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EFFECTS OF NON-CONSERVATIVE FORCES ON LIE SYMMETRIES AND CONSERVED QUANTITIES OF A LAGRANGE SYSTEM *

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Non-conservative forces are exerted on a Lagrange system. Their effects on Lie symmetries, structure equation and conserved quantities of the system are studied. It can be seen that some Lie symmetries disappear and some new Lie symmetries emerge. Under certain conditions, some Lie symmetries will still remain present.

Keywords: analytical mechanics, Lagrange system, non-conserved force, Lie symmetry, conserved quantity.

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I. INTRODUCTION

Monographs of the mathematical research on Lie symmetries have already been published.[1−3] Since Lutzky and others introduced Lie symmetry method into the study on mechanical systems in late 1970's, there have been considerable developments of Lie symmetry method in mechanical studies (see Refs.[4–10]). Effects of non-conservative forces on Lie symmetries and conserved quantities of Lagrange systems are studied in this paper. When non-conservative forces are exerted on a Lagrange system, the determining equations of Lie symmetry, structure equation and conserved quantities change. Some Lie symmetries disappear and some new Lie symmetries emerge; under certain conditions, some Lie symmetries will still remain present. Conditions, under which both a Lagrange system and the system on which non-conservative forces are exerted have the same Lie symmetries, are given. The conditions under which both systems have the same structure equation are given too.

II. DIFFERENTIAL EQUATIONS OF MOTION OF THE SYSTEM

The equations of motion of a Lagrange system can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=0,\quad (s=1,\cdots,n), \tag{1}$$

where $L{=}T-V$, T is the kinetic energy of the system, V is the potential energy, q_s are generalized coordinates. Exerting non-conservative forces $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ on Eq(1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s,\quad (s=1,\cdots,n). \tag{2}$$

Here L in Eq.(2) is the same as that in Eq.(1). Suppose that system (1) is not singular, i.e.

$$\Delta=\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0. \tag{3}$$

Then Eq.(1) can be expressed explicitly[11,12]

$$\ddot{q}_s=\frac{\Delta_{ls}}{\Delta}\left\{-[k,m;l]\dot{q}_k\dot{q}_m+\left(\frac{\partial B_k}{\partial q_l}-\frac{\partial B_l}{\partial q_k}\right)\dot{q}_k-\frac{\partial B_l}{\partial t}+\frac{\partial T_0}{\partial q_l}-\frac{\partial A_{kl}}{\partial t}\dot{q}_k\right\}, \tag{4}$$

where Δ_{ls} is the algebraic complement of the element (l,s) of Δ, $[k,m;l]$ is the Christoffel symbol of second kind of the coefficients of the quadratic form in the kinetic energy,

$$[k,m;l]=\frac{1}{2}\left(\frac{\partial A_{kl}}{\partial q_m}+\frac{\partial A_{ml}}{\partial q_k}-\frac{\partial A_{km}}{\partial q_l}\right), \tag{5}$$

A_{ks} are the coefficients of the quadratic form of the velocity in the kinetic energy, B_s are the coefficients of the linear form of the velocity in the kinetic energy, T_0 are the terms without $\dot{q}_s$ in the kinetic energy. Similarly, Eq.(2) can also be expressed explicitly

$$\ddot{q}_s=\frac{\Delta_{ls}}{\Delta}\left\{-[k,m;l]\dot{q}_k\dot{q}_m+\left(\frac{\partial B_k}{\partial q_l}-\frac{\partial B_l}{\partial q_k}\right)\dot{q}_k-\frac{\partial B_l}{\partial t}+\frac{\partial T_0}{\partial q_l}-\frac{\partial A_{kl}}{\partial t}\dot{q}_k+Q_l\right\}. \tag{6}$$

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Equations (4) and (5) can be simply written as

$$\ddot{q}_s = f_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \cdots, n), \tag{7}$$

$$\ddot{q}_s = g_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \cdots, n), \tag{8}$$

Apparently, it follows that

$$g_s = f_s + \frac{\Delta_{ls}}{\Delta} Q_l \tag{9}$$

III. INFINITESIMAL TRANSFORMATIONS AND DETERMINING EQUATIONS OF LIE SYMMETRY

Choosing the infinitesimal transformations of time and coordinates as

$$\begin{aligned} t^* &= t + \Delta t \\ q_s^*(t^*) &= q_s(t) + \Delta q_s, \quad (s = 1, \cdots, n), \end{aligned} \tag{10}$$

or their expanded forms

$$\begin{aligned} t^* &= t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \cdots, n), \end{aligned} \tag{11}$$

where ε is the infinitesimal parameter and ξ_0, ξ_s are the generators of infinitesimal transformations. Then we have the generator vector of infinitesimal transformations

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}. \tag{12}$$

Its first extended form is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}. \tag{13}$$

According to the invariance of the ordinary differential equations under infinitesimal transformations in Lie theory,[2] the invariance of Eqs.(7) and (8) under infinitesimal transformations can be expressed respectively as the following determining equations

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 f_s = X^{(1)}(f_s) \quad (s = 1, \cdots, n), \tag{14}$$

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 g_s = X^{(1)}(g_s) \quad (s = 1, \cdots, n), \tag{15}$$

If some infinitesimal generators ξ_0, ξ_s satisfy both Eqs.(14) and (15), they are corresponding to the Lie symmetries of systems (1) and (2) respectively, i.e., when non-conservative forces are exerted on the system, these Lie symmetries will still remain present. If some infinitesimal generators ξ_0, ξ_s satisfy Eq.(14) but do not satisfy Eq. (15), they are corresponding to the Lie symmetries of system (1), but not the symmetries of system (2), i.e., these Lie symmetries will disappear because of the non-conservative forces. If some infinitesimal generators ξ_0, ξ_s satisfy Eq.(15) but do not satisfy Eq.(14), they are corresponding to the Lie symmetries of system (2), but not the symmetries of system (1), i.e., these Lie symmetries emerge because of the non-conservative forces.

Conditions under which systems (1) and (2) have the same Lie symmetries are given by the following theorem.

Theorem 1 If some infinitesimal generators ξ_0, ξ_s are corresponding to the Lie symmetries of system (1) and both the generalized forces Q_s and the infinitesimal generators ξ_0, ξ_s satisfy

$$-2\dot{\xi}_0 \frac{\Delta_{ls}}{\Delta} Q_l = X^{(1)} \left(\frac{\Delta_{ls}}{\Delta} Q_l \right) \quad (s = 1, \cdots, n), \tag{16}$$

then these Lie symmetries will still remain present when non-conservative forces are exerted on the system.

Proof. Substituting Eq.(9) into Eq.(14) and taking note of the relation (16), we obtain Eq.(15).

IV. STRUCTURE EQUATION AND CONSERVED QUANTITIES

Lie symmetries do not always generate conserved quantities. The following Theorem 2 gives the condition under which Lie symmetries will generate conserved quantities.

Theorem 2[5] For those infinitesimal generators ξ_0, ξ_s satisfying Eq.(14), if there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation

$$L\dot{\xi}_0 + X^{(1)}(L) + \dot{G} = 0, \tag{17}$$

then system (1) has the conserved quantities as follows

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) + G = \text{const.} \tag{18}$$

Theorem 3[5] For those infinitesimal generators ξ_0, ξ_s satisfying Eq.(15), if there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \dot{q}_s \xi_0) + \dot{G} = 0, \tag{19}$$

then system (2) has the conserved quantity whose form is the same as Eq.(18).

Theorem 4 For those infinitesimal generators ξ_0, ξ_s satisfying Eq.(14), if a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies the structure equation Eq.(17), and both the generalized forces Q_s and the infinitesimal generators ξ_0, ξ_s satisfy

$$Q_s(\xi_s - \dot{q}_s \xi_0) = 0, \tag{20}$$

then systems (2) and (1) have the same structure equation, as well as the same conserved quantities.

V. ILLUSTRATED EXAMPLES

Example 1 Suppose there is a Lagrange system with two degrees of freedom

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}\dot{q}_1^2. \tag{21}$$

Non-conservative forces exerted on the system are

$$Q_1 = -\dot{q}_1, \quad Q_2 = -\dot{q}_2. \tag{22}$$

The determining equations of original system are

$$\begin{aligned} &\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0 q_1 = \xi_1, \\ &\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 = 0. \end{aligned} \tag{23}$$

The determining equations when the generalized forces (22) are exerted on the system, are

$$\begin{aligned} &\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0(-q_1 - \dot{q}_1) = \xi_1 - (\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0), \\ &\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0(-\dot{q}_2) = -(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0). \end{aligned} \tag{24}$$

Equation (16) become

$$\begin{aligned} -2\dot{\xi}_0(-\dot{q}_1) &= -(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) \\ -2\dot{\xi}_0(-\dot{q}_2) &= -(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) \end{aligned} \tag{25}$$

which have solutions

$$\xi_0 = 1, \quad \xi_1 = \xi_2 = 0, \tag{26}$$

$$\xi_2 = 1, \quad \xi_0 = \xi_1 = 0, \tag{27}$$

$$\xi_1 = 1, \quad \xi_0 = \xi_2 = 0. \tag{28}$$

Meanwhile, Eqs.(26) and (27) are also the solutions of the determining equations (23) and they are corresponding to the Lie symmetries of system (21). According to Theorem 1, they are also corresponding to the Lie symmetries of the system expressed by Eqs. (21) and (22). But, Eq.(28) is not the solution of Eqs.(23) and (24). Though both systems have the same Lie symmetries (26) and (27), they have different structure equations. For system (21), the structure equations corresponding to generators (26) and (27) are all

$$\dot{G} = 0. \tag{29}$$

For the system expressed by Eqs.(21) and (22), the structure equation (19) becomes respectively

$$\dot{q}_1^2 + \dot{q}_2^2 + \dot{G} = 0. \tag{30}$$

$$-\dot{q}_2 + \dot{G} = 0. \tag{31}$$

From Eqs.(26) and (29) we obtain the conserved quantity of system (21)

$$I_1 = L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s = \text{const.} \tag{32}$$

From Eqs.(27) and (29) we obtain another conserved quantity of system (21)

$$I_2 = \frac{\partial L}{\partial \dot{q}_2} = \text{const.} \tag{33}$$

From Eq.(27) and (31) we obtain the conserved quantity of the system expressed by Eqs.(21) and (22)

$$I_3 = \frac{\partial L}{\partial \dot{q}_2} + q_2 = \text{const.} \tag{34}$$

There are no corresponding conserved quantities of the system expressed by Eqs.(21) and (22) to generators (26).

Example 2 The generalized forces in Example 1 are changed into

$$Q_1 = -\dot{q}_2, \quad Q_2 = \dot{q}_1. \tag{35}$$

Others keep unchanged. Now, the determining equations (24) become

$$\begin{aligned} &\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0(-q_1 - \dot{q}_2) = \xi_1 - (\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0), \\ &\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0(\dot{q}_1) = \dot{\xi}_1 - \dot{q}_1\dot{\xi}_0. \end{aligned} \tag{36}$$

And Eq.(16) become

$$\begin{aligned} -2\dot{\xi}_0(-\dot{q}_2) &= -(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0), \\ -2\dot{\xi}_0\dot{q}_1 &= \dot{\xi}_1 - \dot{q}_1\dot{\xi}_0. \end{aligned} \tag{37}$$

Equation.(37) still has solutions (26) and (27). According to Theorem 1, they are also corresponding to the symmetries of the system expressed by Eqs.(21) and (35). For generators (26) and the generalized forces (35), Eq.(20) holds apparently. According to Theorem 4, both systems have the same structure equation, as well as the conserved quantity (32). For the generators (27), the structure equation (19) gives

$$\dot{q}_1 + \dot{G} = 0, \tag{38}$$

then

$$G = -q_1. \tag{39}$$

And the conserved quantity (18) gives

$$I = \frac{\partial L}{\partial \dot{q}_2} - q_1 = \text{const.} \tag{40}$$

It is different from formula (33).

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