Volume 9, Number 10
1009-1963/2000/09(10)/0731-06
CHINESE PHYSICS
October, 2000
© 2000 Chin. Phys. Soc.

GENERAL FORMULA AND RECURRENCE FORMULA FOR RADIAL MATRIX ELEMENTS OF N-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR*

CHEN CHANG-YUAN(陈昌远)

Department of Physics, Yancheng Teachers College, Yancheng 224002, China

(Received 3 March 2000; revised manuscript received 24 June 2000)

In this paper, the general formulas and the recurrence formulas for radial matrix elements of N-dimensional isotropic harmonic oscillator are obtained. The relevant results of 2- dimensional and 3- dimensional isotropic harmonic oscillators reported in the reference papers are contained in a more general equations derived in this paper as special cases.

Keywords: N-dimensional isotropic harmonic oscillator, general formulas for radial matrix elements, recurrence formulas for radial matrix elements

PACC: 0365

I. INTRODUCTION

It is well known that the class of analytically soluble systems in quantum mechanics is limited, N-dimensional isotropic harmonic oscillator is one of the exact soluble problems. In the previous works, energy equation and radial wave functions of N-dimensional isotropic harmonic oscillator were given.[1−3] In particular, the normalized radial wave functions and the general formulas and recurrence formulas for radial matrix elements of 2-dimensional and 3-dimensional isotropic harmonic oscillators have been obtained.[4−9] On the basis of the previous works, in this paper, we will give the general formulas and recurrence formulas for radial matrix elements of N-dimensional isotropic harmonic oscillator. The relevant results of 2-dimensional and 3-dimensional isotropic harmonic oscillators in Refs.[4−9] are contained in more general conclusions of this paper as special cases.

II. THE GENERAL FORMULAS FOR RADIAL MATRIX ELEMENTS

Letting

$$\Psi(x) = \frac{\chi(r)}{r^{(N-1)/2}} \cdot Y_{J_{N-2}J_{N-3}\cdots J_1 J_0}(\theta_{N-2}, \theta_{N-3}, \cdots, \theta_1, \theta_0), \quad (1)$$

and then separating variables of the Schrödinger equation for N-dimensional isotropic harmonic oscillator ($\hbar = \omega = m = 1$) in an N-dimensional space, we can get the radial equation of N-dimensional isotropic harmonic oscillator[1−5]

$$\frac{\mathrm{d}^2\chi_{n_r J_{N-2}}}{\mathrm{d}r^2} + \left\{2E_{n_r J_{N-2}} - r^2 - \frac{1}{r^2}\left[J_{N-2}(J_{N-2}+N-2) + \left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]\right\}\chi_{n_r J_{N-2}} = 0. \quad (2)$$

Solving the above equation gives the exact energy equation and radial wave functions of N-dimensional isotropic harmonic oscillator[1,2]

$$E_{n_r J_{N-2}} = 2n_r + |J_{N-2}| + N/2, \ n_r = 0, 1, 2, \cdots, \quad (3)$$

$$\chi_{n_r J_{N-2}} \approx r^{J_{N-2}+(N-1)/2}\mathrm{e}^{-r^2/2} \cdot \mathrm{F}(-n_r, J_{N-2}+N/2, r^2), \quad (4)$$

where J_{N-2} is the quantum number of Casimir operator of SO_N group, n_r is the radial quantum number, and $\mathrm{F}(-n_r, J_{N-2}+N/2, r^2)$ is the confluent hypergeometric function. Using the relation of confluent hypergeometric function with generalized Laguerre polynomials[10]

$$\mathrm{F}(-n, \mu+1, z) = \frac{n!\Gamma(\mu+1)}{\Gamma(n+\mu+1)}\mathrm{L}_n^{\mu}(z), \quad (5)$$

we can re-write the normalized radial wave functions as

$$\chi_{n_r J_{N-2}} = N_{n_r J_{N-2}} \frac{n_r!\Gamma(J_{N-2}+N/2)}{\Gamma(n_r + J_{N-2} + N/2)} \cdot r^{J_{N-2}+(N-1)/2}\mathrm{e}^{-r^2/2}\mathrm{L}_{n_r}^{J_{N-2}+N/2-1}(r^2), \quad (6)$$

*Project supported by the Natural Science Foundation of the Education Commission of Jiangsu Province, China (Grant No:98KJD140003).

where $N_{n_r J_{N-2}}$ is the normalization constant of radial wave functions.

The radial matrix elements of N-dimensional isotropic harmonic oscillators are

$$\begin{aligned}< n_r J_{N-2}|r^p|n'_r J'_{N-2} >=&\int_0^\infty \chi_{n_r J_{N-2}} r^p \chi_{n'_r J'_{N-2}} \mathrm{d}r \\ =&\frac{N_{n_r J_{N-2}} N_{n'_r J'_{N-2}}}{2} \frac{n_r!\Gamma(J_{N-2}+N/2)}{\Gamma(n_r+J_{N-2}+N/2)} \frac{n'_r!\Gamma(J'_{N-2}+N/2)}{\Gamma(n'_r+J'_{N-2}+N/2)} \\ &\cdot \int_0^\infty (r^2)^{(J_{N-2}+J'_{N-2}+N+p-2)/2} \mathrm{e}^{-r^2} \mathrm{L}_{n_r}^{J_{N-2}+N/2-1}(r^2) \mathrm{L}_{n'_r}^{J'_{N-2}+N/2-1}(r^2) \mathrm{d}r^2. \end{aligned} \tag{7}$$

The integral formulas of containing generalized Laguerre polynomials are[10]

$$\int_0^\infty z^\lambda \mathrm{e}^{-z} \mathrm{L}_n^\mu(z) \mathrm{L}_{n'}^{\mu'}(z) \mathrm{d}z = (-1)^{n+n'} \Gamma(\lambda+1) \sum_k \begin{pmatrix} \lambda-\mu \\ n-k \end{pmatrix} \begin{pmatrix} \lambda-\mu' \\ n'-k \end{pmatrix} \begin{pmatrix} \lambda+k \\ k \end{pmatrix}, \tag{8}$$

where $\mathrm{Re}(\lambda) > -1$, then the integral convergence is ensured. Substituting Eq.(8) into Eq.(7), we obtain

$$\begin{aligned}< n_r J_{N-2}|r^p|n'_r J'_{N-2} >=&\frac{(-1)^{n_r+n'_r} N_{n_r J_{N-2}} N_{n'_r J'_{N-2}}}{2} \frac{n_r!\Gamma(J_{N-2}+N/2)}{\Gamma(n_r+J_{N-2}+N/2)} \frac{n'_r!\Gamma(J'_{N-2}+N/2)}{\Gamma(n'_r+J'_{N-2}+N/2)} \\ &\cdot \Gamma\left(\frac{J_{N-2}+J'_{N-2}+N+p}{2}\right) \sum_k \begin{pmatrix} (J'_{N-2}+p-J_{N-2})/2 \\ n_r-k \end{pmatrix} \begin{pmatrix} (J_{N-2}+p-J'_{N-2})/2 \\ n'_r-k \end{pmatrix} \\ &\cdot \begin{pmatrix} (J_{N-2}+J'_{N-2}+N+p+2k-2)/2 \\ k \end{pmatrix}. \end{aligned} \tag{9}$$

The expression (9) is the general formulas for radial matrix elements of N-dimensional isotropic harmonic oscillators. The application condition of the above expression is $p > -N$, since $\mathrm{Re}(\lambda) > -1$.

In formula (9), by letting $n'_r = n_r, J'_{N-2} = J_{N-2}$, we can get the general formula for the radial average value of N-dimensional isotropic harmonic oscillator, i.e.,

$$\begin{aligned}< n_r J_{N-2}|r^p|n_r J_{N-2} >=&\frac{[N_{n_r J_{N-2}}]^2}{2} \left[\frac{n_r!\Gamma(J_{N-2}+N/2)}{\Gamma(n_r+J_{N-2}+N/2)}\right]^2 \Gamma\left(\frac{2J_{N-2}+N+p}{2}\right) \\ &\cdot \sum_k \begin{pmatrix} p/2 \\ n_r-k \end{pmatrix} \begin{pmatrix} p/2 \\ n_r-k \end{pmatrix} \begin{pmatrix} (2J_{N-2}+N+p+2k-2)/2 \\ k \end{pmatrix}. \end{aligned} \tag{10}$$

Further letting p=0 in the above expression, it is considered that the summation of formula (10) on the right hand side only has the term $k = n_r$, and the result is

$$\begin{aligned}< n_r J_{N-2}|n_r J_{N-2} >=&\frac{[N_{n_r J_{N-2}}]^2}{2} \left[\frac{n_r!\Gamma(J_{N-2}+N/2)}{\Gamma(n_r+J_{N-2}+N/2)}\right]^2 \Gamma(J_{N-2}+N/2) \begin{pmatrix} (2J_{N-2}+N+2n_r-2)/2 \\ n_r \end{pmatrix} \\ =&\frac{[N_{n_r J_{N-2}}]^2}{2} \frac{n_r![\Gamma(J_{N-2}+N/2)]^2}{\Gamma(n_r+J_{N-2}+N/2)} = 1. \end{aligned} \tag{11}$$

Hence, the normalization constant for radial wave functions of N-dimensional isotropic harmonic oscillator is

$$N_{n_r J_{N-2}} = \sqrt{\frac{2\Gamma(J_{N-2}+N/2+n_r)}{n_r![\Gamma(J_{N-2}+N/2)]^2}}. \tag{12}$$

For 3-dimensional and 2-dimensional isotropic harmonic oscillators, the quantum numbers are $J_{N-2} = J_1 = l = 0, 1, 2, \cdots$ and $J_{N-2} = J_0 = |m| = 0, 1, 2, \cdots$,[1,2] respectively. Hence formulas (9) and (12) reduce to the general formulas for radial matrix elements, and the normalization constants for radial wave functions of 3-dimensional and 2-dimensional isotropic harmonic oscillators in Refs.[4–7], respectively.

Ⅲ. RECURRENCE FORMULA FOR RADIAL MATRIX ELEMENTS

In the above section, we gave the general formulas for radial matrix elements and radial average values of N-dimensional isotropic harmonic oscillator. In order to simplify the calculation of radial matrix elements and radial average values in the case of high power-orders, the recurrence formulas for them are derived in this section .

For the state $n_r J_{N-2}$ and the state $n'_r J'_{N-2}$, substituting Eq.(3) into Eq.(2), we have

$$\frac{\mathrm{d}^2\chi_{n_r J_{N-2}}}{\mathrm{d}r^2}+\left\{(4n_r+2|J_{N-2}|+N)-r^2-\frac{1}{r^2}\left[J_{N-2}(J_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]\right\}\chi_{n_r J_{N-2}}=0, \quad (13)$$

$$\frac{\mathrm{d}^2\chi_{n'_r J'_{N-2}}}{\mathrm{d}r^2}+\left\{(4n'_r+2|J'_{N-2}|+N)-r^2-\frac{1}{r^2}\left[J'_{N-2}(J'_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]\right\}\chi_{n'_r J'_{N-2}}=0. \quad (14)$$

Multiply $r^p\chi_{n'_r J'_{N-2}}$ and each terms of Eq.(13) together and then take integration $\int_0^\infty \cdots \mathrm{d}r$. Obviously, the three terms in the braces are matrix elements of r^p, r^{p+2} and r^{p-2}, the first term is integrated twice by parts, giving

$$\begin{aligned}
&-\int_0^\infty r^p\frac{\mathrm{d}\chi_{n_r J_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r+\int_0^\infty pr^{p-1}\chi_{n_r J_{N-2}}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
=&\left\{\left[J_{N-2}(J_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]-p(p-1)\right\}<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}>\\
&-(4n_r+2|J_{N-2}|+N)<n_r J_{N-2}|r^p|n'_r J'_{N-2}>+<n_r J_{N-2}|r^{p+2}|n'_r J'_{N-2}>. \qquad (15)
\end{aligned}$$

Multiply $r^p\chi_{n_r J_{N-2}}$ and each terms of Eq.(14) together and then take integration $\int_0^\infty \cdots \mathrm{d}r$. Obviously, the three terms in the braces are matrix elements of r^p, r^{p+2} and r^{p-2}, the first term is integrated once by parts, giving

$$\begin{aligned}
&-\int_0^\infty r^p\frac{\mathrm{d}\chi_{n_r J_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r-\int_0^\infty pr^{p-1}\chi_{n_r J_{N-2}}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
=&\left[J'_{N-2}(J'_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}>\\
&-(4n'_r+2|J'_{N-2}|+N)<n_r J_{N-2}|r^p|n'_r J'_{N-2}>+<n_r J_{N-2}|r^{p+2}|n'_r J'_{N-2}>. \qquad (16)
\end{aligned}$$

Comparing Eqs.(15) and (16), we have

$$\begin{aligned}
&-\int_0^\infty r^p\frac{\mathrm{d}\chi_{n_r J_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
=&\frac{1}{2}\left\{\left[J_{N-2}(J_{N-2}+N-2)+J'_{N-2}(J'_{N-2}+N-2)+2\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]-p(p-1)\right\}\\
&\cdot<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}>-(2n_r+2n'_r+|J_{N-2}|+|J'_{N-2}|+N)<n_r J_{N-2}|r^p|n'_r J'_{N-2}>\\
&+<n_r J_{N-2}|r^{p+2}|n'_r J'_{N-2}>, \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty pr^{p-1}\chi_{n_r J_{N-2}}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
=&\frac{1}{2}\{[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)]-p(p-1)\}<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}>\\
&-(2n_r-2n'_r+|J_{N-2}|-|J'_{N-2}|)<n_r J_{N-2}|r^p|n'_r J'_{N-2}>. \qquad (18)
\end{aligned}$$

Multiply each term of Eq.(13) by $r^{p+1}\dfrac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}$ and then take integration $\int_0^\infty \cdots \mathrm{d}r$. The first term is integrated once by parts, the result is

$$
\begin{aligned}
&-\int_0^\infty (p+1)r^p \frac{\mathrm{d}\chi_{n_rJ_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r-\int_0^\infty r^{p+1}\frac{\mathrm{d}\chi_{n_rJ_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}^2\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r^2}\mathrm{d}r\\
=&\left[J_{N-2}(J_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]\int_0^\infty r^{p-1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
&-(4n'_r+2|J_{N-2}|+N)\int_0^\infty r^{p+1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r+\int_0^\infty r^{p+3}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r. \qquad (19)
\end{aligned}
$$

Multiply each terms of Eq.(14) by $r^{p+1}\dfrac{\mathrm{d}\chi_{n_rJ_{N-2}}}{\mathrm{d}r}$ and then take integration $\int_0^\infty \cdots \mathrm{d}r$. Each term in the braces is integrated once by parts, and the result is

$$
\begin{aligned}
&\int_0^\infty r^{p+1}\frac{\mathrm{d}\chi_{n_rJ_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}^2\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r^2}\mathrm{d}r\\
=&-\left[J'_{N-2}(J'_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right](p-1)<n_rJ_{N-2}|r^{p-2}|n'_rJ'_{N-2}>\\
&-\left[J'_{N-2}(J'_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]\int_0^\infty r^{p-1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r+(4n'_r+2|J'_{N-2}|+N)\\
&\cdot(p+1)<n_rJ_{N-2}|r^p|n'_rJ'_{N-2}>+(4n'_r+2|J'_{N-2}|+N)\int_0^\infty r^{p+1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
&-(p+3)<n_rJ_{N-2}|r^{p+2}|n'_rJ'_{N-2}>-\int_0^\infty r^{p+3}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r. \qquad (20)
\end{aligned}
$$

Substituting Eq.(20) into (19) and then simplifying, we have

$$
\begin{aligned}
&-\int_0^\infty (p+1)r^p\frac{\mathrm{d}\chi_{n_rJ_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
=&-\left[J'_{N-2}(J'_{N-2}+N-2)+\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right](p-1)<n_rJ_{N-2}|r^{p-2}|n'_rJ'_{N-2}>\\
&+[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)]\int_0^\infty r^{p-1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r+(4n'_r+2|J'_{N-2}|+N)\\
&\cdot(p+1)<n_rJ_{N-2}|r^p|n'_rJ'_{N-2}>-2(2n_r-2n'_r+|J'_{N-2}|-|J_{N-2}|)\int_0^\infty r^{p+1}\chi_{n_rJ_{N-2}}\frac{\mathrm{d}\chi_{n'_rJ'_{N-2}}}{\mathrm{d}r}\mathrm{d}r\\
&-(p+3)<n_rJ_{N-2}|r^{p+2}|n'_rJ'_{N-2}>. \qquad (21)
\end{aligned}
$$

The two-term integrals on the right hand side of the above expression may be replaced by the recurrence formula

(18), and by simplifying, we have

$$
\begin{aligned}
&-\int_0^\infty r^p \frac{\mathrm{d}\chi_{n_r J_{N-2}}}{\mathrm{d}r}\frac{\mathrm{d}\chi_{n'_r J'_{N-2}}}{\mathrm{d}r}\mathrm{d}r \\
=&\left\{-\frac{\left[J_{N-2}(J_{N-2}+N-2)+J'_{N-2}(J'_{N-2}+N-2)+2\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right](p-1)}{2(p+1)}\right. \\
&\left.+\frac{[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)]^2}{2p(p+1)}\right\}<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}> \\
&+\left\{-\frac{2[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)](2n_r-2n'_r+|J_{N-2}|-|J'_{N-2}|)}{p(p+2)}\right. \\
&\left.+(2n_r+2n'_r+|J_{N-2}|+|J'_{N-2}|+N)\right\}<n_r J_{N-2}|r^{p}|n'_r J'_{N-2}> \\
&+\left\{\frac{2(2n_r-2n'_r+|J_{N-2}|-|J'_{N-2}|)^2}{(p+1)(p+2)}-\frac{p+3}{p+1}\right\}<n_r J_{N-2}|r^{p+2}|n'_r J'_{N-2}>. \qquad (22)
\end{aligned}
$$

Comparing Eq.(17) with Eq.(22), we have the recurrence formulas for the radial matrix elements, i.e.,

$$
\begin{aligned}
&\left\{\frac{(2n_r-2n'_r+|J_{N-2}|-|J'_{N-2}|)^2}{(p+1)(p+2)}-\frac{p+2}{p+1}\right\}<n_r J_{N-2}|r^{p+2}|n'_r J'_{N-2}> \\
=&\left\{\frac{[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)](2n_r-2n'_r+|J_{N-2}|-|J'_{N-2}|)}{p(p+2)}\right. \\
&\left.-(2n_r+2n'_r+|J_{N-2}|+|J'_{N-2}|+N)\right\}<n_r J_{N-2}|r^{p}|n'_r J'_{N-2}> \\
&+\left\{\frac{p\left[J_{N-2}(J_{N-2}+N-2)+J'_{N-2}(J'_{N-2}+N-2)+2\left(\frac{N-1}{2}\right)\left(\frac{N-3}{2}\right)\right]}{2(p+1)}\right. \\
&\left.-\frac{p(p-1)}{4}-\frac{[J_{N-2}(J_{N-2}+N-2)-J'_{N-2}(J'_{N-2}+N-2)]^2}{4p(p+1)}\right\}<n_r J_{N-2}|r^{p-2}|n'_r J'_{N-2}>. \qquad (23)
\end{aligned}
$$

The above expression shows that if the values of $< n_r J_{N-2}|r|n'_r J'_{N-2} >$, $< n_r J_{N-2}|r^2|n'_r J'_{N-2} >$ and $< n_r J_{N-2}|r^3|n'_r J'_{N-2} >$ are known, the matrix elements of any power-order may be calculated by using formula (23) .

In formula (23), by letting $n'_r = n_r, J'_{N-2} = J_{N-2}$, we can get the recurrence formula for the radial average value, i.e.,

$$
\begin{aligned}
&(p+2)<n_r J_{N-2}|r^{p+2}|n_r J_{N-2}> \\
=&(4n_r+2|J_{N-2}|+N)(p+1)<n_r J_{N-2}|r^{p}|n_r J_{N-2}> \\
&-\frac{p[(2J_{N-2}+N-2)^2-p^2]}{4} \\
&\cdot<n_r J_{N-2}|r^{p-2}|n_r J_{N-2}>. \qquad (24)
\end{aligned}
$$

For 3-dimensional and 2-dimensional isotropic harmonic oscillators, the quantum numbers are $J_{N-2} = J_1 = l = 0, 1, 2, \cdots$ and $J_{N-2} = J_0 = |m| = 0, 1, 2, \cdots$,[1,2] respectively. Hence the recurrence formulas (23) and (24) reduce to the recurrence formulas for radial matrix elements and the radial average value of 3-dimensional and 2-dimensional isotropic harmonic oscillators in Refs.[8,9].

IV. CONCLUSION

In this paper, the general formulas and the recurrence formulas for the radial matrix elements and the radial average value of N-dimensional isotropic harmonic oscillator have been derived. Using these formulas, we can calculate values of various power-order of radial matrix elements and radial average values for N-dimensional isotropic harmonic oscillator. The relevant results of 2-dimensional and 3-dimensional isotropic harmonic oscillators in Refs.[4–9] are contained in the more general formulas derived in this paper as special cases.

REFERENCES

[1] J. D. Louck, W. H. Schaffer, *J. Mol. Spec.,* **4**(1960),285; J. D. Louck, *J. Mol. Spec.,* **4**(1960),298;334.

[2] Y. F. Liu, J. Y. Zeng, *Science in China,* **A40**(1997),1110.

[3] Y. K. Qian, J. Y. Zeng, *Science in China,* **A36**(1993),595.

[4] J. Y. Zeng, Quantum Mechanics, Vol.**I** (2nd ed) (Science Press, Beijing, 1997), Chap.6(in Chinese).

[5] J. Y. Zeng, Quantum Mechanics, Vol.**II** (2nd ed) (Science Press, Beijing, 1997), Chap.7(in Chinese).

[6] C. F. Hou, X. D. Sun, Z. X. Zhou, Y. Li, *Acta Physica Sinica,* **48**(1999),385(in Chinese).

[7] P. Goldhammer, *Rev. Mod. Phys.,* **35**(1963)40.

[8] C. Y. Chen, *Acta Physica Sinica,* **49**(2000),607 (in Chinese).

[9] C. Y. Chen, *Acta Photonica Sinica,* **28**(1999),588 (in Chinese).

[10] Z. X. Wang, D. R. Guo, An Introduction to Special Function (Science Press, Beijing, 1979), pp.361–365(in Chinese).