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PERTURBATION METHOD SOLVING ELASTIC PROBLEMS OF ICOSAHEDRAL QUASICRYSTALS CONTAINING A CIRCULAR CRACK*

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Perturbation method for solving elastic three-dimensional (3D) problems for 3D icosahedral quasicrystals is proposed. Considering an infinite 3D icosahedral quasicrystal weakened by a circular crack, we obtain the uniformly valid asymptotic solutions up to $O(R^2)$ for the mode I loading, where R is the elastic constant of phonon-phason coupling.

Keywords: icosahedral quasicrystal, circular crack, perturbation method
PACC: 6140M, 6170G

I. INTRODUCTION

A three-dimensional (3D) quasicrystal refers to a 3D solid structure with quasiperiodic arrangement in all the three directions, where phonon and phason fields exist simultaneously. As far as elastic properties are concerned, therefore, 3D quasicrystals are much more complicated than 1D and 2D ones. Icosahedral quasicrystals (IQs) in 3D were discovered earlier.[1] As in conventional crystals, many structural defects have already been observed experimentally, and experiments show that quasicrystals are quite brittle.[2] Based on Landau theory, quasicrystal elasticity theory was formulated.[3−6] According to this theory, some elastic problems, mainly concerning dislocation and crack, have been widely considered.[7−11] As is well known, elastic problems for quasicrystals are much more complicated than classical ones. Many techniques have been devised to study them and among these methods decomposition procedure, complex variable function method, Green function method and integral transformation[8−11] have been particularly successful. However, all these authors consider only elastic two-dimensional (2D) problems, i.e., they suppose the elastic fields induced in quasicrystals are independent of the variable z. In other words, they consider only the elastic plane or anti-plane problems for quasicrystals. Here it should be stated that Yang *et al.*[10] studied dislocation problems for IQs under the conditions that the elastic constant R of phonon-phason coupling is zero apart from this common supposition. In addition, we and Fan[12] solved the 3D elastic problem of cubic quasicrystal.

Although the methods mentioned above are powerful tools for the study of 2D problems, they are not easily applied to study elastic 3D problems for quasicrystals. Moreover, elastic equations for 3D IQs are so complex that it is almost impossible to obtain their exact solutions. So perturbation method for studying elastic 3D problems for IQs is proposed in this paper. We consider the problem of a circular crack embedded in an infinite 3D IQ. By regarding phason field as a perturbation and extending the stress intensity factors (SIFs) to quasicrystals, the mode I SIFs on the front of the circular crack are obtained up to $O(R^2)$.

II. THE BASIC EQUATIONS

According to quasicrystal elasticity theory,[6] stress-strain relations are (here listed only a part of them)

$$\begin{aligned}
\sigma_{zz} &= \lambda\theta + 2\mu\varepsilon_{zz} - 2Rw_{zz}, \\
\sigma_{zx} &= 2\mu\varepsilon_{zx} + R(w_{xx} - w_{yy} + w_{zx}) = \sigma_{xz}, \\
\sigma_{zy} &= 2\mu\varepsilon_{yz} + R(w_{zy} - w_{xy} - w_{yx}) = \sigma_{yz}, \\
H_{zz} &= R(\varepsilon_{xx} + \varepsilon_{yy} - 2\varepsilon_{zz}) + (K_1 + K_2)w_{zz}, \\
H_{xz} &= R(\varepsilon_{xx} - \varepsilon_{yy}) + K_2(w_{xx} + w_{yy}) \\
&\quad + (K_1 - K_2)w_{xz}, \\
H_{yz} &= -2R\varepsilon_{xy} + (K_1 - K_2)w_{yz} + K_2(w_{xy} - w_{yx}),
\end{aligned} \tag{1}$$

where

$$\theta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}, \varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2, w_{ij} = \partial_j w_i$$

and the equilibrium equations in terms of displacements, in the absence of body forces, are

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$$
\begin{aligned}
&\mu\nabla^2 u_x + (\lambda+\mu)\partial_x e + R(\partial_x^2 w_x + 2\partial_x\partial_z w_x - \partial_y^2 w_x + 2\partial_x\partial_y w_y - 2\partial_y\partial_z w_y + 2\partial_x\partial_z w_z) = 0,\\
&\mu\nabla^2 u_y + (\lambda+\mu)\partial_y e + R(-2\partial_x\partial_y w_x - 2\partial_y\partial_z w_x + \partial_x^2 w_y - 2\partial_x\partial_z w_y - \partial_y^2 w_y + 2\partial_y\partial_z w_z) = 0,\\
&\mu\nabla^2 u_z + (\lambda+\mu)\partial_z e + R(\partial_x^2 w_x - \partial_y^2 w_x - 2\partial_x\partial_y w_y + \partial_x^2 w_z + \partial_y^2 w_z - 2\partial_z^2 w_z) = 0,\\
&K_1\nabla^2 w_x + K_2(2\partial_x\partial_z w_x - \partial_z^2 w_x + 2\partial_y\partial_z w_y + \partial_x^2 w_z - \partial_y^2 w_z)\\
&+ R(\partial_x^2 u_x - \partial_y^2 u_x + 2\partial_x\partial_z u_x - 2\partial_x\partial_y u_y - 2\partial_y\partial_z u_y + \partial_x^2 u_z - \partial_y^2 u_z) = 0,\\
&K_1\nabla^2 w_y + K_2(2\partial_y\partial_z w_x - \partial_x\partial_z w_y - 2\partial_x\partial_y w_z - \partial_z^2 w_y)\\
&+ R(2\partial_x\partial_y u_x - 2\partial_y\partial_z u_x + \partial_x^2 u_y - \partial_y^2 u_y - 2\partial_x\partial_z u_y - 2\partial_x\partial_y u_z) = 0,\\
&(K_1-K_2)\nabla^2 w_z + K_2(\partial_x^2 w_x - \partial_y^2 w_x - 2\partial_x\partial_y w_y + 2\partial_z^2 w_x)\\
&+ R(2\partial_x\partial_z u_x + 2\partial_x\partial_y u_y + \partial_x^2 u_z + \partial_y^2 u_z - 2\partial_z^2 u_z) = 0,
\end{aligned}
\tag{2}
$$

where the z-axis is assumed to be the fivefold symmetry axis, u_i, w_i phonon and phason displacements in the physical and perpendicular spaces, respectively; $e = \partial_x u_x + \partial_y u_y + \partial_z u_z$; λ, μ Lamé constants of phonon field; K_1, K_2 the elastic constants of phason field; R the elastic constant of phonon-phason coupling.

Eqs.(2) are very complicated, and it is almost impossible to solve them exactly under definite boundary conditions. Therefore the perturbation method for solving Eqs. (2) is proposed in this paper, which paves the way for tackling the elastic 3D problems of IQs.

Ⅲ. PERTURBATION METHOD

Let us consider an infinite 3D IC weakened by a circular crack with radius a in the plane z=0, with uniform loads applied normal to the crack faces. For convenience, rectangular and cylindrical polar coordinates will be used simultaneously in the sequel. Introducing the following dimensionless quantities:

$$
\begin{aligned}
&\overline{x} = x/a..., \quad \overline{r} = r/a, \quad \overline{u}_x = u_x/a..., \quad \overline{w}_x = w_x/a..., \quad \overline{u}_r = u_r/a..., \quad \overline{w}_r = w_r/a,\\
&\varepsilon = R/\mu, \quad (\lambda+\mu)/\mu = (1-2\nu)^{-1}, \quad \overline{\sigma}_{ij} = \sigma_{ij}/\mu, \quad \overline{H}_{ij} = H_{ij}/\mu, \quad \overline{K}_i = K_i/\mu,
\end{aligned}
$$

we can rewrite (2) as (omitting the sign "—" on each of the letters)

$$
\begin{aligned}
&\mu\nabla^2 u_x + (1-2\nu)^{-1}\partial_x e + \varepsilon(\partial_x^2 w_x + 2\partial_x\partial_z w_x - \partial_y^2 w_x + 2\partial_x\partial_y w_y - 2\partial_y\partial_z w_y + 2\partial_x\partial_z w_z) = 0,\\
&\mu\nabla^2 u_y + (1-2\nu)^{-1}\partial_y e + \varepsilon(-2\partial_x\partial_y w_x - 2\partial_y\partial_z w_x + \partial_x^2 w_y + 2\partial_x\partial_z w_y - \partial_y^2 w_y + 2\partial_y\partial_z w_z) = 0,\\
&\mu\nabla^2 u_z + (1-2\nu)^{-1}\partial_z e + \varepsilon(\partial_x^2 w_x - \partial_y^2 w_x - 2\partial_x\partial_y w_y + \partial_x^2 w_z + \partial_y^2 w_z - 2\partial_z^2 w_z) = 0,\\
&K_1\nabla^2 w_x + K_2(2\partial_x\partial_z w_x - \partial_z^2 w_x + 2\partial_y\partial_z w_y + \partial_x^2 w_z - \partial_y^2 w_z)\\
&+ \varepsilon(\partial_x^2 u_x - \partial_y^2 u_x + 2\partial_x\partial_z u_x - 2\partial_x\partial_y u_y - 2\partial_y\partial_z u_y + \partial_x^2 u_z - \partial_y^2 u_z) = 0,\\
&K_1\nabla^2 w_y + K_2(2\partial_y\partial_z w_x - 2\partial_x\partial_z w_y - 2\partial_x\partial_y w_z - \partial_z^2 w_y)\\
&+ \varepsilon(2\partial_x\partial_y u_x - 2\partial_y\partial_z u_x + \partial_x^2 u_y - \partial_y^2 u_y - 2\partial_x\partial_z u_y - 2\partial_x\partial_y u_z) = 0,\\
&(K_1-K_2)\nabla^2 w_z + K_2(\partial_x^2 w_x - \partial_y^2 w_x - 2\partial_x\partial_y w_y + 2\partial_z^2 w_x)\\
&+ \varepsilon(2\partial_x\partial_z u_x + 2\partial_x\partial_y u_y + \partial_x^2 u_z + \partial_y^2 u_z - 2\partial_z^2 u_z) = 0.
\end{aligned}
\tag{3}
$$

Then the problem can be formulated as follows:find the solution to the set of differential equations (3) for a half-space $z \geq 0$, subject to the mixed boundary conditions on the plane z=0:

$$
\begin{aligned}
&\sigma_{zz} = -\sigma, \quad H_{zz} = 0, \quad 0 < r < 1; \quad u_z = 0, \quad w_z = 0, \quad r > 1;\\
&\sigma_{zr} = \sigma_{z\theta} = 0, \quad H_{rz} = H_{\theta z} = H_{\theta z} = 0, \quad r \geq 0;\\
&\sigma_{ij} \to 0, \quad H_{ij} \to 0, \quad \sqrt{r^2+z^2} \to \infty.
\end{aligned}
\tag{4}
$$

We look for the solution of the form

$$
\begin{aligned}
u_x &= u_x^{(0)} + \varepsilon u_x^{(1)} + ..., \quad u_y = u_y^{(0)} + \varepsilon u_y^{(1)} + ..., \quad u_z = u_z^{(0)} + \varepsilon u_z^{(1)} + ...,\\
w_x &= w_x^{(0)} + \varepsilon w_x^{(1)} + ..., \quad w_y = w_y^{(0)} + \varepsilon w_y^{(1)} + ..., \quad w_z = w_z^{(0)} + \varepsilon w_z^{(1)} + ...,
\end{aligned}
\tag{5}
$$

where $u_x^{(0)},...w_z^{(0)}$ satisfy the degenerate equations of (3), i.e., the equations when $\varepsilon = 0$ in (3), while $u_x^{(1)},...w_z^{(1)}$ satisfy the following equations:

$$
\begin{aligned}
&\nabla^2 u_x^{(1)} + (1-2\nu)^{-1}\partial_x e^{(1)} = -(\partial_x^2 w_x^{(0)} + 2\partial_x\partial_z w_x^{(0)} - \partial_y^2 w_x^{(0)} + 2\partial_x\partial_y w_y^{(0)} - 2\partial_y\partial_z w_y^{(0)} + 2\partial_x\partial_z w_z^{(0)}),\\
&\nabla^2 u_y^{(1)} + (1-2\nu)^{-1}\partial_y e^{(1)} = -(-2\partial_x\partial_y w_x^{(0)} - 2\partial_y\partial_z w_x^{(0)} + \partial_x^2 w_y^{(0)} - 2\partial_x\partial_z w_y^{(0)} - \partial_y^2 w_y^{(0)} + 2\partial_y\partial_z w_z^{(0)}),\\
&\nabla^2 u_z^{(1)} + (1-2\nu)^{-1}\partial_z e^{(1)} = -(\partial_x^2 w_x^{(0)} - \partial_y^2 w_x^{(0)} - 2\partial_x\partial_y w_y^{(0)} + \partial_x^2 w_z^{(0)} + \partial_y^2 w_z^{(0)} - 2\partial_z^2 w_z^{(0)}),\\
&K_1\nabla^2 w_x^{(1)} + K_2(2\partial_x\partial_z w_x^{(1)} - \partial_z^2 w_x^{(1)} + 2\partial_y\partial_z w_y^{(1)} + \partial_x^2 w_z^{(1)} - \partial_y^2 w_z^{(1)})\\
=&-(\partial_x^2 u_x^{(0)} - \partial_y^2 u_x^{(0)} + 2\partial_x\partial_z u_x^{(0)} - 2\partial_x\partial_y u_y^{(0)} - 2\partial_y\partial_z u_y^{(0)} + \partial_x^2 u_z^{(0)} - \partial_y^2 u_z^{(0)}) \qquad (6)\\
&K_1\nabla^2 w_y^{(1)} + K_2(2\partial_y\partial_z w_x^{(1)} - 2\partial_x\partial_z w_y^{(1)} - 2\partial_x\partial_y w_z^{(1)} - \partial_z^2 w_y^{(1)})\\
=&-(2\partial_x\partial_y u_x^{(0)} - 2\partial_y\partial_z u_x^{(0)} + \partial_x^2 u_y^{(0)} - \partial_y^2 u_y^{(0)} - 2\partial_x\partial_z u_y^{(0)} - 2\partial_x\partial_y u_z^{(0)})\\
&(K_1-K_2)\nabla^2 w_z^{(1)} + K_2(\partial_x^2 w_x^{(1)} - \partial_y^2 w_x^{(1)} - 2\partial_x\partial_y w_y^{(1)} + 2\partial_z^2 w_z^{(1)})\\
=&-(2\partial_x\partial_z u_x^{(0)} + 2\partial_x\partial_y u_y^{(0)} + \partial_x^2 u_z^{(0)} + \partial_y^2 u_z^{(0)} - 2\partial_z^2 u_z^{(0)}).
\end{aligned}
$$

It follows from (5), (4) and (1) (equating the coefficients of the like powers of ε) that

$$
\begin{aligned}
&\sigma_{zz}^{(0)} = -\sigma, \quad \sigma_{zz}^{(i)} = 0 \quad (i \geq 1),\\
&H_{zz}^{(i)} = 0, \quad 0 < r < 1; \quad u_z^{(i)} = 0, \quad w_z^{(i)} = 0 \quad r > 1;\\
&\sigma_{zr}^{(i)} = \sigma_{z\theta}^{(i)} = 0, \quad H_{rz}^{(i)} = H_{\theta z}^{(i)} = 0 \quad r \geq 0; \qquad (7)\\
&\sigma_{ij}^{(i)} \to 0, \quad H_{ij}^{(i)} \to 0, \quad \sqrt{r^2+z^2} \to \infty.
\end{aligned}
$$

Note that here the superscripts are in correspondence to those in (5). The solution to the degenerate equation of (3) under the condition (7) for $u_x^{(0)}$, $u_y^{(0)}$ and $u_z^{(0)}$ can be found in classical elasticity theory, for example Ref.[13], while it is obvious that $w_x^{(0)} = w_y^{(0)} = w_z^{(0)}$=0. If we are only interested in stress intensity factors (SIFs)–the most important physical quantity in fracture theory, care should be taken with the property of $u_r^{(0)}$, $u_\theta^{(0)}$ and $u_z^{(0)}$ i.e., $u_\theta^{(0)} \equiv 0$ and $\partial_\theta u_r^{(0)} = \partial_\theta u_z^{(0)} \equiv 0$. Thus the expression on the right-hand side of the first three equations in (6) become zero and this is not true for the rest. It is easy to see that the solution to the problem (6) and (7) for i=1 is $u_r^{(1)} = u_\theta^{(1)} = u_z^{(1)} \equiv 0$, but $w_r^{(1)}, w_\theta^{(1)}$ and $w_z^{(1)}$ non-zero. To derive the expression for SIFs in front of the crack, the first thing is to extend SIFs in conventional crystals to quasicrystals. We define

$$
\begin{aligned}
K_{\mathrm{I}}^{\parallel} &= \lim_{r\to 1+} \sqrt{2\pi(r-1)}\,\sigma_{zz}(r,0),\\
K_{\mathrm{I}}^{\perp} &= \lim_{r\to 1+} \sqrt{2\pi(r-1)}\,H_{zz}(r,0).
\end{aligned} \qquad (8)
$$

According to the analysis above, we have

$$
\sigma_{zz}(r,0) = \begin{cases} -\sigma & 0 < r < 1;\\ -(2\sigma/\pi)[-(r^2-1)^{-1/2} & \\ \quad +\arcsin r^{-1}] + O(\varepsilon^2) & r > 1, \end{cases} \qquad (9)
$$

$$
H_{zz}(r,0) = O(\varepsilon).
$$

The substitution of (9) into (8) yields

$$
K_{\mathrm{I}}^{\parallel} = 2\sigma/\sqrt{\pi} + O(\varepsilon^2), \quad K_{\mathrm{I}}^{\perp} = O(\varepsilon).
$$

No space is here available to compute the stress and displacement fields.

IV. CONCLUTION

Perturbation method for solving elastic 3D problems for 3D IQs is developed. For a 3D IQ weakened by a circular crack, the influence of phason field on phonon field is only $O(R^2)$.

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