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FLUCTUATIONS OF FLUX LINES ON THE SURFACE OF SUPERCONDUCTOR*

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In this paper, we study the magnetic structure of the half-ring flux line ended on the surface of a superconductor. The elemental modes for these half-ring flux lines are the half-ellipse flux lines ended on and perpendicular to the surface. The statistics of these half-ring flux lines is, in a certain degree, similar to the vortex-antivortex pairs in thin superconducting films. An abrupt change of the thermodynamic properties for this system takes place at a temperature lower than T_c. It is analogous to the KT transition in thin superconducting films in certain sense.

Keywords: Kosterlitz-Thouless transition, 3-dimensional superconductor, surface of superconductor, flux lines.

PACC: 7430C, 7430E, 7440, 6480, 6890

I. INTRODUCTION

In a thin superconducting film, it is believed that there exists a population of equal numbers of oppositely aligned vortices due to thermal fluctuations even in a zero applied field. When the transverse penetration depth λ^2/d (with λ the London penetration depth and d the thickness of the film) for this film can be comparable to the sample size, the system of these thermally excited vortex pairs should exhibit some phenomena of the Kosterlize- Thouless phase transition.[1−5] Similarly, in the surface of a 3-dimensional superconductor, it is also believed that some half-rings of flux lines(FLs), which are ended on the surface, are present due to thermal fluctuation even in a zero applied field. If taking into account the boundary conditions, these vortex rings must be perpendicular to the surface. It is clear that the elemental mode of these half- ring flux lines is half-ellipse which is ended on and perpendicular to the surface. Because the energy of the FL parallel to the surface of the superconductors approaches to zero when this FL is approaching to the surface (due to the Bean-Livingston barrier[6]), these thermal-excited half-ring FLs exist mostly in a thin layer only a few coherence lengths from the surface. Therefore, in spite of the vortex loop inside the superconductor, these thermal-excited half-ring FLs, in a sense, are analogous to the vortex -antivortex pairs in a thin film.

What kinds of physical property do these half -ring FLs have? How do they influence the physical characteristics of the superconductor? In particular, it is very interesting to consider these fluctuations in the surface of strongly anisotropic superconductors, for example, the high T_c superconductors. This is the purpose of our work. In order to simplify the discussion, we consider a typical type-II superconductor occupying the region $y < 0$ and assume its Ginzburg-Landau parameter $\kappa = \lambda/\xi >> 1$ (here λ is the London penetration depth, ξ is the coherence length), so the London approximation is applicable for such situation. As shown in Fig.1, the half-ellipse FL is laid on the x-y plane for $y < 0$.

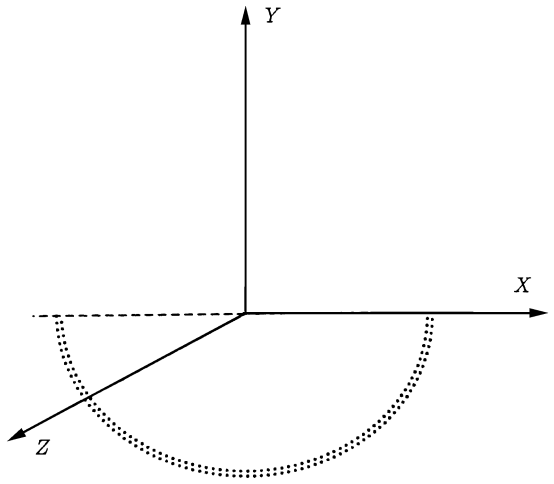


Fig.1. Peretration of flux line in the form of semi-ellipse into the 3D type-II superconductor. The flux line is placed in x-y plane and the superconductor occupies the half space ($y < 0$).

The magnetic structure of this FL is determined by the solutions of the London equation for $y < 0$,

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the static magnetic equation in vacuum for $y > 0$, and the boundary condition in the plane $y = 0$. It is easy to prove that the solution of the problem for $y < 0$ can be expressed as the sum of the solution of the inhomogeneous London equation for a closed ellipse FL on x-y plane in an unbounded superconductor and the solution of the homogeneous London equation for $y < 0$. In Sec. II, a simplest case, a closed circular FL in an unbounded superconductor, is considered. In fact, this kind of FL has been studied by Samokohvlov *et al.*[7−10], in this section we only give a simpler representation of the solution. Then in Sec. III, this solution is generalized to the situation for a closed ellipse FL in an unbounded superconductor. In Sec. IV, we consider the magnetic field in vacuum $y > 0$, the boundary condition in plane $y = 0$ and the solution for the homogeneous London equation. As a result, the complete solution for a half-ellipse flux line is obtained. In Sec.V, we study the free energy of the half-ellipse flux line ended on the surface of the superconductor. In Sec.VI, we discuss the results obtained in the context of strongly layered superconductor. After the discussion of a solitary half-ellipse flux line, the statistics of these half-ellipses in the surface of a superconductor is considered, and some analogies with the KT transition in thin film are also discussed. These are the objects of Sec.VII and VIII.

Ⅱ. THE MAGNETIC STRUCTURE OF A CIRCULAR FLUX LINE IN AN UNBOUNDED SUPERCONDUCTOR.

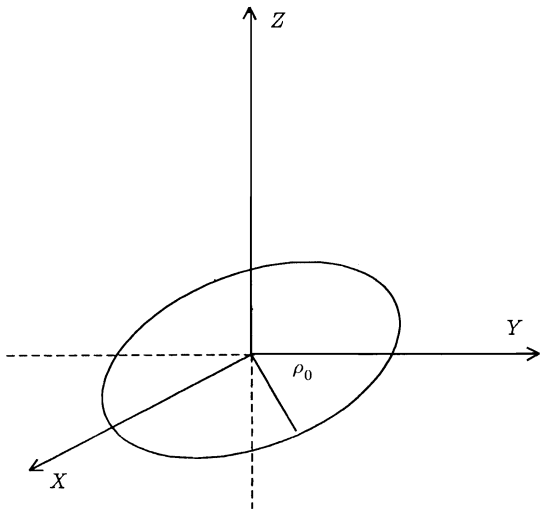


Fig.2. In an unbounded type-II superconductor, the losed circular flux line with radius ρ_0 is displaced in $z = 0$ plane.

First, we consider the simplest case: the circular FL with radius ρ_0 in an unbounded superconductor. As shown in Figs.2 and 3, in view of the symmetry of the problem, using the cylindrical coordinates (ρ, θ, z) is appropriate.

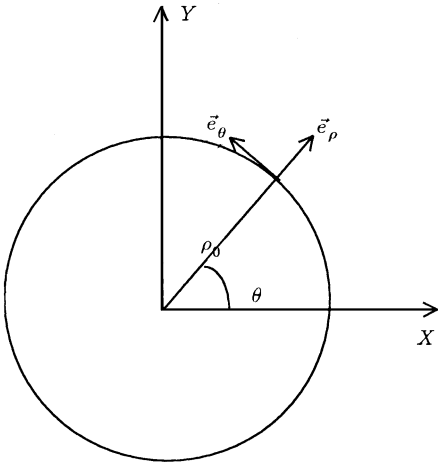


Fig.3. In view of the symmetry of the closed circular flux line, only the azimuthal component of the magnetic field exists.

Then the London equation has the following form[11]:

$$\boldsymbol{h} + \lambda^2 \nabla \times \nabla \times \boldsymbol{h} = \phi_0 \boldsymbol{e}_\theta \delta(\rho - \rho_0)\delta(z), \qquad (1)$$

where ϕ_0 is the quantum magnetic flux, $\boldsymbol{e}_\theta$ is the base vector of the cylindrical coordinate. Taking into account the symmetry of the closed circular flux line, only the azimuthal component of the magnetic field is present. To introduce the dimensionless units for the field and the spatial coordinate by $h = (\phi_0/\lambda^2)H$ and $r_0 = \rho_0/\lambda$, $r = \rho/\lambda$, $\xi = z/\lambda$, Eq.(1) becomes:

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \xi^2} - \left(1 + \frac{1}{r^2}\right)\right] H = -\delta(r - r_0)\delta(\xi). \qquad (2)$$

The solution of Eq.(2) has been obtained in Refs.[7] by Fourier-Bessel transform, and can be represented as follows:

$$H(r,\xi) = \frac{r_0}{2}\int_0^\infty \mathrm{d}q \frac{1}{\sqrt{1+q^2}} J_1(qr_0)J_1(qr)\exp(-\sqrt{1+q^2}\xi), \qquad (3)$$

where J_1 is the first order Bessel function. According to the addition theorems of Bessel functions,

$$J_1(qr_0)J_1(qr) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} J_0(qR)\cos\varphi \mathrm{d}\varphi. \qquad (4)$$

where R is

$$R^2 = r_0^2 + r^2 - 2rr_0\cos\varphi, \qquad (5)$$

Substitute Eq.(4) into Eq.(3) and use the integral formula

$$\int_1^\infty \exp(-\alpha x) J_0(\beta\sqrt{(x^2-1)})\mathrm{d}x$$
$$=\exp(-\sqrt{(\alpha^2+\beta^2)})/\sqrt{(\alpha^2+\beta^2)},$$

a simpler expression of the magnetic structure of the FL is obtained

$$H(R,\xi)=\frac{r_0}{4\pi}\int_{-\pi}^{\pi}\cos\varphi \mathrm{d}\varphi\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}}. \quad (6)$$

The physical meaning of this expression is clear: the integral is the sum of individual contributions from every segment $r_0\mathrm{d}\varphi$ of this circle to the magnetic field at the $A(r,\theta,\xi)$, $(\sqrt{R^2+\xi^2})$, is the distance between (r,θ,ξ) and $(r_0,\varphi,0)$, $\cos\varphi$ is the projection of $\boldsymbol{e}_\varphi$ on $\boldsymbol{e}_\theta$. This is illustrated in Fig.4.

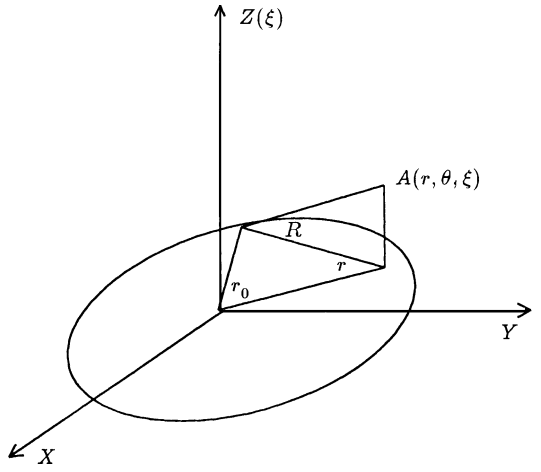


Fig.4. The magnetic field at point A, composed of the contribution around the closed circular flux line.

It is evident that usually the majority of the thermally excited half-ring FLs are shorter than the penetration depth λ. In other words, we can assume $r, r_0, \xi << 1$, so we have $\exp(-\sqrt{(R^2+\xi^2)}) \cong 1$, and Eq.(6) can be approximated by

$$H(r,\xi)=\frac{1}{\pi}\left(\frac{r_0}{\sqrt{r_0^2+r^2+\xi^2}}\right)\frac{\sqrt{1+\kappa^2}}{\kappa}(K(\kappa)-E(\kappa)), \quad (7)$$

where K and E are the complete elliptic integrals of the first kind and the second kind respectively; and κ is defined by:

$$\kappa=\frac{1-\sqrt{1-\mu^2}}{\mu},\ \ \mu=\frac{2rr_0}{r_0^2+r^2+\xi^2}. \quad (8)$$

Now we consider the magnetic field in the plane $\xi=0$, from Eq.(8) we have

$$\kappa = r/r_0(\text{if}\quad r<r_0))\quad \text{or}\quad r_0/r(\text{if}\quad r_0<r).$$

If we consider the case $r_0<r$ and $r-r_0<<r_0$, from the expansion of functions K and E for $\kappa<1$[12], we can obtain the following approximation

$$H(r,0)\cong\left(\frac{1}{\pi}\right)(K(r_0/r)-E(r_0/r))$$
$$\cong\left(\frac{1}{2\pi}\right)\left(\ln\left(\frac{r_0}{r-r_0}\right)+0.079+\cdots\right). \quad (9)$$

If $r<r_0$ and $r_0-r<<r_0$, similarly, we can obtain:

$$H(r,0)\cong\left(\frac{1}{\pi\kappa}\right)(K(r_0/r)-E(r_0/r))$$
$$\cong\left(\frac{1}{2\pi}\right)\left(\frac{r_0}{r}\right)\left(\ln\left(\frac{r_0}{r-r_0}\right)+0.079+\cdots\right). \quad (10)$$

Another case is $r/r_0<<1$, using the expansion of functions K and E for $\kappa<<1$[12], we can obtain

$$H(r,0)\cong(1/4)(r/r_0)+\cdots \quad (11)$$

Equations (9) and (10) show that the magnetic field has logarithmic divergence when $|r-r_0|$ approaches to zero. It is also easy to prove from Eq.(7) that there is a similar divergence when $\sqrt{(r-r_0)^2}+\xi^2$ approaches to zero.

In order to calculate the free energy of the flux line, we should calculate the integral of the function $((\nabla\times\boldsymbol{h})\times\boldsymbol{h})\cdot\boldsymbol{e}_n\mathrm{d}\sigma$ on the torus of this flux line with radius ξ (shown in Fig.5), where ξ is coherence length, and $\boldsymbol{e}_n$ is the normal vector of the surface element $\mathrm{d}\sigma$.

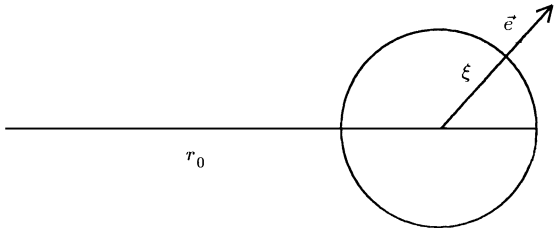


Fig.5. On the torus of the flux line: ξ is the radius of the hard core, $\vec{e}_n$ is the normal vector of the surface element $\mathrm{d}\sigma$.

When $\rho_0>>\xi$, it is easy to obtain the free energy of this circle, which can be approximated by:

$$\left(\frac{\phi_0}{4\pi\lambda}\right)_2^2\pi\rho_0\ln\left(\frac{\rho_0}{\xi}\right). \quad (12)$$

Ⅲ. THE MAGNETIC STRUCTURE OF THE ELLIPTICAL FLUX LINE IN AN UNBOUNDED SUPERCONDUCTOR

The ellipse is set on the x-y plane, the lengths of its major axis and minor axis are a and b respectively. In cylindrical coordinate, the ellipse is described by

$$\rho = \rho(\theta) = a\frac{\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2\theta}}, \tag{13}$$

where $e^2 = (a^2-b^2)/a^2$, the London equation is[11]

$$\boldsymbol{h} + \lambda^2 \nabla \times \nabla \times \boldsymbol{h} = \boldsymbol{e}_t \phi_0 \delta^{(2)}, \tag{14}$$

where $\boldsymbol{e}_t(\theta)$ is the tangent vector of the ellipse, and $\delta^{(2)}$ is the two-dimensional (2D) δ function in the plane perpendicular to $\boldsymbol{e}_t(\theta)$, as shown in Fig.6

$$\boldsymbol{e}_t(\theta) = \cos\varphi \boldsymbol{e}_\theta + \sin\varphi \boldsymbol{e}_\rho, \tag{15}$$

here, $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\theta$ are the base vectors of the cylindrical corrdinates, and φ is the angle between $\boldsymbol{e}_t$ and $\boldsymbol{e}_\theta$. It is easy to prove that φ is determined by

$$\mathrm{tg}\,\varphi = -e^2 \sin\theta\cos\theta/(1-e^2\cos^2\theta). \tag{16}$$

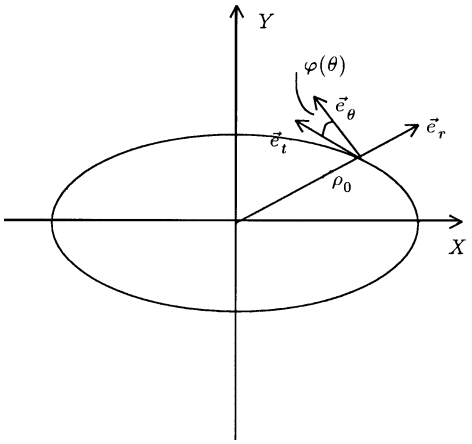


Fig.6. Elliptical flux line on x-y plane. In cylindrical corrdination, $\vec{e}_t$ is the unit vector directed along the flux line, $\vec{e}_\rho$ and $\vec{e}_\theta$ are the basic vectors of the ellipse. $\varphi(\theta)$ is the angle from $\vec{e}_t$ to $\vec{e}_\theta$.

From Eq.(15), the London equation (14) can be rewritten as:

$$(\lambda^2\nabla^2 - 1)\boldsymbol{h} = -\phi_0(\boldsymbol{e}_\theta + \mathrm{tg}\,\varphi \boldsymbol{e}_\rho)\delta(z)\delta(\rho-\rho_0). \tag{17}$$

If we use the dimensionless units of the magnetic field and the spatial coordinates as in Sec.II, then Eq.(17) can be further rewritten as:

$$\begin{aligned}&(\nabla^2-1)\boldsymbol{H}(r,\theta,\xi)\\ &= -\left[\frac{1}{2}(1-j\mathrm{tg}\,\varphi)\boldsymbol{e}_+ + \frac{1}{2}(1+j\mathrm{tg}\,\varphi)\boldsymbol{e}_-\right]\delta(r-r_0)\delta(\xi),\end{aligned} \tag{18}$$

where $\boldsymbol{e}_+ = \boldsymbol{e}_\theta + j\boldsymbol{e}_\rho$ and $\boldsymbol{e}_- = \boldsymbol{e}_\theta - j\boldsymbol{e}_\rho$. The solution of Eq.(18) can be expressed in the following form:

$$\boldsymbol{H}(r,\theta,\xi) = H^{(+)}(r,\theta,\xi)\boldsymbol{e}_+ + H^{(-)}(r,\theta,\xi)\boldsymbol{e}_-, \tag{19}$$

and evidently we have

$$H^{(+)} = [H^{(-)}]^*. \tag{20}$$

If we express the functions $H^{(+)}(r,\theta,\xi)$ and $H^{(-)}(r,\theta,\xi)$ in the form of 2D Fourier transformation:

$$\begin{aligned}H^{(+)}(r,\theta,\xi) = &\sum_{m=-\infty}^{\infty}\exp(jm\theta)\\ &\times\int_{-\infty}^{\infty}\mathrm{d}p\exp(jp\xi)H_m^{(+)}(r,p),\end{aligned} \tag{21}$$

$$\begin{aligned}H^{(-)}(r,\theta,\xi) = &\sum_{m=-\infty}^{\infty}\exp(-jm\theta)\\ &\times\int_{-\infty}^{\infty}\mathrm{d}p\exp(-jp\xi)H_m^{(-)}(r,p),\end{aligned} \tag{22}$$

and take the Hankel transform for $H_m^{(+)}(r,p)$ and $H_m^{(-)}(r,p)$:

$$\begin{aligned}H_m^{(+)}(r,p) &= \int_0^\infty \mathrm{d}q q H_m^{(+)}(q,p)J_{m+1}(qr),\\ H_m^{(-)}(r,p) &= \int_0^\infty \mathrm{d}q q H_m^{(-)}(q,p)J_{m+1}(qr).\end{aligned} \tag{23}$$

Here, J_{m+1} is the (m+1)th order Bessel function, and substituting these expressions into Eq.(18), we obtain the solutions:

$$\begin{aligned}H_m^{(+)}(q,p) &= \frac{1}{1+q^2+p^2}F_m^{(+)}(q,p)\\ &= (H_m^{(-)}(q,p))^*\end{aligned} \tag{24}$$

and

$$\begin{aligned}F_m^{(+)}(q) = &\left(\frac{1}{2\pi}\right)^2\int_{-\pi}^{\pi}\mathrm{d}\theta' r_0(\theta')\exp(-jm\theta')\\ &\cdot J_{m+1}(qr_0(\theta'))\left(\frac{1-j\mathrm{tg}\,\varphi(\theta')}{2}\right).\end{aligned} \tag{25}$$

From Eq. (16), the arc element of the ellipse $\mathrm{d}s(\theta)$ can be written as:

$$\begin{aligned}\mathrm{d}s(\theta) &= r_0\mathrm{d}\theta/\cos\varphi(\theta)\\ &= \exp(j\varphi(\theta))(1-j\theta\varphi(\theta))r_0\mathrm{d}\theta.\end{aligned} \tag{26}$$

Thus Eq. (25) is rewritten as:

$$\begin{aligned}F_m^{(+)}(q) = &\left(\frac{1}{8\pi^2}\right)^2\int_{-\pi}^{\pi}\mathrm{d}s(\theta')\exp(-jm\theta')\\ &\cdot J_{m+1}(qr_0(\theta'))\exp(-j\varphi(\theta')).\end{aligned} \tag{27}$$

Similar to Sec.II, use the addition theorems of Bessel functions:

$$J_0(qR) = \sum_{m=-\infty}^{\infty}\exp(j(m+1)\theta)J_{m+1}(qr)j_{m+1}(qr_0)$$

(where $R^2 = r^2 + r_0^2 - 2rr_0\cos\theta$), and substitute Eqs.(24) and (27) in Eq.(21), the function $H(r,\theta,\xi)$ can be given by

$$H^{(+)}(r,\theta,\xi) = \frac{1}{8\pi}\int_{-\pi}^{\pi} \mathrm{d}s(\theta')\exp\{-j(\theta-\theta'+\varphi(\theta'))\} \cdot \frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}}. \quad (28)$$

Therefore, from Eqs.(19) and (20), the magnetic field of the elliptical flux line is expressed as

$$\boldsymbol{H}(r,\theta,\xi) = 2R_e[H^{(+)}(r,\theta,\xi)(\boldsymbol{e}_\theta + j\boldsymbol{e}_r)],$$

or in cylindrical coordinates:

$$\begin{aligned} H^{(+)}(r,\theta,\xi) =& \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}s(\theta')\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}} \\ &\times \cos(\theta-\theta'+\varphi(\theta'))\boldsymbol{e}_\theta \\ &+ \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}s(\theta')\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}} \\ &\times \sin(\theta-\theta'+\varphi(\theta'))\boldsymbol{e}_r \end{aligned} \quad (29)$$

where $R^2 = r_0^2(\theta') + r^2 - 2rr_0(\theta')\cos(\theta-\theta')$. Since $(\theta-\theta'+\varphi(\theta'))$ is the angle from $\boldsymbol{e}_t(\theta')$ to $\boldsymbol{e}_\theta(\theta)$ (see Fig.7), we have:

$$\begin{aligned} \boldsymbol{e}_t(\theta')\cdot\boldsymbol{e}_\theta(\theta) &= \cos(\theta-\theta'+\varphi(\theta')) \\ \boldsymbol{e}_t(\theta')\cdot\boldsymbol{e}_r(\theta) &= \sin(\theta-\theta'+\varphi(\theta')), \end{aligned} \quad (30)$$

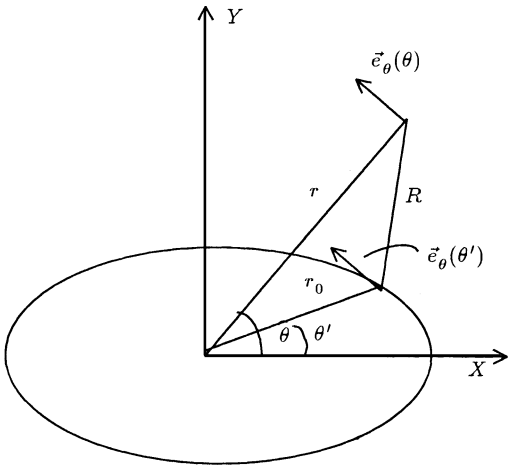


Fig.7. Relationshps between the defined angles, where $(\theta-\theta'-\varphi(\theta'))$ is the angle from $\vec{e}_t(\theta')$ to $\vec{e}_\theta(\theta)$.

So, the solution of the London equation is:

$$\begin{aligned} \boldsymbol{H}(r,\theta,\xi) =& \frac{1}{4\pi}\Big\{\int_{-\pi}^{\pi} \mathrm{d}s(\theta')\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}} \\ &\times [\boldsymbol{e}_\theta(\theta)\cdot\boldsymbol{e}_t(\theta')]\Big\}\boldsymbol{e}_\theta(\theta) \\ &+ \frac{1}{4\pi}\Big\{\int_{-\pi}^{\pi} \mathrm{d}s(\theta')\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}} \\ &\times [\boldsymbol{e}_r(\theta)\cdot\boldsymbol{e}_t(\theta')]\Big\}\boldsymbol{e}_r(\theta). \end{aligned} \quad (31)$$

Compared with the circular case, this representation has a clear physical meaning. The integral is the sum of the contributions of individual segments $\mathrm{d}s(\theta')$ of this ellipse to the magnetic field at point A with coordinates(r,θ,ξ), $\mathrm{d}s(\theta')$ is the arc element of the ellipse. $\sqrt{R^2+\xi^2}$ is the distance between this segment and the point A, $\frac{1}{4\pi}\frac{\exp(-\sqrt{R^2+\xi^2})}{\sqrt{R^2+\xi^2}}$ is the contribution factor of this segment to point A, $(\boldsymbol{e}_\theta(\theta)\cdot\boldsymbol{e}_t(\theta'))$ and $(\boldsymbol{e}_r(\theta)\cdot\boldsymbol{e}_t(\theta'))$ are projections on $\boldsymbol{e}_\theta(\theta)$ and $\boldsymbol{e}_r(\theta)$ respectively. This is illustrated in Fig.(8).

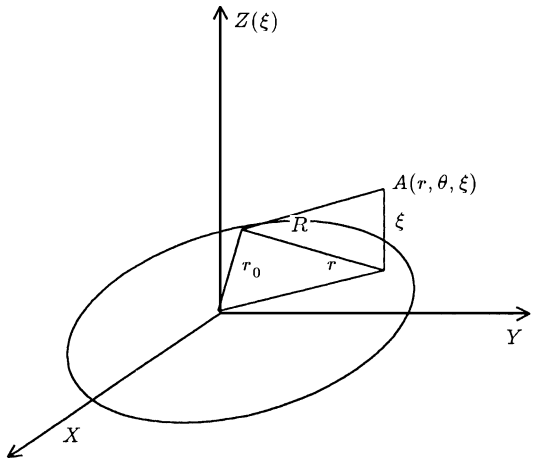


Fig.8. The Magnetic field at point $A,(r,\theta,\xi)$, which consists of the contributions of individual segments around the elliptical flux line.

IV. THE COMPLETE SOLUTION FOR THE HALF-ELLIPSE FLUX LINE ENDED ON THE SURFACE.

The complete solution for the half-ellipse FL ended on the surface $y = 0$ can be represented by the superposition of two parts (illustrated in Fig.9): one part is the solution $\boldsymbol{h}(x,y,z)$ for a closed elliptical FL set on the x-y plane in an unbounded superconductor. This solution has been obtained in Sec.III, and it is clear that the field $\boldsymbol{h}$ is perpendicular to x-z plane at $y = 0$; in other words, $\boldsymbol{h}_{x,z}(x,0,z)$, and

$\boldsymbol{h}_y(x,0,z) = \boldsymbol{h}'(x,z) \neq 0$. Another part is the solution for the homogeneous London equation for $y < 0$, i.e.,

$$\boldsymbol{h}_0 - \lambda^2 \nabla \times \nabla \times \boldsymbol{h}_0 = 0 \tag{32}$$

and

$$\nabla \cdot \boldsymbol{h}_0 = 0, \quad \boldsymbol{h}_0|_{y\to-\infty} = 0. \tag{33}$$

The field describes the penetration of the magnetic field from the outside of the superconductor ($y > 0$) to the inside of the superconductor ($y < 0$). From Eqs.(32) and (33) ,the Fourier transform of $\boldsymbol{h}_0$ is given as follows:

$$\begin{aligned}\boldsymbol{h}_0 =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}q_x \mathrm{d}q_z \tilde{\boldsymbol{h}}_0'(q_x,q_z) \\ &\cdot \exp\left(y\sqrt{\frac{1}{\lambda^2}+q_x^2+q_z^2} + i(q_x x + q_z z)\right)\end{aligned} \tag{34}$$

and

$$\tilde{h}'_{0y} = -i(q_x \tilde{h}'_{0x} + q_z \tilde{h}'_{0z})/Q' \tag{35}$$

Here

$$Q' = \sqrt{\frac{1}{\lambda^2}+q_x^2+q_y^2}.$$

Then, we assume the complete solution for $y < 0$ be

$$\boldsymbol{h}_{\mathrm{in}}(x,y,z) = \boldsymbol{h}_0(x,y,z) + \boldsymbol{h}(x,y,z). \tag{36}$$

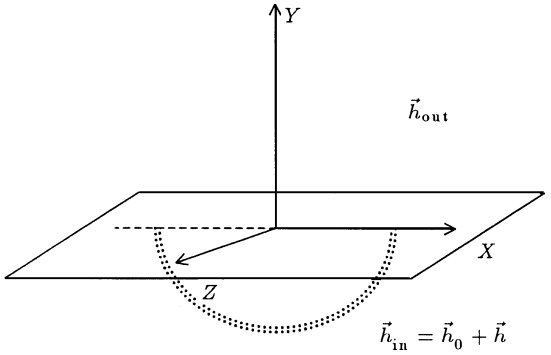


Fig.9. The magnetic structure of the half-elliptical flux line near the surface of the superconductor. The sample occupies the region ($y < 0$), the flux ring is placed in the x-y plane.

Outside the superconductor($y > 0$), the magnetic field $\boldsymbol{h}_{\mathrm{out}}$ must satisfy the static magnetic field equation. To introduce the potential function $V(x,y,z)$ by

$$\boldsymbol{h}_{\mathrm{out}}(x,y,z) = -\nabla V(x,y,z), \tag{37}$$

it must obey the Laplace equation:

$$\Delta V(x,y,z) = 0, \tag{38}$$

and the boundary conditions:

$$\boldsymbol{h}_{\mathrm{in}}(x,0,z) = \boldsymbol{h}_{\mathrm{out}}(x,0,z), \tag{39}$$

$$V(x,y,z)|_{y\to+\infty} = 0. \tag{40}$$

Similar to Eq.(34), the potential function V is given by:

$$\begin{aligned}V(x,y,z) =& \frac{1}{2\pi}\iint \mathrm{d}q_x \mathrm{d}q_z \exp(-\sqrt{q_x^2+q_z^2}\,y \\ &+ i(q_x x + q_z z))\tilde{V}_0(q_x,q_z),\end{aligned} \tag{41}$$

If the function $h'(x,z)$ is also represented by its Fourier transform

$$\begin{aligned}h'(x,z) =& \frac{1}{2\pi}\iint \mathrm{d}q_x \mathrm{d}q_z \\ &\cdot \exp(i(q_x x + q_z z))\tilde{h}'(q_x,q_z),\end{aligned} \tag{42}$$

then using the continuous boundary condition at $y = 0$ (Eq.(39)), we can obtain the algebraic equations for the components of $\tilde{\boldsymbol{h}}_0'(q_x,q_z)$ and $\tilde{V}_0(q_x,q_z)$:

$$\begin{aligned}\tilde{V}_0 &= \frac{Q'}{Q(Q+Q')}\tilde{h}'(q_x,q_z), \\ \tilde{h}'_{0y} &= -\frac{Q}{Q+Q'}\tilde{h}'(q_x,q_z), \\ \tilde{h}'_{0x} &= -iq_x\frac{\tilde{h}'(q_x,q_z)}{Q+\dfrac{Q^2}{Q'}}, \\ \tilde{h}'_{0z} &= -iq_z\frac{\tilde{h}'(q_x,q_z)}{Q+\dfrac{Q^2}{Q'}},\end{aligned} \tag{43}$$

where

$$Q = \sqrt{q_x^2+q_z^2}, \quad Q' = \sqrt{\frac{1}{\lambda^2}+Q^2}.$$

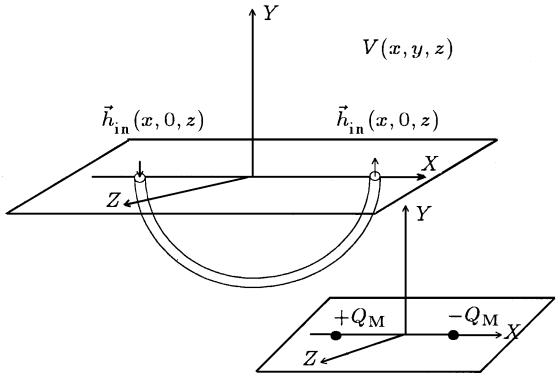


Fig.10. The plot of V, which is the static magnetic field in the upper half-space with surface magnetic charge distribution $h_{\mathrm{in}}(x,0,z)$ in the plane $y = 0$. This potential is the potential of the dipole. The total charge of the dipole is Q_M.

The physical meaning of these expressions is clear: From Eq.(36), the Fourier transform of function $h_{\rm in}(x,0,z)$ is $(Q'/(Q+Q'))\tilde{h}'(q_x,q_z)$. The expression of V is just the static magnetic field in the upper half-space with surface magnetic charge distribution $h_{\rm in}(x,0,z)$ in the plane $y=0$. This is illus trated in Fig.(10).

It is evident that if $\lambda \to 0$, or in other words, if $|Q| << (1/\lambda)$, then $\tilde{h}_{\rm in}(q_x,q_z) \cong \tilde{h}'(q_x,q_z)$, and if $|Q| >> (1\lambda)$, then $\tilde{h}_{\rm in}(q_x,q_z) \cong (1/2)\tilde{h}'(q_x,q_z)$. Thus in this case we are interested in

$$\tilde{h}_{\rm in}(q_x,q_z) \subseteq ((1/2)\tilde{h}'(q_x,q_z),\tilde{h}'(q_x,q_z)). \tag{44}$$

The total charge of this "dipole" is:

$$\begin{aligned} Q_M &= \int_0^\infty {\rm d}x \int_{-\infty}^\infty {\rm d}z h_{in,y}(x,0,z) \\ &= 2\int_0^\infty {\rm d}x \int_0^\infty {\rm d}z h_{in,y}(x,0,z) \end{aligned} \tag{45}$$

It is a function of r_0. In the case of circular FL, an analytical expression for $Q'_M = 2\int_0^\infty {\rm d}x \int_0^\infty {\rm d}z h'(x,z)$ has been obtained and $Q_M = (0.5 \sim 1)Q'_M$,

$$Q'_M = \pi\phi_0 r_0 [(j_v(r_0) - j_v(r_0))/\sin(v\pi)]_{v\to 1}, \tag{46}$$

where J_v is the Anger function and j_v is the Bessel function. It is easy to prove that if $r_0 \to \infty$, then $Q'_M \to \phi_0$; and if $r_0 \to 0$, then:

$$\begin{aligned} Q'_M =& \frac{\pi}{2}\phi_0 r_0 \sum_0^\infty (-1)^n \\ & \frac{1}{\left(n+\frac{1}{2}\right)\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{r_0}{2}\right)^{2n}, \end{aligned}$$

or

$$Q'_M = \phi_0 r_0 (1 - (1/3)r_0^2 + (1/45)r_0^4 - \cdots). \tag{47}$$

V. THE FREE ENERGY OF THE HALF-ELLIPSE FLUX LINE ENDED ON THE SURFACE OF THE SUPERCONDUCTOR.

The free energy of the complete solution discussed in the last section consistes of two parts. One is the static magnetic field energy in the upper half-space $y>0$, expressed as

$$\int_{y>0} |\boldsymbol{h}_{\rm out}|^2 {\rm d}\tau. \tag{48}$$

The other is the Ginzburg—Landau free energy in the half-space $y<0$:

$$\left(\frac{1}{8\pi}\right)\int_{y<0} \left(|\boldsymbol{h}_{\rm in}|^2 + \lambda^2 |\nabla \times \boldsymbol{h}_{\rm in}|^2\right) {\rm d}\tau. \tag{49}$$

By using the Gause theorem, the integrals (48) and (49) are transformed to surface integrals, which are the integral on the interface between vacuum and the superconductor, which is the plane $y=0$; and the integral on the interface between the superconducting and the normal states, which is the "tube" of the half-ellipse FL with radius ξ, as shown in Fig.(11).

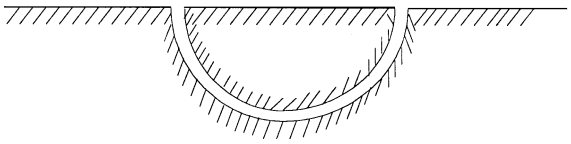


Fig.11. The surface integrals, including two parts: One is the integral on the interface between vacuum and the superconductor, which is the plane $y=0$; the other is the integral on the interface between the superconducting state and the normal state, which is the "tube" of the half-ellipse with radius ξ.

After some lengthy but straight calculations, the integral for $y=0$ is

$$F_1 = \left(\frac{1}{8\pi}\right)\int \frac{Q'}{Q(Q+Q')}[\tilde{h}'(q_x,q_z)]^2 {\rm d}q_x,{\rm d}q_z. \tag{50}$$

It can be rewritten as

$$F_1 = \left(\frac{1}{2}\right)\iint V(x,0,z)\left(\frac{1}{4\pi}\right)h(x,0,z){\rm d}x{\rm d}z, \tag{51}$$

where the expressions of $Q, Q', V, \tilde{h}'$ and h have been shown in the last section. The integral on the surface of the tube is

$$F_2 = \frac{\lambda^2}{8\pi}\iint [(\nabla \times \boldsymbol{h}_{\rm in}) \times \boldsymbol{h}_{\rm in}] \cdot \boldsymbol{e}_n {\rm d}\sigma, \tag{52}$$

where ${\rm d}\sigma$ is the surface element of this tube and $\boldsymbol{e}_n$ is the normal vector of ${\rm d}\sigma$. The estimation of the integral F_1 for the half-circle FL have been carried out. When $\rho_0 << \lambda$, it can be approximated as

$$F_1 \approx (\rho_0^2/\lambda)(\phi_0/4\pi\lambda)^2. \tag{53}$$

From the discussion in Sec.IV, generally, when $\lambda >> \rho_0$, $\boldsymbol{h}_{\rm in}$ is smaller than $\boldsymbol{h}$, and the integral F_2 can be estimated as:

$$F_2 = (0.5 \sim 1)\frac{\lambda^2}{8\pi}\iint [(\nabla \times \boldsymbol{h}) \times \boldsymbol{h}] \cdot \boldsymbol{e}_n {\rm d}\sigma. \tag{54}$$

In Sec.III, we have calculated the integral in Eq.(54) for the circular FL, i.e.,

$$F_2 = (0.5 \sim 1)\varepsilon_0 \pi(\rho_0/\xi)\ln(\rho_0/\xi), \tag{55}$$

where $\varepsilon_0 = (\phi_0/4\pi\lambda)^2\xi$. So, if $\lambda >> \rho_0$, then $F_1 << F_2$. Consequently, the free energy of the half-ellipse FL can be estimated by using the following approximate expression :

$$F = F_1 + F_2 \cong (0.5 \sim 1)\frac{\lambda^2}{8\pi}\iint[(\nabla \times \boldsymbol{h}) \times \boldsymbol{h}] \cdot \boldsymbol{e}_n \mathrm{d}\sigma. \tag{56}$$

When the radius of the ellipse $\rho_0 >> \xi$, the integral in Eq.(56) can be approximated as:

$$\frac{1}{8\pi}\frac{\phi_0}{4\pi\lambda^2}\int_0^\pi \mathrm{d}s(\theta')\int_0^\pi \mathrm{d}s(\theta) \frac{1}{R}\cos(\theta' - \theta + \varphi(\theta) - \varphi(\theta')), \tag{57}$$

where,

$$R^2 = r_0^2(\theta') + r^2(\theta) - 2r(\theta)r_0(\theta')\cos(\theta - \theta'),$$

$r_0(\theta')$ is the radius of the ellipse (shown in Eq.(13)), $r(\theta)$ is $(r_0(\theta) - \xi), \mathrm{d}s(\theta')$, corresponds to $r_0(\theta')$, $\mathrm{d}s(\theta)$ corresponds to $r(\theta)$ And the function $\varphi(\theta)$ is Eq.(16). The integral (57) can only be calculated by numerical method. But it can be described by the following approximate expression:

$$F = \varepsilon_0\xi A(a,b)\left[\left(\frac{a}{\xi}\right)\ln\left(\frac{b}{\xi}\right) + \left(\frac{b}{\xi}\right)\ln\left(\frac{a}{\xi}\right)\right]. \tag{58}$$

Here, $\varepsilon_0 = (\phi_0/4\pi\lambda)^2$, A(a,b) is a factor determined from the values of a and b. In this case $a = b$, we have $A \approx \pi$.

VI. THE STRONGLY LAYERED SUPERCONDUCTORS.

The vortex structure in a strongly layered superconductor can be viewed as an array of pancake vortices threading the individual superconducting layers and interconnected by Josephson strings.[13] If the coordinate is chosen as the x-z plane parallel to the superconducting layers, the usual normal core of the FL is replaced by the phase core which has the dimensions d along y and $\Lambda = d/\varepsilon$ along x or z, where d is the layer spacing of this material, $\varepsilon = (m/M)^{1/2}$ is the anisotropy parameter, m and M are the effective masses in the layer and perpendicular to the layer respectively, usually, we have $\varepsilon << 1$. The region outside the phase core is roughly equivalent to the corresponding regime in a FL with screening currents extending a distance λ/ε along z (if the FL is parallel to x) and λ along y, where λ is the Ginzberg-Landau (GL) penetration depth in the layer. This picture is illustrated in Fig.12 for the section of a FL parallel to x.

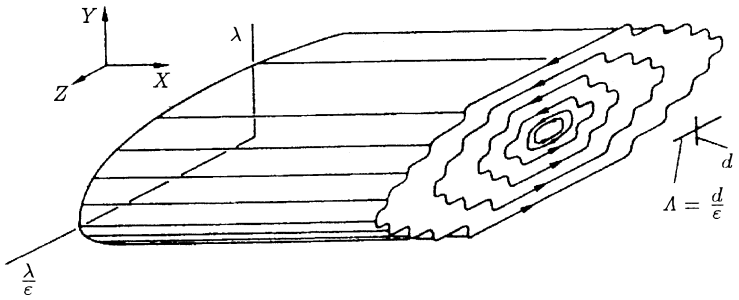


Fig.12. Josephson vortex in a strongly layered superconductor. The usual normal core of the Abrikosov vortex (dimensions $\varepsilon\xi$ and ξ along y and z, respectively) is replaced by the phase core (dimensions d and $\Lambda = d/\varepsilon$ along y and z, respectively) within which the nonlinearity and the discreteness of the situation are relevant. The region outside the phase core is roughly equivalent to the corresponding regime in an Abrikosov vortex with screening currents extending a distance λ along y and λ/ε along z.

From this picture, in the continuous anisotropic description, the linear energy of the FL (the free energy of unit length) paralleled to the $x - z$ plane is:

$$E_L = \varepsilon\varepsilon_0\ln(2b/d), \quad d < b < \alpha, \tag{59}$$

where b is the distance between the FL and the surface, and $\varepsilon_0 = (\phi_0/4\pi\lambda)^2$. For two FLs paralleled to y-axis:

$$E_L = \varepsilon_0\ln(a/\xi), \quad \xi < a < \lambda, \tag{60}$$

where a is the distance between these FLs, ξ and λ are the coherence length and GL penetration depth in this layer. These results are in agreement with the calculation for the pancake picture in weak coupling for the layers. (see Refs.[13]–[15]).

It is evident that because of $\varepsilon << 1$, if Eq.(58) is used to describe the behaviors of the half-ellipse FL ended on the surface of a strongly layered superconductor, the contributions from the segments of the FL

nearly parallel to the surface are over-estimated.

VII. THE STATISTICS OF THE HALF-RING FLUX LINES ENDED ON THE SURFACE OF A SUPERCONDUCTOR.

In order to get a bird's eye view of the surface of a superconductor, the thermally excited half-ring FLs ended on the surface are like a population of equal numbers of oppositely aligned vortices in such 2D system. Now we consider the statistics for these vortex - antivotex pairs. The elemental mode of these half-ring FLs is the half- ellipse which is ended on and perpendicular to the surface; other forms of the half- ring FL are regarded as a half-ellipse FL combined with some forms of closed loop FL inside the superconductor. For this reason only the statistics of these half - ellipse FLs is discussed. If we ignore the interaction between these half-ellipse FLs, the density of the thermally excited half - ellipse FL (the average number of these FLs in unit area of the surface) is:

$$\overline{n} = \frac{1}{\xi^5}\int_{\xi}^{\infty} 2\pi a\mathrm{d}a \int_{\xi}^{\infty} db \exp[-\beta E(a,b)], \qquad (61)$$

where $\beta = \dfrac{1}{k_B T}$, and $E(a,b)$ is the free energy of a half - ellipse FL ended on the surface, a and b are the lengths of the major and minor axes of this ellipse. From the discussions made in Sec. V and VI, we assume

$$E(a,b) = \varepsilon_0 \xi A(s\ln(t) + t\ln(s)) + \mu_0, \qquad (62)$$

where the factor, $A \approx \pi(1/2 \sim 1), s = a/\xi$ and $t = b/\xi$. So Eq. (61) becomes

$$\overline{n} = \exp(-\beta\mu_0)\frac{2\pi}{\xi^2}\int_1^{\infty} s\mathrm{d}s \int_1^{\infty} \mathrm{d}t \exp(-\beta\overline{E}(s\ln t + t\ln s)), \qquad (63)$$

where $\overline{E} = \varepsilon_0 \xi A$.

The integral (63) can be approximated by:

$$\overline{n} = \frac{2\pi}{\xi^2}\frac{1}{\beta\overline{E}}\exp(-\beta\mu_0)\sum_0^{\infty}(-1)^n \frac{\Gamma(n+1)}{(\beta\overline{E}+n-1)^{n+1}}. \qquad (64)$$

The average value of the lengths of these "dipoles" (in unit of ξ) is

$$< s^2 > = \frac{2\pi}{\xi^2}\exp(-\beta\mu_0)\frac{1}{\overline{n}}\int_1^{\infty} s^3\mathrm{d}s \int_1^{\infty} \mathrm{d}t \cdot \exp(-\beta\overline{E}(s\ln t + t\ln s)). \qquad (65)$$

Consequently, we have

$$< s^2 > = \left[\sum_0^{\infty}(-1)^n \frac{\Gamma(n+1)}{(\beta\overline{E}+n-3)^{n+1}}\right] \left[\sum_0^{\infty}(-1)^n \frac{\Gamma(n+1)}{(\beta\overline{E}+n-1)^{n+1}}\right]^{-1} \approx \frac{\beta\overline{E}-1}{\beta\overline{E}-3} \qquad (66)$$

Therefore, if the temperature of the system satisfies

$$\beta\overline{E} - 3 = 0, \qquad (67)$$

then $< s^2 >$ is divergent. In Ref.[16], Kosterlitz and Thouless discussed the thermodynamics of a 2D system and by similar heuristic method suggested that a topological phase transition takes place at the temperature the divergence of $< s^2 >$ occurs. But in their discussions the condition (67) is replaced by

$$\beta\overline{E} - 4 = 0, \qquad (68)$$

In order to understand this difference, we take following approximations for Eq.(62):

$$s\ln t + t\ln s \approx \ln s + (t-1)(s + \ln s),$$

and take the integrals with respect to t in Eqs.(64) and (65), as the results, the integral with respect to s becomes:

$$\int_1^{\infty} \frac{s^m}{s+\ln s}\exp(-\beta\overline{E}\ln s)\mathrm{d}s.$$

To compare this integral with the statistical sum in the 2D system, we can understand the difference between Eqs.(67) and (68). Therefore, it can be expected, at least, that a remarkable change for the statistics of the vortex and antivortex existing on the surface of the superconductor will appear at the temperature satisfying Eq.(67); in certain sense, it is like a KT phase transition in thin superconducting films. Of course, detailed analyses are necessary, in particular, those analyses for the interactions between these half-ring FLs, and the screening effects for these interactions are necessary. It is evident that this change is enhanced in the layered superconductors. We shall make these analyses in another paper.

Because of

$$\lambda = \lambda_0\left(1 - \frac{T}{T_c}\right)^{-(1/2)} \qquad \xi = \xi_0\left(1 - \frac{T}{T_c}\right)^{-(1/2)},$$

the condition (67) can be represented as

$$\frac{t^*}{t}\sqrt{1-t} = 1, \qquad (69)$$

where $t = (T/T_c)$, and

$$t^* = \frac{1}{3k_B T_c} A\xi_0 \left(\frac{\phi_0}{4\pi\lambda_0}\right)^2. \qquad (70)$$

Thus, the temperature satisfying Eq.(67) is

$$T_p = T_c \frac{(t^*)^2}{2}\left(\sqrt{1+\frac{4}{(t^*)^2}}-1\right). \qquad (71)$$

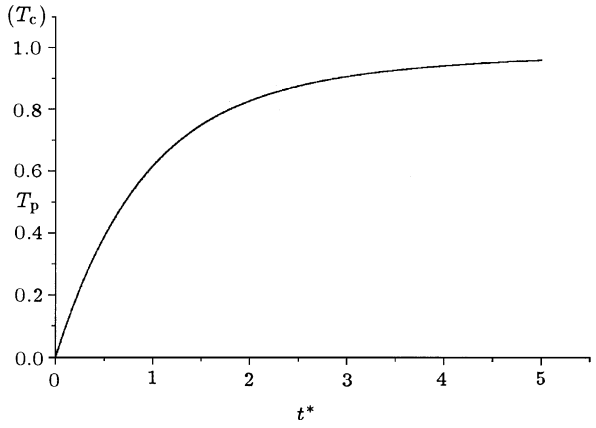


Fig.13. The relation between T_p and t^*. It is evident that in all values of $t^*, T_p < T_c$ and only when $t^* \to \infty, T_p \to T_c$.

Figure (13) shows the relation between T_p and t^*. It is evident that for all values of t^*, we have $T_p < T_c$ and only when $t^* \to \infty$, $T_p \to T_c$. As an example, if ξ=1.0 nm, λ=400 nm, and T_c=60 K, $t^* \approx 1$, then $T_p \approx 0.62T_c$. In other words, when the temperature rises to higher than $0.62T_c$, even though the superconductivity is maintained inside of the superconductor, some remarkable changes of the thermodynamics of the vortex - antivortex existing in the surface take place.

VIII. CONCLUSION.

We have discussed the magnetic structure of the half-ring flux line ended on the surface of a superconductor. The elemental modes for this half-ring FLs are the half-ellipse FLs ended on and perpendicular to the surface. We have obtained the complete solution for this half-ellipse FL and studied the free energy of it. The statistics of these half-ellipse FLs, in certain sense, is similar to that of the vortex-antivortex pairs in thin superconducting films. An abrupt change in the thermodynamic propertie s of these FLs on surface occurs at a temperature lower than T_c. In certain degree, it is like the KT phase transition in a thin superconducting film. Some physical effects will appear at this temperature, for example, the Van der Waals interaction between two superconductors, the phase fluctuations at the interface of superconductors, etc. The detailed discussions about this "transition" will be given in another paper.

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