam quality factor.[17] Substitution of Eq.(14) into Eq.(8) yields

$$\Gamma(x_1,x_2;z) = \sum_n \alpha_n \phi_n(x_1,z)\phi_n^*(x_2,z). \tag{17}$$

In deriving Eq.(17), we use the fact that the modes ϕ_n are uncorrelated and take the form

$$\phi_n(x,z) = \sqrt{\frac{\sqrt{2}}{\sqrt{\pi}2^n n!\omega_0 u}}\left(\frac{v}{u}\right)^{n/2} \mathrm{H}_n\left(\frac{\sqrt{2}(x-e/2)}{\omega_0\sqrt{uv}}\right)\exp\left\{-\mathrm{i}\frac{\pi}{\lambda}\left(\frac{x^2\tau}{u}+\frac{x}{B}\left(\frac{e}{u}+f\right)\right)\right\}\exp(-\mathrm{i}\varphi'), \tag{18}$$

where $\omega_0 = 2\sigma_{\mathrm{i}0}/M$, $q_0 = \mathrm{i}\pi\omega_0^2/\lambda$,

$$u = A + B/q_0,\ v = A + B/q_0^*,\ \tau = C + D/q_0,\ \text{and}\ \varphi' = \varphi - \pi e^2/4\lambda B u. \tag{19}$$

With the relation[14]

$$\sum_n \frac{\mathrm{H}_n(x_1)\mathrm{H}_n(x_2)t^n}{n!2^n} = \frac{\exp\left\{\dfrac{2x_1x_2t-(x_1^2+x_2^2)t^2}{1-t^2}\right\}}{\sqrt{1-t^2}}, \tag{20}$$

and after some algebra, Eq.(17) becomes

$$\Gamma(x_1,x_2;z) = A(z)\exp\left[-\frac{x_1^2+x_2^2}{4\sigma_{\mathrm{i}}^2(z)} - \frac{(x_1-x_2)^2}{2\sigma_{\mathrm{c}}^2(z)} - \mathrm{i}\frac{\pi}{\lambda R(z)}(x_1^2-x_2^2)\right]\exp[\Delta_{\mathrm{i}}(x_1+x_2) - \mathrm{i}\Delta_{\mathrm{c}}(x_1-x_2) + \Delta_{\mathrm{const}}], \tag{21}$$

with

$$A(z) = \frac{2\alpha_0\exp[2\mathrm{Im}(\varphi')]}{\pi\omega_0^2(|u|^2-p^2|v|^2)},$$

$$\sigma_{\mathrm{i}}^2(z) = \frac{(|u|^2-p^2|v|^2)|u^2|\omega_0^2}{4\left\{2p^2\mathrm{Re}(uv) + \frac{\pi\omega_0^2}{\lambda}(|u|^2-p^2|v|^2)\mathrm{Im}(\tau^* u) - 2p|u|^2\right\}},$$

$$\sigma_{\mathrm{c}}^2(z) = \frac{(|u|^2-p^2|v|^2)\omega_0^2}{4p}, \qquad R(z) = \frac{(|u|^2-p^2|v|^2)|u^2|\omega_0^2}{-\dfrac{2\lambda}{\pi}p^2\mathrm{Im}(uv) + \omega_0^2(|u|^2-p^2|v|^2)\mathrm{Re}(\tau^* u)},$$

$$\Delta_{\mathrm{i}} = \frac{2}{\omega_0^2(|u|^2-p^2|v|^2)}\mathrm{Re}(-pe^* + p^2 ev^*/u) + \frac{2\pi}{\lambda}\mathrm{Im}\left(\frac{(e/u+f)}{2B}\right),$$

$$\Delta_{\mathrm{c}} = -\frac{2}{\omega_0^2(|u|^2-p^2|v|^2)}\mathrm{Im}(-pe^* + p^2 ev^*/u) + \frac{2\pi}{\lambda}\mathrm{Re}\left(\frac{(e/u+f)}{2B}\right),$$

and

$$\Delta_{\rm const} = \frac{p|e|^2 - p^2{\rm Re}(e^2v^*/u)}{\omega_0^2(|u|^2 - p^2|v|^2)}. \tag{22}$$

Equations (21) and (22) describe the transformation law of a GSM beam through a complex $ABCDGH$ system. It is readily seen that the output beam is still in the GSM form but the position is displaced, and there is an additional phase shift. By using Eqs.(11) and (12), one obtains the beam radius $W(z) = 2\sigma_{\rm i}(z)$, the transverse displacement $X_0 = 2\sigma_{\rm i}^2\Delta_{\rm i}$, the far-field divergence $\Theta = \sqrt{(\lambda/\pi)^2(1/4\sigma_{\rm i}^2 + 1/\sigma_{\rm c}^2) + 4\sigma_{\rm i}^2/R^2}$, and the deflection angle $\theta_{\rm def} = 2\sigma_{\rm i}^2\Delta_{\rm i}/R + \lambda\Delta_{\rm c}/2\pi$.

B. Propagation of the mutual coherence function through any *ABCDGH* system in the presence of random media

The study of waves in random media is important in some fields such as astronomy, optical communication and remote sensing. For the case of weak fluctuation, the extended Huygens-Fresnel principle[18] is a fairly effective method for describing paraxial beams propagation through random media. Along with this principle and on the basis of the $ABCDGH$ diffraction integral discussed in Sec.Ⅱ, we can develop a general expression for the MCF of paraxial beams propagating through any optical system characterized by complex $ABCDGH$ matrices in the presence of random media.

Following the extended Huygens-Fresnel principle in weak-fluctuation media, we have

$$E(x,z) = \int {\rm d}\zeta {\rm g}(x,\zeta)E_{\rm in}(\zeta,z_0)\exp[\psi(x,\zeta)], \tag{23}$$

where g() is defined by Eq.(9). If considering the second-order Rytov approximation, one has

$$\psi = \psi_1 + \psi_2, \quad \psi_1 = U_1/U_0, \quad \psi_2 = U_2/U_0 - \psi_1^2/2, \tag{24}$$

where U_0 is the output field of a spherical wavelet in the absence of random media given by Eq.(5). U_1 and U_2 are the perturbation solutions, respectively, corresponding to the first- and second-order Born approximation, namely,

$$U_1(x,\zeta) = -k^2\int {\rm d}\rho_1 {\rm g}(x,\rho_1)\varepsilon(\rho_1)U_0(\rho_1,\zeta), \tag{25}$$

$$U_2(x,\zeta) = k^4\iint {\rm d}\rho_1{\rm d}\rho_2 {\rm g}(x,\rho_2)\varepsilon(\rho_2){\rm g}(\rho_2,\rho_1)\varepsilon(\rho_1)U_0(\rho_1,\zeta). \tag{26}$$

Here $\varepsilon = (n_0+n_1)^2 - n_0^2 = 2n_0n_1 + n_1^2$, n_1 represents the fluctuation of the refraction index. Substitution of Eq.(23) into Eq.(7) gives

$$\begin{aligned}\Gamma(x_1,x_2;z) = &\iint {\rm d}\zeta_1{\rm d}\zeta_2 {\rm g}(x_1,\zeta_1){\rm g}^*(x_2,\zeta_2)\Gamma_{\rm in}(\zeta_1,\zeta_2;z_0)\\ &\times <\exp[\psi_1(x_1,\zeta_1) + \psi_2(x_1,\zeta_1) + \psi_1^*(x_2,\zeta_2) + \psi_2^*(x_2,\zeta_2)]>.\end{aligned} \tag{27}$$

In deriving Eq.(27), we use the fact that the fluctuations in the source and the random medium are statistically independent. For Gaussian random variables or correcting through terms of the order of ε^2, we can deduce

$$\begin{aligned}&<\exp[\psi_1(x_1,\zeta_1) + \psi_2(x_1,\zeta_1) + \psi_1^*(x_2,\zeta_2) + \psi_2^*(x_2,\zeta_2)]>\\ &= \exp[<\phi_2(x_1,\zeta_1) + \phi_2^*(x_2,\zeta_2)> + <\psi_1(x_1,\zeta_1)\psi_1^*(x_2,\zeta_2)>],\end{aligned} \tag{28}$$

where $\phi_2 = U_2/U_0$. After a lengthy derivation, one obtains

$$<\phi_2(x_1,\zeta_1)> = -\pi k^2\int_{-\infty}^{+\infty}{\rm d}K\int_0^L {\rm d}z\,\Phi_\varepsilon(K,z), \tag{29a}$$

$$\begin{aligned}<\psi_1(x_1,\zeta_1)\psi_1^*(x_2,\zeta_2)> = &2\pi k^2\int_{-\infty}^{+\infty}{\rm d}K\int_0^L {\rm d}z\,\Phi_\varepsilon(K,z)\exp[-{\rm Im}(B_{0z}B_{zL}/B)K^2/k]\\ &\times\exp\{-{\rm i}K[(x_1-x_2){\rm Re}(B_{0z}/B) + (\zeta_1-\zeta_2){\rm Re}(B_{zL}/B) + {\rm i}(x_1+x_2){\rm Im}(B_{0z}/B)\\ &+ {\rm i}(\zeta_1+\zeta_2){\rm Im}(B_{zL}/B)] - K[{\rm Im}(B_{0z}e_{zL}/B) + {\rm Im}(B_{zL}f_{0z}/B)]\},\end{aligned} \tag{29b}$$

where

$$\Phi_\varepsilon(K,z) = \int_0^\infty {\rm d}(\Delta z)\int_{-\infty}^{+\infty}{\rm d}(\zeta_1-\zeta_2)C_\varepsilon(\zeta_1-\zeta_2;\Delta z,z)\exp[{\rm i}K(\zeta_1-\zeta_2)]$$

can be viewed as the two-dimensional Fourier transform of the correlation function of the dielectric fluctuation C_ε evaluated at $K_z = 0$. C_ε is represented by

$$C_\varepsilon(\zeta_1 - \zeta_2; \Delta z, z) = < \varepsilon(\zeta_1, z_1)\varepsilon^*(\zeta_2, z_2) > .$$

Here $\Delta z = z_1 - z_2$, $z = (z_1 + z_2)/2$, and the transverse correlation is merely related to the difference of coordinates for the case of a stationary random medium.

Thus we obtain an extended expression for the MCF in random media. The extension to the expressions for other statistical moments including the mean irradiance and the log-amplitude, phase and wave-structure functions as discussed in Ref.[18] is straightforward by means of the generalized $ABCDGH$ diffraction integral. Here we will not study in detail.

IV. CONCLUSION

We have derived an extended version of the $ABCDGH$ diffraction integral, which is suitable for the case of a curved optical axis, from the paraxial wave equation. On the basis of this result, some applications have been further investigated. This paper again shows that the generalized Fresnel diffraction integral in terms of matrix formalism is really a valuable and convenient tool for the study of paraxial beams propagation.

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