Volume 9, Number 10
1009-1963/2000/09(10)/0753-04
CHINESE PHYSICS
October, 2000
© 2000 Chin. Phys. Soc.

THE NONLINEAR SCHRÖDINGER EQUATION AND THE CROSS- PHASE MODULATION IN ERBIUM-DOPED FIBER AMPLIFIERS*

Liu Ning(刘 宁), Liu Song-hao(刘颂豪), Liao Chang-jun (廖常俊),
Guo Qi(郭 旗), and Xu Wen-cheng(徐文成)
Institute of Quantum Electronics, South China Normal University, Guangzhou 510631, *China*

(Received 24 February 2000; revised manuscript received 3 June 2000)

The nonlinear Schrödinger equation(NLSE) in erbium-doped fiber(EDF) was obtained. The cross-phase modulation (XPM) in the erbium-doped fiber amplifiers (EDFA) was studied based on this NLSE and the rate equations. A more generalized form of the propagation equation in EDFA was obtained which included the phase shifts the EDFA induced. An analytical expression was given to the XPM in the EDFA. It was found that the XPM in the EDFA dose not change very much with the wavelength except at the neighboring wavelengths (around 1531 nm) where the absorption and emission cross-sections of the erbium ions reach their maxima, and the XPMs have opposite signs on the two sides around 1531 nm. Furthermore, it was found that the XPM increases with the increase of the length of EDF.

Keywords: cross-phase modulation, erbium-doped fiber amplifier, optical fiber communication, wavelength-division multiplexing.
PACC: 4265, 4255N

I. INTRODUCTION

Cross-phase modulation (XPM) has an important impact on the performance of high-speed wavelength division multiplexing (WDM) optical fiber communication system.[1] It will cause crosstalk between simultaneously transmitted channels in WDM systems and is believed to be one of the most significant factor limiting the transmission capacity. XPM is usually considered to originate from the Kerr effect in optical fibers, and the contribution of erbium-doped fiber amplifiers (EDFA) is usually neglected. However, this is incorrect for some cases.[2,3] Recent experimental studies showed that the XPM in EDFA is likely to cause significant impairments to the performance of WDM systems.[2,3] The reason is that the XPM can be converted into intensity modulation by the chromatic dispersion of the transmission optical fiber. The crosstalk caused by EDFA has previously been focused on the gain dynamics of the EDFA, and has been studied based on a EDFA model which consists of the rate equations and the power propagation equations. The phase effects in EDFA are neglected. Recently an extended EDFA model has been developed to study the phase effects in EDFA[4] and the nonlinear dispersion in EDFA is studied based on this model.[5] In this paper, we derive the nonlinear Schrödinger equation (NLSE) in erbium-doped fiber (EDF) which includes the refractive index of the host medium. Based on this NLSE and the rate equations, we studied the XPM in the EDFA in detail.

II. NONLINEAR SCHRÖDINGER EQUATION IN ERBIUM-DOPED FIBER

In the slowly-varying-envelope approximation, the Maxwell equation in optical fiber can be written as[6]

$$\nabla^2 \tilde{E}(\boldsymbol{r},\omega) + \varepsilon(\omega) k_0^2 \tilde{E}(\boldsymbol{r},\omega) = 0, \tag{1}$$

where $\tilde{E}(\boldsymbol{r},\omega)$ is the Fourier transform of the slowly varying part of the electric field, $k_0 = 2\pi/\lambda_0, \varepsilon(\omega)$ is the dielectric constant. In the case of erbium-doped fiber, $\varepsilon(\omega)$ consists of two parts, one is the contribution of the host medium $\chi_{\rm h}$ and the other is the contribution of the erbium ions $\chi_{\rm e}$. So $\varepsilon(\omega)$ has the following form:

$$\varepsilon(\omega) = 1 + \chi_{\rm h} + \chi_{\rm e}, \tag{2}$$

where

$$\chi_{\rm h} = \chi'_{\rm h} + {\rm j}\chi''_{\rm h} \quad \text{and} \quad \chi_{\rm e} = \chi'_{\rm e} + {\rm j}\chi''_{\rm e} \tag{3}$$

are the complex atomic susceptibility. For the host medium the real and imaginary parts of $\chi_{\rm h}$ are related to the refractive index and absorption coefficient

*Project supported by the Outstanding Young Scholar Foundation of Guangdong Province, China.

of the host medium by[6]

$$n_0^2 + n_2|\tilde{E}|^2 = 1 + \chi_{\rm h}', \quad \alpha_0 = \frac{k_0\chi_{\rm h}''}{n_0}, \tag{4}$$

where n_0 is the linear refractive index, n_2 is the nonlinear-index coefficient, and α_0 is the absorption coefficient. In the regime of far from saturation in the EDFA, the real and imaginary parts of $\chi_{\rm e}$ can be expressed as a linear combination of the absorption and emission cross sections multiplied by the dopant (erbium ions) density in the lower and upper states respectively,[7,8]

$$\chi_{\rm e}''(\omega) = -\frac{n_0 c}{\omega}[N_2\sigma_{\rm e}(\omega) - N_1\sigma_{\rm a}(\omega)], \tag{5}$$

$$\chi_{\rm e}'(\omega) = \frac{n_0 c}{\omega}\Big[N_2{\rm KKR}(\sigma_{\rm e}(\omega)) - N_1{\rm KKR}(\sigma_{\rm a}(\omega))\Big], \tag{6}$$

where N_1 and N_2 are the lower and upper state populations, $\sigma_{\rm e}(\omega)$ and $\sigma_{\rm a}(\omega)$ are the emission and absorption cross sections respectively; KKR is the Kramers-Kronig transform. So Eq.(2) can be written as

$$\varepsilon(\omega) = n_0^2 + n_2|\tilde{E}|^2 + {\rm j}\frac{n_0\alpha_0}{k_0} + \chi_{\rm e}. \tag{7}$$

Equation(1) can be solved using the method of separation of variables.[6] If we assume a solution of the form

$$\tilde{E}(\boldsymbol{r}, \omega - \omega_0) = F(x, y)\tilde{A}(z, \omega - \omega_0)\exp({\rm j}\beta_0 z), \tag{8}$$

where $\tilde{A}(z, \omega)$ is a slowly varying function of z and β_0 is the wavenumber. Equation(1) leads to the following two equations for $F(x, y)$ and $\tilde{A}(z, \omega)$:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \left[\varepsilon(\omega)k_0^2 - \overline{\beta}^2\right]F = 0, \tag{9}$$

$$2{\rm j}\beta_0\frac{\partial\tilde{A}}{\partial z} + \left(\overline{\beta}^2 - \beta_0^2\right)\tilde{A} = 0. \tag{10}$$

Wavenumber $\overline{\beta}$ is determined by solving the eigenvalue equation(9) for the fiber modes. The dielectric constant $\varepsilon(\omega)$ in Eq.(9) can be approximated by

$$\varepsilon(\omega) = (n_0 + \Delta n)^2 \approx n_0^2 + 2n_0\Delta n, \tag{11}$$

where Δn is a small perturbation, from Eqs.(7) and (11) it is given by

$$\Delta n = \frac{n_2}{2n_0}|\tilde{E}|^2 + {\rm j}\frac{\alpha_0}{2k_0} + \frac{1}{2n_0}\chi_{\rm e}. \tag{12}$$

Now Eq.(9) can be solved by using the first-order perturbation theory. It was first solved by replacing ε by n_0^2, to obtain the modal distribution $F(x, y)$ and corresponding wavenumber $\beta(\omega)$. For a single-mode fiber, $F(x, y)$ corresponds to the fundamental fiber mode. Then the effect of Δn is included in Eq.(9), in the first-order perturbation theory, Δn dose not affect the modal distribution $F(x, y)$. However, the eigenvalue $\overline{\beta}$ is given by

$$\overline{\beta} = \beta(\omega) + \Delta\beta, \tag{13}$$

where

$$\Delta\beta = \frac{k_0\iint\limits_{-\infty}^{\infty}\Delta n|F(x, y)|^2{\rm d}x{\rm d}y}{\iint\limits_{-\infty}^{\infty}|F(x, y)|^2{\rm d}x{\rm d}y}. \tag{14}$$

So Eq.(10) can be written as

$$\frac{\partial\tilde{A}}{\partial z} = {\rm j}\big[\beta(\omega) + \Delta\beta - \beta_0\big]\tilde{A}. \tag{15}$$

Expanding $\beta(\omega)$ in a Taylor series about the carrier frequency ω_0, i.e.,

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}\big(\omega - \omega_0\big)^2\beta_2 + \frac{1}{6}\big(\omega - \omega_0\big)^3\beta_3 + \cdots, \tag{16}$$

where

$$\beta_n = \left[\frac{{\rm d}^n\beta}{{\rm d}\omega^n}\right]_{\omega=\omega_0} \qquad n = 1, 2, 3, \cdots. \tag{17}$$

Substituting Eq.(16) in Eq.(15) and taking the inverse Fourier transform by using

$$A(z, t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\tilde{A}(z, \omega - \omega_0)\exp[-{\rm j}(\omega - \omega_0)t]{\rm d}\omega, \tag{18}$$

the result is

$$\frac{\partial A}{\partial z} = -\beta_1\frac{\partial A}{\partial t} - \frac{\rm j}{2}\beta_2\frac{\partial^2 A}{\partial t^2} + {\rm j}\Delta\beta A. \tag{19}$$

By using Eqs.(12) and (14), $\Delta\beta$ can be evaluated and substituted in Eq.(19). The result is

$$\frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} + \frac{\rm j}{2}\beta_2\frac{\partial^2 A}{\partial t^2} + \frac{\alpha_0}{2}A = {\rm j}\gamma|A|^2 A + \frac{{\rm j}k_0}{2n_0}\Gamma\chi_{\rm e}A, \tag{20}$$

where γ is the nonlinear coefficient and

$$\Gamma = \frac{\int\limits_{\rm doped\ region}\int|F(x, y)|^2{\rm d}x{\rm d}y}{\int\limits_{-\infty}^{\infty}\int|F(x, y)|^2{\rm d}x{\rm d}y} \tag{21}$$

is the overlap integral. Equation (20) is the nonlinear Schrödinger equation in erbium-doped fiber. It is actually applicable to other doped optical fiber.

Ⅲ. CROSS-PHASE MODULATION IN EDFA

For an EDFA, the length of EDF is not very long. So the absorption coefficient α_0, the second order dispersion β_2 and the nonlinearity γ of the host can be neglected. Equation (20) becomes

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}=\frac{\mathrm{j}k_0}{2n_0}\Gamma\chi_{\mathrm{e}}A. \tag{22}$$

By using the retarded time frame $T=t-\beta_1 z$ and substituting Eqs.(5) and (6) in Eq.(22), one has

$$\begin{aligned}\frac{\partial A}{\partial z}=&\frac{\Gamma}{2}[N_2\sigma_{\mathrm{e}}(\omega_0)-N_1\sigma_{\mathrm{a}}(\omega_0)]A\\&+\frac{\mathrm{j}\Gamma}{2}[N_2\mathrm{KKR}(\sigma_{\mathrm{e}}(\omega_0))\\&-N_1\mathrm{KKR}(\sigma_{\mathrm{a}}(\omega_0))]A.\end{aligned} \tag{23}$$

The first term on the right hand side of Eq.(23) is the gain of the EDFA and the second term describes the phase shift the EDFA induced. We can see that the first term is identical with the power propagation equation in the EDFA model.[9] Equation (23) is the more generalized form of the propagation equation in EDFA which has included the phase shifts the EDFA induced. Equation (23) can be integrated along the length of EDF, the result is

$$\begin{aligned}A(L)=&A(0)\exp\Bigg\{\int_0^L\frac{\Gamma}{2}\Big[N_2\sigma_{\mathrm{e}}(\omega_0)-N_1\sigma_{\mathrm{a}}(\omega_0)\Big]\mathrm{d}z\\&+\mathrm{j}\int_0^L\frac{\Gamma}{2}\Big[N_2\mathrm{KKR}(\sigma_{\mathrm{e}}(\omega_0))\\&-N_1\mathrm{KKR}(\sigma_{\mathrm{a}}(\omega_0))\Big]\mathrm{d}z\Bigg\},\end{aligned} \tag{24}$$

where L is the length of EDF.

By using Eq.(24) and the rate equation[9] of EDFA

$$\begin{aligned}\frac{\mathrm{d}N_2}{\mathrm{d}t}=&\sum_k\frac{I_k\sigma_{\mathrm{a}k}}{h\nu_k}N_1(r,\phi,z)\\&-\sum_k\frac{I_k\sigma_{\mathrm{e}k}}{h\nu_k}N_2(r,\phi,z)-\frac{N_2(r,\phi,z)}{\tau},\end{aligned} \tag{25}$$

$$N_{\mathrm{t}}(r,\phi,z)=N_1(r,\phi,z)+N_2(r,\phi,z), \tag{26}$$

where $I_k=|A_k|^2$ is the light intensity, we can study the behavior of the EDFA in great detail. Because the gain properties of the EDFA have been studied extensively, we will study the phase shift the EDFA will induce according to Eqs.(24),(25) and (26).

From Eq.(24) we know that the phase shift the EDFA induced is

$$\varphi_k=\frac{\Gamma_k}{2}\int_0^L\Big[N_2\mathrm{KKR}(\sigma_{\mathrm{e}k})-N_1\mathrm{KKR}(\sigma_{\mathrm{a}k})\Big]\mathrm{d}z. \tag{27}$$

It can be seen that the phase shift is intensity-dependent, just like the XPM in optical fiber. So it will induce a frequency chirp to the optical pulses passing the EDFA.

$$\begin{aligned}\delta\omega_k(T)=&-\frac{\partial\varphi_k}{\partial T}=-\frac{\Gamma_k}{2}\Big[\mathrm{KKR}(\sigma_{\mathrm{e}k})\\&+\mathrm{KKR}(\sigma_{\mathrm{a}k})\Big]\int_0^L\frac{\partial N_2}{\partial T}\mathrm{d}z.\end{aligned} \tag{28}$$

From Eq.(28) we can see that the time dependent term is the same for every channel, this means that the time variation of the frequency chirp is the same for every channel, but the magnitude of the frequency chirp is wavelength dependent for each channel. Figure 1 shows the wavelength dependence of $\mathrm{KKR}(\sigma_{\mathrm{e}k})+\mathrm{KKR}(\sigma_{\mathrm{a}k})$. It can be seen that except at the neighboring wavelengths (around 1531 nm) where σ_{ek} and $\sigma_{\mathrm{a}k}$ reach their maxima, $\mathrm{KKR}(\sigma_{ek})+\mathrm{KKR}(\sigma_{\mathrm{a}k})$ does not change very much with the wavelength. This means that the frequency chirp for every channel is not wavelength dependent, so does the XPM for each channel. This result is consistent with the experimental results.[2,3] But it can be seen that the signs of the frequency chirp on the two sides around 1531 nm are opposite, i.e., the frequency chirp will have opposite effects on the two sides.

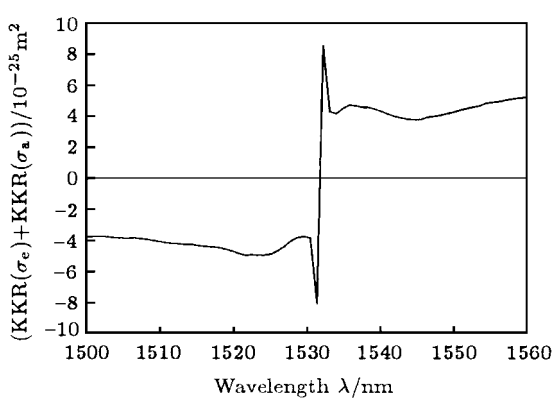


Fig.1. The diagram of$(\mathrm{KKR}(\sigma_{ek})+\mathrm{KKR}(\sigma_{\mathrm{a}k}))$changing with wavelength.

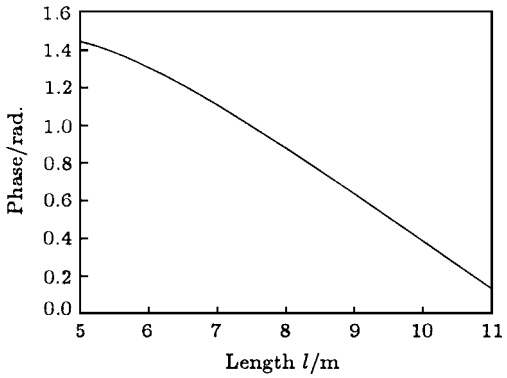


Fig.2(a). The diagram of the phase changing with the length of EDF for a Gaussian pulse.

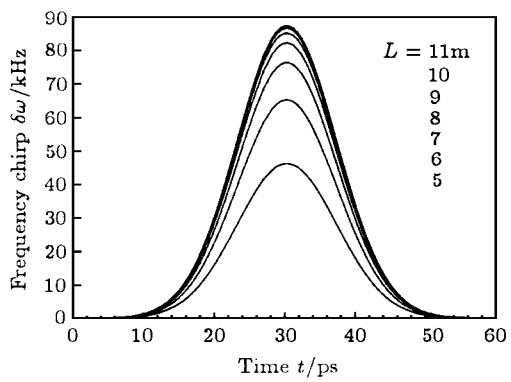


Fig.2(b). The diagram of the temporal change of the frequency chirp $\delta\omega$ changing with the length of EDF for a Gaussian pulse.

We calculated the phase and the frequency chirp the EDFA induced to a Gaussian pulse for different length of EDF, the result is shown in Fig.2, from which it can be seen that with the increase of the length of EDF the temporal variation of the frequency chirp will increase. This means that XPM will increase with the increase of the length of EDF. This is consistent with the experimental results.[3] But this increase will stop at the optimal length of the EDFA, because after this length the EDFA begins to absorb the signals.

IV. CONCLUSION

The nonlinear Schrödinger equation in erbium-doped fiber has been obtained which includes the gain and phase shift the dopant induces. This NLSE is actually applicable to the optical fibers doped with other elements. The XPM in the EDFA has benn studied based on this NLSE and the rate equations. A more generalized form of the propagation equation in EDFA is obtained which includes the phase shifts the EDFA induces. An analytical expression is given to the XPM in the EDFA. It was found that the XPM in the EDFA is not wavelength dependent except at the neighboring wavelengths (around 1531 nm) where σ_{ek} and σ_{ak} reach their maxima, and the XPM has opposite sign on the two sides around 1531 nm. Furthermore, it was found that the XPM will increase with the increase of the length of EDF.

REFERENCES

[1] R. Q. Hui, Kenneth R. Demarest, Christopher T. Allen, *J. Lightwave Technol.,* **17**(1999),1018.

[2] M. Shtaif, M. Eiselt, R. W. Tkach, R. H. Stolen, A. H. Gnauck, *IEEE Photon. Technol. Lett.,* **10**(1998),1796.

[3] Michael Eiselt, Mark Shtaif, R. W. Tkach, Felton A. Flood, Sergey Ten, Douglas Butler, *IEEE Photon. Technol. Lett.,* **11**(1999)1575.

[4] S. Reichel, W. Lenz, R. Zengerle, *IEEE Photon. Technol. Lett.,* **10**(1998),1724.

[5] Steffen Reichel, Remingius Zengerle, *J. Lightwave Technol.,* **17**(1999),1152.

[6] G. P. Agrawal, Nonlinear Fiber Optics (2nd ed) (San Diego, CA: Academic, 1995), Chap. 2.

[7] Emmanuel Desurvire, *J. Lightwave Technol.,* **8**(1990), 1517.

[8] Mark Janos, Stephen C. Guy, *J. Lightwave Technol.,* **16**(1998),542.

[9] C. Randy Giles, Emmanuel Desurvire, *J. Lightwave Technol.,* **9**(1991),271.