Volume 9, Number 2
1009-1963/2000/09(02)/0081-05
CHINESE PHYSICS
February, 2000
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PARTICULAR SOLITONS IN NONLINEAR LATTICE*

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(Received 23 December 1998; revised manuscript received 18 May 1999)

One soliton of particle velocity and its amplitude (maximum particle velocity of one soliton) in Toda lattice is given analytically. It has also been known numerically that the maximum particle velocity (when the collision of two solitons reaches their maximum, we define V_n at this time as its maximum particle velocity) during the collision of two solitons moving in the same direction is equal to the difference between the amplitudes of two solitons if the difference is large enough; however, the maximum particle velocity is equal to the adding up of the amplitudes of two solitons moving in the opposite directions. The relationship between the maximum value of $e^{-r_n}-1$ and their initial amplitude of $e^{-r_n}-1$ is also given analytically in Toda lattice if the amplitudes of the two solitons are the same and their moving directions are opposite. Compared with the Boussinesq equation, there are differences between the Toda lattice equation and the Boussinesq equation for solitons during the collision.

PACC: 0340K

I. INTRODUCTION

The concept of solitons has now become ubiquitous in modern nonlinear science and indeed can be found in various branches of physics.[1−6] In optics,[1,2] and nonlinear lattice,[3,4] for surface water waves[5,6] and the nonlinear waves in a fluid-filled elastic tube,[7,8] both KdV solitons and nonlinear Schrödinger solitons are found. In the present investigation, we mainly study the solitons in nonlinear Toda lattice. Although solitons have been reported previously for the function $(e^{-r_n}-1)$, we study the magnitude of the particle velocity for the solitons in Toda lattice and report some interesting results in this paper.

II. PARTICLE VELOCITY AND ITS AMPLITUDE FOR ONE SOLITON

The dimensionless Toda lattice equation is[4]

$$\frac{\partial^2 Q_n}{\partial t^2} = e^{Q_{n-1}-Q_n} - e^{Q_n-Q_{n+1}}. \tag{1}$$

Letting

$$Q_n = \ln\frac{f_{n-1}}{f_n}, \qquad f_n = 1 + e^{\alpha n-\omega t+\delta}, \tag{2}$$

and

$$\omega^2 = 4\sinh^2\frac{\alpha}{2}, \tag{3}$$

we obtain a solution of Eq.(1)

$$Q_n = \ln\frac{1+e^{\alpha n-\omega t+\delta-\alpha}}{1+e^{\alpha n-\omega t+\delta}}, \tag{4}$$

where Q_n is the displacement from equilibrium position of the nth particle. The nth particle velocity V_n can be got by differentiating Q_n with respect to time t

$$V_n = \frac{\omega}{2}\sinh\frac{\alpha}{2}\mathrm{sech}\frac{1}{2}(\alpha n-\omega t+\delta)\mathrm{sech}\frac{1}{2}(\alpha n-\omega t+\delta-\alpha). \tag{5}$$

Equation (5) represents a soliton of the nth particle velocity. The maximum value of particle velocity can be got if we maximize Eq.(5)

$$V_n|_{\max} = \frac{\omega}{2}\sinh\frac{\alpha}{2}\mathrm{sech}^2\frac{\alpha}{4}. \tag{6}$$

III. PARTICLE VELOCITY Vn FOR TWO SOLITONS

Letting

$$\begin{aligned} f_n =& 1 + A_1\exp(\alpha_1 n-\omega_1 t) + A_2\exp(\alpha_2 n-\omega_2 t) \\ &+ \exp((\alpha_1+\alpha_2)n-(\omega_1+\omega_2)t), \end{aligned} \tag{7}$$

and $\alpha_i, A_i > 0\ (i=1,2,), \alpha_2 > \alpha_2, \omega_i = 4\sinh^2\dfrac{\alpha_i}{2}$,

$$A_1A_2 = \frac{4\sinh^2\dfrac{1}{2}(\alpha_1+\alpha_2)-(\omega_1+\omega_2)^2}{(\omega_1-\omega_2)^2-4\sinh^2\dfrac{1}{2}(\alpha_1-\alpha_2)},$$

*Project supported by the Natural Science Foundation of Gansu Province, China (Grant No. ZS981-A25-008-Z).

we obtain a solution of Eq.(1) for Q_n

$$Q_n = \ln \frac{1 + A_1 \exp(\alpha_1 n - \omega_1 t - \alpha_1) + A_2 \exp(\alpha_2 n - \omega_2 t - \alpha_2) + \exp((\alpha_1 + \alpha_2)n - (\omega_1 + \omega_2)t - (\alpha_1 + \alpha_2))}{1 + A_1 \exp(\alpha_1 n - \omega_1 t) + A_2 \exp(\alpha_2 n - \omega_2 t) + \exp((\alpha_1 + \alpha_2)n - (\omega_1 + \omega_2)t)}. \tag{8}$$

Letting $x_1 = \alpha_1 n - \omega_1 t, x_2 = \alpha_2 n - \omega_2 t, K_1 = 1 - \mathrm{e}^{-\alpha_1}, K_2 = 1 - \mathrm{e}^{-\alpha_2}$ and differentiating Q_n with respect to t, we get the nth particle velocity

$$\begin{aligned} V_n =& [A_1\omega_1 K_1 \mathrm{e}^{x_1} + A_2\omega_2 K_2 \mathrm{e}^{x_2} + (\omega_1 + \omega_2)\mathrm{e}^{x_1+x_2}(1 - \mathrm{e}^{-(\alpha_1+\alpha_2)}) \\ &+ A_1 A_2 \mathrm{e}^{x_1+x_2}(\mathrm{e}^{-\alpha_1} - \mathrm{e}^{-\alpha_2})(\omega_2 - \omega_1) + A_1\omega_2 K_2 \mathrm{e}^{2x_1+x_2}\mathrm{e}^{-\alpha_1} + A_2\omega_1 K_1 \mathrm{e}^{x_1+2x_2}\mathrm{e}^{-\alpha_2}]/ \\ &[(1 + A_1\mathrm{e}^{x_1} + A_2\mathrm{e}^{x_2} + \mathrm{e}^{x_1+x_2})(1 + A_1\mathrm{e}^{x_1-\alpha_1} + A_2\mathrm{e}^{x_2-\alpha_2} + \mathrm{e}^{x_1+x_2-\alpha_1-\alpha_2})]. \end{aligned} \tag{9}$$

IV. NUMERICAL RESULTS

It have been known from numerical results that in Eq.(9) V_n is the nth particle velocity for two solitons in Toda lattice. The numerical results of two solitons moving in the same and in the opposite directions are given in the following context respectively.

A. Two solitons moving in the same direction ($\omega_1\omega_2 > 0$)

1. $\alpha_1 \gg \alpha_2$.

Figure 1 represents the wave form when the maximum value of V_n is got during the collision of two solitons, where we choose $\alpha_1 = 10, \alpha_2 = 5, A_2 = 2$. The amplitudes of both solitons before the collision are

$$V^1_{\max} = 146.42, \quad V^2_{\max} = 10.2646,$$

respectively. The maximum particle velocity during the collision is $V_{\max} = 136.1554$, or $V_{\max} = V^1_{\max} - V^2_{\max}$. Similar results can also be obtained for other parameters. Table 1 gives the numerical results for other parameters.

Table 1.

α_1	α_2	A_1	$V^1_{\max}$	$V^2_{\max}$	$V^1_{\max} - V^2_{\max}$	$V_{\max}$
10	6	2.0	146.42	18.1353	128.2846	128.2846
10	7	2.0	146.42	31.1456	115.2742	115.2742
10	5	2×10	146.42	10.26958	136.1553	136.1553
10	4	2×10	146.42	5.524	140.896	140.896

It suggests that the maximum amplitude of particle velocity during the collision is equal to the difference between the amplitudes of the two solitons before collision.

2. $\alpha_1 \sim \alpha_2$.

Figure 2 is the wave form of numerical results of Eq.(9) when the maximum amplitude of particle velocity is got during the collision of two solitons, where we choose $\alpha_1 = 10, \alpha_2 = 8$. It indicates, for V_n, that there are one minimum $V_{\min} = 93.8034$, and two maxima of the same value $V_{\max} = 103.4552$. The amplitudes of two solitons before collision are

$$V^1_{\max} = 146.416,$$
$$V^2_{\max} = 52.61647.$$

It can be easily got

$$V_{\min} = V^1_{\max} - V^2_{\max}.$$

Same results can also be got for other parameters. Table 2 is the numerical results of Eq.(9) for other parameters.

Fig.1. Wave form of particle velocity when the maximum particle velocity is got during the collision of two solitons, their moving directions are same and the difference between their amplitude is large.

Fig.2. Wave form of particle velocity when the maximum particle velocity is got during the collision of two solitons, their moving directions are same and the difference between their amplitude is small.

Table 2.

α_1	α_2	A_1	$V^1_{\max}$	$V^2_{\max}$	$V^1_{\max} - V^2_{\max}$	$V_{\max}$
10	9	2.0	146.42	88.07842	58.39167	58.39166
10	8	2.0	146.42	52.61647	93.80344	93.80343
10	7.5	2.0	146.42	40.5446	105.8754	105.875

From numerical results we conclude that if $\alpha_1 \sim \alpha_2$ and $\alpha_1/\alpha_2 > 3/4$, there are one minimum whose magnitude is just equal to the difference of amplitudes of the two initial solitons, and two symmetrical maxima whose magnitudes are the same when the collision reaches the maximum value.

B. Two solions moving in opposite directions ($\omega_1\omega_2 < 0$)

If $\omega_1 = 2\sinh\dfrac{\alpha_1}{2}$ and $\omega_2 = -2\sinh\dfrac{\alpha_2}{2}$,

then two solitons will move in opposite directions. Figure 3 is the wave form when the maximum particle velocity is got during the collision, where $\alpha_1 = 10, \alpha_2 = 8$. There are two symmetrical minima and one maximum of $V_{\max} = 93.80344$. The amplitudes of the two initial solitons before collision are

$$V^1_{\max} = 146.416,$$
$$V^2_{\max} = -52.61647.$$

Fig.3. Wave form of particle velocity when the maximum particle velocity is got during the collision of two solitons, their moving directions are opposite.

Table 3.

α_1	α_2	A_1	$V^1_{\max}$	$V^2_{\max}$	$V^1_{\max} - V^2_{\max}$	$V_{\max}$
10	6	2.0	146.42	−88.0284	58.39165	58.39166
10	7	2.0	146.42	−10.26458	136.1554	136.1553
10	5	2.0	146.42	−52.61647	93.80343	93.80344
10	4	2.0	146.42	−31.14565	115.2752	115.2742

We have

$$V_{\max} = V^1_{\max} + V^2_{\max}.$$

Same results are obtained for other parameters (Table 3). We conclude that the maximum particle velocity is equal to the sum of the amplitudes of the two initial solitons before collision if the two solitons move in opposite directions.

V. COLLISION OF TWO SOLITONS WHOSE AMPLITUDES ARE THE SAME AND MOVING DIRECTIONS ARE OPPOSITE

Now we study the collision of two solitons moving in opposite directions and having the same amplitude. In other words $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = -\omega_2 = 2\sinh(\alpha/2)$. For this case we can obtain Q_n and V_n

$$Q_n = \ln\frac{1+\exp(\alpha n-\omega t-\alpha+A)+\exp(\alpha n+\omega t-\alpha+B)+\exp(2\alpha n-2\alpha)}{1+\exp(\alpha n-\omega t+A)+\exp(\alpha n+\omega t+B)+\exp 2\alpha n}, \tag{10}$$

where $A = \ln A_1, B = \ln A_2$;

$$V_n = \frac{\omega(\exp(A-\omega t)-\exp(B+\omega t))(\exp\alpha n+\exp(3\alpha n-2\alpha)-\exp(\alpha n-\alpha)-\exp(3\alpha n-\alpha))}{(1+\exp(\alpha n-\omega t-\alpha+A)+\exp(\alpha n+\omega t-\alpha+B)+\exp(2\alpha n-2\alpha))(1+\exp(\alpha n-\omega t+A)+\exp(\alpha n+\omega t+B)+\exp 2\alpha n)}. \tag{11}$$

we get from Eq.(11)

$$V_{1/2} = 0, \qquad V_n(t_0) = 0, \tag{12}$$

where $t_0 = \frac{1}{2\omega}\ln(A_1/A_2)$. Letting $r_n = Q_n - Q_{n-1}$, which is the difference of displacements between two nearest neighbours, we get from Eq.(10)

$$\begin{aligned} e^{-r_n}-1 =& 4\frac{\sinh^2(\alpha/2)(\exp(\alpha n-\alpha-\omega t+A)+\exp(\alpha n-\alpha+\omega t+B)+\exp(3\alpha n-3\alpha-\omega t+A)+\exp(3\alpha n-3\alpha+\omega t+B))}{(1+\exp(\alpha n-\alpha-\omega t+A)+\exp(\alpha n-\alpha+\omega t+B)+\exp(2\alpha n-2\alpha))^2} \\ &+ 4\frac{\sinh^2[\alpha(\exp(2\alpha n-2\alpha))]}{(1+\exp(\alpha n-\alpha-\omega t+A)+\exp(\alpha n-\alpha+\omega t+B)+\exp(2\alpha n-2\alpha))^2}. \end{aligned} \tag{13}$$

In order to get the maximum value of $(e^{-r_n}-1)$, we differentiate the function $(e^{-r_n}-1)$ with respect to t and n, and let

$$\frac{\partial(e^{-r_n}-1)}{\partial t} = 0, \qquad \frac{\partial(e^{-r_n}-1)}{\partial n} = 0, \tag{14}$$

then if

$$t = t_0 = \frac{1}{2\omega}\ln\frac{A_1}{A_2}, \qquad n = 1, \tag{15}$$

$(e^{-r_n}-1)$ will reach its maxmum value expressed as

$$(e^{-r_n}-1)_{\max} = 4C + 4(1-\sqrt{1+C}), \tag{16}$$

where $C = \sinh^2\frac{\alpha}{2}$ is the amplitude of initial solitons of function $(e^{-r_n}-1)$[4] when the effect of collision of two solitons can be completely neglected. If C is small enough, we get from Eq.(16)

$$(e^{-r_n}-1)_{\max} = 2C + C^2 + \cdots \tag{17}$$

The maximum value of $e^{-r_n}-1$ is larger than the sum of amplitudes of the two initial solitons. This is different from the linear one and also different from the solitons represented by Boussinesq equation. We will see this difference in next section.

VI. COLLISION OF TWO BOUSSINESQ-TYPE SOLITONS WHOSE AMPLITUDES ARE THE SAME AND MOVING DIRECTIONS ARE OPPOSITE

Boussinesq equation can be written in the form[9]

$$u_{tt} - u_{xx} - u_{xxxx} - 6(u^2)_{xx} = 0. \tag{18}$$

In order to obtain a solution for Eq.(18) we write

$$u(x,t) = \frac{ff_{xx} - f_x^2}{f^2}, \tag{19}$$

if

$$f = 1 + A_1 e^{kx-qt} + A_2 e^{kx+qt} + e^{2kx}, \tag{20}$$

then we get a solution of Eq.(18). This solution can be expressed by

$$u(x,t) = k^2\frac{A_1 e^{kx-qt} + A_2 e^{kx+qt} + 4e^{2kx} + A_1 e^{3kx-qt} + A_2 e^{3kx+qt}}{(1 + A_1 e^{kx-qt} + A_2 e^{kx+qt} + e^{2kx})^2}. \tag{21}$$

It can be known both from the numerical and asymptotical results of Eq.(21) that it is a solution of two solitons whose amplitudes are the same and moving directions are opposite.[9]

Differentiating $u(x,t)$ with respect to x and t, and letting the derivatives equal to zero

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} = 0, \tag{22}$$

we have

$$t = t_0 = \frac{1}{q}\ln(A_1/A_2)^{1/2}, \qquad x = x_0 = 0, \tag{23}$$

and finally we obtain the maximum value of $u(x,t)$

$$u(x,t)_{\max} = \frac{4D}{1+\sqrt{(1+16D)/(1+4D)}}, \tag{24}$$

where $D = \frac{1}{4}k^2$ is the initial amplitude of the solitons before collision takes place.[9] We now emphasize the following cases:

1. If D is small enough,

$$u(x,t) \sim 2D; \tag{25}$$

it indicates that the maximum value of $u(x,t)$ is equal to the sum of the amplitudes of the two initial solitons when the amplitudes of them are small enough.

2. If D is large enough,

$$u(x,t) \sim \frac{2}{3}(D+D); \tag{26}$$

it suggests that the maximum value of function $u(x,t)$ is equal to $\frac{2}{3}$ times that of the sum of the amplitudes of the two initial solitons if the amplitudes of them are large enough.

3. In general, for $x > 0$, function $y = \frac{2}{1+\sqrt{(1+16D)/(1+4D)}}$ will be always positive, or $y > 0$. It indicates that the maximum value of $u(x,t) < 2D$. The results are different from the interaction of Toda lattice-type solitons, which have been studied in the last section.

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