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NORMALLY ORDERED FORMS OF SHAPIRO-WAGNER PHASE OPERATORS AND THEIR APPLICATION

LIANG XIAN-TING(梁先庭)

Department of Physics and Institute of Mathematics, Huaihua Teachers College, Huaihua 418008, *China*

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Based on the quantum mechanical representation $|\xi\rangle = \exp\left[-\frac{1}{2}|\xi| + \xi a_1^+ + \xi^* a_2^+ - a_1^+ a_2^+\right]|00\rangle$ constructed by Fan Hong-yi and with the help of the technique of integration within an ordered product of operators, we derive the normally ordered expressions of the Shapiro-Wagner (SW) phase operators. They are in terms of the Bessel functions. As their application we discuss the minimal uncertain relation regarding the two-mode phase and number-difference operators.

Keywords: SW phase operator, Bessel function, normally ordered product
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I. INTRODUCTION

How to define a correct quantum phase operator has been a great challenge to physicists for many years, but is still an enigma hitherto. Based on Dirac's pioneer work,[1] Susskind and Glogower[2] proposed the so-called SG quantum phase operators $e^{i\varphi} \equiv (N+1)^{-1/2}\hat{a}$, $e^{-i\varphi} \equiv \hat{a}^+((N+1)^{-1/2}$, which have been widely used in quantum optics, but they are nonunitary. By using an alternative mathematical model, Pegg and Barnett *et al.*[3] constructed a unitary phase operator in finite but arbitrarily large state space. Ban[4] also defined a phase operator for a two-mode photon system in terms of the relative-number states. One of the most attractive achievements in the field is the concept of feasible phase measurement proposed by Shapiro and Wagner (SW) in 1984,[5] According to the propositions Hradil[6] put forward a two-mode phase operator (called SW phase operator), i.e.,

$$\widehat{\exp(i\varphi)} = \sqrt{\frac{a_1 + a_2^+}{a_1^+ + a_2}}, \quad \widehat{\exp(-i\varphi)} = \sqrt{\frac{a_1^+ + a_2}{a_1 + a_2^+}}, \quad (1)$$

where a_1 and a_2 are annihilation operators of the two modes, respectively. This phase operators are nonlinear and can hardly be employed in practical use such as calculating their expectation values in miscillaneous states. Thus, deriving the normally ordered forms of the SW phase operator is of key importance for their applications. In this paper, we employ Fan's $|\xi\rangle$ representation to get the normally ordered expression of SW phase operators in terms of the Bessel functions. As an example of its application, we calculate the minimal uncertain relation of the SW phase and number-difference operators. In the appendixes we give some details of derivation.

II. NORMALLY ORDERED FORMS OF SW OPERATORS

The common eigenvector $|\xi\rangle$ of $a_1 + a_2^+$ and $a_1^+ + a_2$ is given by[7]

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + \xi a_1^+ + \xi^* a_2^+ - a_1^+ a_2^+\right]|00\rangle, \quad (2)$$

which obeys

$$(a_1 + a_2^+)|\xi\rangle = \xi|\xi\rangle, \quad (a_1^+ + a_2) = \xi^*|\xi\rangle. \quad (3)$$

With the help of the normally ordered form of projection operator of the vacuum state

$$|00\rangle\langle 00| =: \exp(-a_1^+ a_1 - a_2^+ a_2) :, \quad (4)$$

where :: denotes normal ordering, and using the technique of integration within an ordered product (IWOP) of operators,[8] one can prove that $|\xi\rangle$ satisfies the completeness relation

$$\int \frac{d^2\xi}{\pi}|\xi\rangle\langle\xi| = \int \frac{d^2\xi}{\pi} : \exp[-|\xi|^2 + \xi(a_1^+ + a_2) + \xi^*(a_1 + a_2^+) - (a_1^+ + a_2)(a_1 + a_1^+)] := 1, \quad (5)$$

and the orthonormal property

$$\langle\xi'|\xi\rangle = \pi\delta^{(2)}(\xi' - \xi). \tag{6}$$

Letting $\xi \equiv re^{i\varphi}$ and using Eq.(5), we have

$$\begin{aligned}
\widehat{\exp(i\varphi)} &= \int \frac{d^2\xi}{\pi}\sqrt{\frac{a_1 + a_2^+}{a_1^+ + a_2}}|\xi\rangle\langle\xi| \\
&= \int \frac{d^2\xi}{\pi}\sqrt{\frac{\xi}{\xi^*}} : \exp[-|\xi|^2 + \xi(a_1^+ + a_2) + \xi^*(a_1 + a_2^+) - (a_1^+ + a_2)(a_1 + a_1^+)] : \\
&= \int \frac{d^2\xi}{\pi}\sqrt{\frac{\xi}{\xi^*}} : \exp[-|\xi|^2 + \xi A + \xi^* A^+ - A^+A] : \\
&= \frac{1}{\pi} : \exp(-A^+A)\int_0^\infty r dr \int_0^{2\pi} d\varphi\, e^{i\varphi}\exp[-r^2 + r(Ae^{i\varphi} + A^+e^{i\varphi})] : \\
&= \frac{1}{\pi} : \exp(-A^+A)\int_0^{2\pi} d\varphi \exp(i\varphi + \omega^2)\int_0^\infty r\exp[-(r-\omega)^2]dr :,
\end{aligned}$$

where we have defined $A \equiv a_1^+ + a_2$ and $\omega = (Ae^{i\varphi} + A^+e^{i\varphi})/2$. To proceed, we use the identity

$$\int_0^\infty r\exp[-(r-\omega)^2]dr = \frac{1}{2}\exp(-\omega^2) + \frac{\omega\sqrt{\pi}}{2}(1 + \mathrm{erf}\omega), \tag{7}$$

where $\mathrm{erf}\omega = \frac{2}{\sqrt{\pi}}\exp(-\omega^2)\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!!}\omega^{2n+1}$ is the Error function.[9] Thus we have

$$\begin{aligned}
\widehat{\exp(i\varphi)} &= \frac{1}{\pi} : \exp(-A^+A)\int_0^{2\pi} d\varphi \exp(i\varphi + \omega^2)\Big[\frac{1}{2}\exp(-\omega^2) + \frac{\omega\sqrt{\pi}}{2}(1 + \mathrm{erf}\omega)\Big] : \\
&= \frac{1}{2\pi} : \exp(-A^+A)\Big[\int_0^{2\pi} d\varphi \exp(i\varphi + \sqrt{\pi}\int_0^{2\pi} d\varphi \exp(i\varphi + \omega^2)\omega(1 + \mathrm{erf}\omega)\Big] : \\
&= \frac{1}{2\sqrt{\pi}} : \exp(-A^+A)\Big[\int_0^{2\pi} d\varphi\omega\exp(i\varphi + \omega^2) + \int_0^{2\pi} d\varphi \exp(i\varphi + \omega^2)\omega\mathrm{erf}\omega\Big] : \\
&= \frac{1}{4\sqrt{\pi}} : \exp(-A^+A/2)(AY + A^+X) :,
\end{aligned} \tag{8}$$

where

$$X =: \int_0^{2\pi}\exp[(A^2e^{2i\varphi} + A^{+2}e^{-2i\varphi})/4] : =: \pi I_0(A^+A/2) :, \tag{9}$$

$$Y =: \int_0^{2\pi} e^{2i\varphi}\exp[(A^2e^{2i\varphi} + A^{2+}e^{-2i\varphi})/4] : =: \frac{\pi A^+}{A}I_1(A^+A/2) : . \tag{10}$$

Here $I_v(A^+A/2) =: \Big[\frac{1}{2}(A^+A/2)\Big]^v\sum_{n=0}^{\infty}\frac{\Big[\frac{1}{4}(A^+A/2)\Big]^k}{k!\Gamma(v+k+1)}$: $(v = 0, 1)$ are the zero and the first kinds of modified Bessel functions. $\Gamma(v+k+1) = (v+k)!$ is the Gamma function.[9] The detail of derivation of Eqs.(9) and (10) is given in the Appendixes. Following relations are used in Eq.(8):

$$\int_0^{2\pi} d\varphi \exp(i\varphi) = 0, \tag{11}$$

$$\int_0^{2\pi} d\varphi \exp(i\varphi + \omega^2)\omega\mathrm{erf}\omega = 0. \tag{12}$$

The proof of Eq.(12) is given in Appendix C. Therefore, we can express the SW phase operators in a compact form as

$$\begin{aligned}
\widehat{\exp(i\varphi)} =& \frac{\sqrt{\pi}}{4} : A^+\exp(-A^+A/2) \\
&\cdot [I_0(A^+A/2) + I_1(A^+A/2)] :,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\widehat{\exp(-i\varphi)} =& \frac{\sqrt{\pi}}{4} : A\exp(-A^+A/2) \\
&\cdot [I_0(A^+A/2) + I_1(A^+A/2)] : .
\end{aligned} \tag{14}$$

Similarly, we can obtain

$$\widehat{\exp(2i\varphi)} =: \frac{1}{2}\{[A^+A - 1 - (A^2+1)\exp(-A^+A)]/A^2\} :, \tag{15}$$

$$\widehat{\exp(-2\mathrm{i}\varphi)} =: \frac{1}{2}\{[A^+A - 1 - (A^{+2}+1) \cdot \exp(-A^+A)]/A^{+2}\} : . \quad (16)$$

Ⅲ. MINIMAL UNCERTAINTY RELATION OF PHASE AND NUMBER-DIFFERENCE OPERATORS

Using Eqs.(13) and (14), we can discuss the uncertain relation of SW phase and number-difference operators. Due to the fact that the exponential phase operators $\widehat{\exp(\mathrm{i}\varphi)}$ and $\widehat{\exp(-\mathrm{i}\varphi)}$ are not Hermitian, it is necessary to define the Hermitian phase operators $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ by

$$\widehat{\cos\varphi} = \frac{1}{2}[\widehat{\exp(\mathrm{i}\varphi)} + \widehat{\exp(-\mathrm{i}\varphi)}],$$
$$\widehat{\sin\varphi} = \frac{1}{2\mathrm{i}}[\widehat{\exp(\mathrm{i}\varphi)} - \widehat{\exp(-\mathrm{i}\varphi)}]. \quad (17)$$

The number-difference operator[10] $\hat{D}$ is

$$\hat{D} = a_1^+a_1 - a_2^+a_2. \quad (18)$$

Using the canonical commutation relation

$$[\hat{\varphi}, \hat{D}] = \mathrm{i}, \quad (19)$$

and the Baker-Hausdoff formula, we observe that

$$\widehat{\exp(\mathrm{i}\varphi)}\hat{D}\widehat{\exp(-\mathrm{i}\varphi)} = \hat{D} + 1. \quad (20)$$

So we have

$$[\widehat{\exp(\mathrm{i}\varphi)}, \hat{D}] = \widehat{\exp(\mathrm{i}\varphi)},$$
$$[\widehat{\exp(-\mathrm{i}\varphi)}, \hat{D}] = \widehat{\exp(-\mathrm{i}\varphi)}, \quad (21)$$

and

$$[\widehat{\cos\varphi}, \hat{D}] = \mathrm{i}\,\widehat{\sin\varphi}, \quad [\widehat{\cos\varphi}, \hat{D}] = \mathrm{i}\,\widehat{\sin\varphi}, \quad (22)$$

from which follows the uncertainty relation:

$$\Delta\hat{D}\Delta\widehat{\cos\varphi} \geq \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|,$$
$$\Delta\hat{D}\Delta\widehat{\sin\varphi} \geq \frac{1}{2}|\langle\widehat{\cos\varphi}\rangle|, \quad (23)$$

where

$$(\Delta\widehat{\cos\varphi})^2 = \langle(\widehat{\cos\varphi})^2\rangle - \langle\widehat{\cos\varphi}\rangle^2,$$
$$(\Delta\widehat{\sin\varphi})^2 = \langle(\widehat{\sin\varphi})^2\rangle - \langle\widehat{\sin\varphi}\rangle^2. \quad (24)$$

Now we use Eqs.(13)–(16), (23) and (24) to discuss the uncertainty relation of $\widehat{\cos\varphi}$ and $\hat{D}$ in two-mode coherent state $|z_1, z_2\rangle$. For the two-mode coherent state $|z_1, z_2\rangle$

$$a_1|z_1, z_2\rangle = z_1|z_1, z_2\rangle,$$
$$a_2|z_1, z_2\rangle = z_2|z_1, z_2\rangle, \quad (25)$$

we can obtain

$$\langle\widehat{\sin\varphi}\rangle = \chi_1\chi_4, \quad (26)$$

and

$$\Delta\hat{D}\Delta\widehat{\cos\varphi} = \sqrt{\chi_6 - \chi_4^2\chi_3^2}, \quad (27)$$

where we have set $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}, z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$ and

$$\chi_1 = 2(r_2\sin\theta_2 - r_1\sin\theta_1),$$
$$\chi_2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1+\theta_2) + 1,$$
$$\chi_3 = 2(r_2\sin\theta_2 + r_1\sin\theta_1),$$
$$\chi_4 = \frac{\sqrt{\pi}}{4}\exp\left(-\frac{\chi_2}{2}\right)\left[\mathrm{I}_0\left(\frac{\chi_2}{2}\right) + \mathrm{I}_1\left(\frac{\chi_2}{2}\right)\right], \quad (28)$$
$$\chi_5 = r_1^2\cos 2\theta_1 + 2r_1r_2\cos(\theta_1-\theta_2) + r_2^2\cos 2\theta_2,$$
$$\chi_6 = \frac{1}{2} + [(\chi_2-1)\chi_1 + (2\chi_2^2+\chi_5)\exp\chi_2]/\chi_2^2.$$

The uncertainty relation of $\widehat{\cos\varphi}$, and $\hat{D}$ in Eq.(23) can be shown clearly as in Figs.1 and 2, namely $\Delta\hat{D}\Delta\widehat{\cos\varphi} \geq \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$.

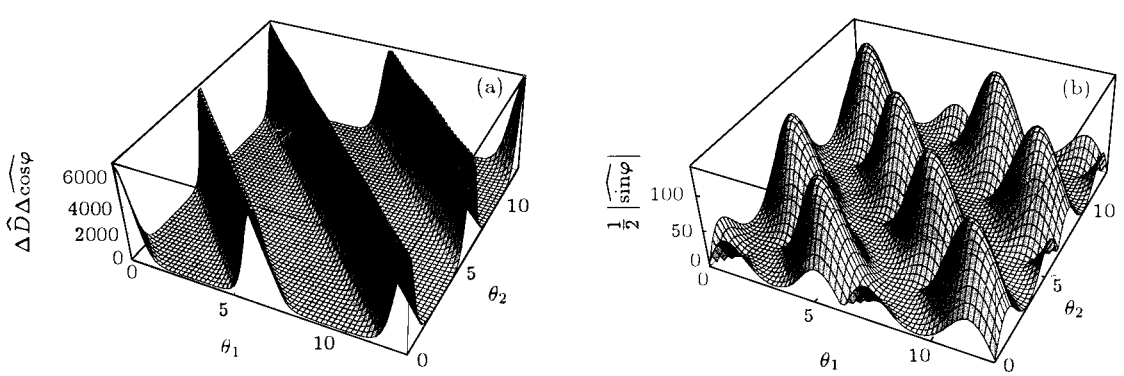


Fig.1. The values of $\Delta\hat{D}\Delta\widehat{\cos\varphi}$ (a) and $\frac{1}{2}|\widehat{\sin\varphi}|$ (b) for r_1=10, r_2=10 and different $\theta_1+\theta_2$. When $\theta_1+\theta_2 = (2n+1)\pi$, $\hat{D}$ and $\widehat{\cos\varphi}$ have their minimal uncertainty relation.

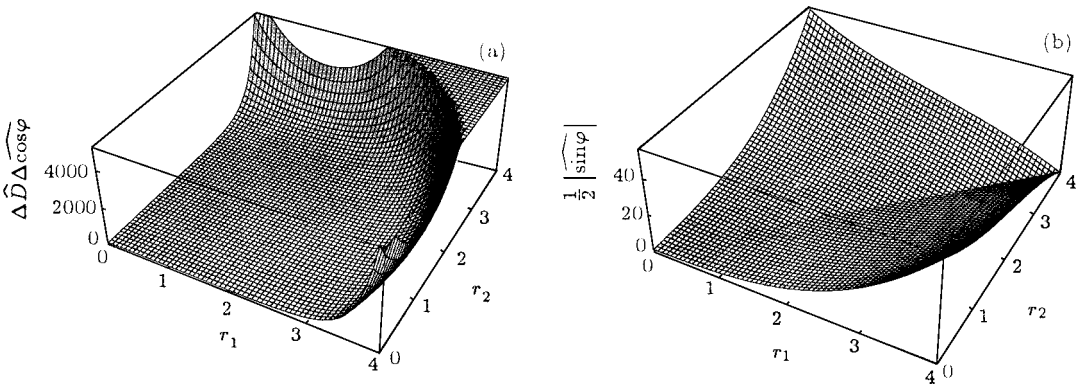


Fig.2. The values of $\Delta\hat{D}\Delta\widehat{\cos\varphi}$ (a) and $\frac{1}{2}|\widehat{\sin\varphi}|$ (b) for θ_1=2.2, θ_2=2.2 and different r_1, r_2. When $r_1 \to 0, r_2 \to 0$, $\hat{D}$ and $\widehat{\cos\varphi}$ have their minimal uncertainty relation..

In Figs.1(a) and 1(b) we plot the $\Delta\hat{D}\Delta\widehat{\cos\varphi}$ and $\frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$ for $r_1 = 10, r_2 = 10$ and different values of θ_1 and θ_2. When $\theta_1 + \theta_2 = (2n+1)\pi$ $(n = 0, 1, 2, ...)$, we have $\Delta\hat{D}\Delta\widehat{\cos\varphi} \to \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$, for other values of $\theta_1 + \theta_2$, $\Delta\hat{D}\Delta\widehat{\cos\varphi} > \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$. It is indicated that $\Delta\hat{D}\Delta\widehat{\cos\varphi} \geq \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$, which is in agreement with the first term in Eq.(23). The figure further indicate when $\theta_1 + \theta_2 = (2n+1)\pi$ $(n = 0, 1, 2, ...)$, $\hat{D}$ and $\widehat{\cos\varphi}$ have their minimal uncertainty relation. In Figs.2(a) and 2(b) we plot the $\Delta\hat{D}\Delta\widehat{\cos\varphi}$ and $\frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$ for θ_1=2.2, θ_2=2.2 and different values of r_1 and r_2. When $\theta_1 + \theta_2 \neq (2n+1)\pi$, if and only if $r_1 \to 0$ and $r_2 \to 0$, we have $\Delta\hat{D}\Delta\widehat{\cos\varphi} \to \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle| = 0$, and for other values of r_1 and r_2, $\Delta\hat{D}\Delta\widehat{\cos\varphi} > \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$, namely $\Delta\hat{D}\Delta\widehat{\cos\varphi} \geq \frac{1}{2}|\langle\widehat{\sin\varphi}\rangle|$, which also agrees with the first term in Eq.(23). The figures indicate when $r_1 = 0, r_2 = 0$, $\hat{D}$ and $\widehat{\cos\varphi}$ have minimal uncertainty relation. In a word, from Figs.1 and 2 we obtain that when $r_1 \to 0$ or $r_2 \to 0$ (for arbitrary θ_1 and θ_2), or $\theta_1 + \theta_2 = (2n+1)\pi$ (for arbitrary r_1 and r_2), $\hat{D}$ and $\widehat{\cos\varphi}$ have their minimal uncertainty, which has not been pointed out in the literature we knew before. Similarly, we can discuss the uncertainty relation of $\hat{D}$ and $\widehat{\sin\varphi}$ and can obtain similar result.

In summary, in this paper we have introduced the normally ordered form of SW phase operators in terms of Bessel functions. A suitable form is helpful for the investigation of the phase properties of two-mode quantum systems.

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APPENDIX A

In this appendix we prove Eq.(9). Set $z = \mathrm{e}^{2\mathrm{i}\varphi}$, then $2\mathrm{i}\mathrm{e}^{2\mathrm{i}\varphi}\mathrm{d}\varphi = \mathrm{d}z$, so we have

$$X = \int_0^{2\pi} \mathrm{d}\varphi \exp\left(\frac{A^2}{4}\mathrm{e}^{2\mathrm{i}\varphi} + \frac{A^{+2}}{4}\mathrm{e}^{-2\mathrm{i}\varphi}\right) = \frac{1}{2\mathrm{i}}\oint_{|z|=1} \mathrm{d}z \sum_{n=0}^{\infty} \frac{C_n^m (A^2/4)^m z^{2m} (A^{+2}/4)^{n-m}}{n! z^{n+1}}. \tag{A.1}$$

At $z = 0$, only when $2m = n$, $X \neq 0$. So we have

$$X = 2\pi\mathrm{i}\frac{1}{2\mathrm{i}}\sum_{n=0}^{\infty} \frac{C_{2n}^m (A^2/4)^m (2m)! (A^{+2}/4)^m}{(2m)!(2m)!} = \pi \sum_{n=0}^{\infty} \frac{\left[\frac{1}{4}\left(\frac{A^+A}{2}\right)^2\right]^m}{(m)!(m)!} = \pi \mathrm{I}_0\left(\frac{A^+A}{2}\right). \tag{A.2}$$

APPENDIX B

In this appendix we prove Eq.(10).

$$Y = \int_0^{2\pi} \mathrm{d}\varphi \mathrm{e}^{2\mathrm{i}\varphi} \exp\left(\frac{A^2}{4}\mathrm{e}^{2\mathrm{i}\varphi} + \frac{A^{+2}}{4}\mathrm{e}^{-2\mathrm{i}\varphi}\right) = \frac{1}{2\mathrm{i}}\oint_{|z|=1} \mathrm{d}z \left(\frac{A^2}{4}z + \frac{A^{+2}}{4}z^{-1}\right) = \frac{1}{2\mathrm{i}}\oint_{|z|=1} \mathrm{d}z \sum_{n=0}^{\infty} \frac{(A^2 z^2/4 + A^{+2}/4)^n}{n! z^n}. = \frac{1}{2\mathrm{i}}\oint_{|z|=1} \mathrm{d}z \sum_{n=0}^{\infty} \frac{C_n^m (A^2/4)^m z^{2m} (A^{+2}/4)^{n-m}}{n! z^n}. \tag{B.1}$$

At z=0, only when $2m = n-1, Y \neq 0$. So we have

$$\begin{aligned} Y &= 2\pi \mathrm{i}\frac{1}{2\mathrm{i}}\sum_{n=0}^{\infty}\frac{C_{2m+1}^{m}(A^2/4)^m(2m)!(A^{+2}/4)^{m+1}}{(2m)!(2m+1)!} \\ &= \pi\sum_{n=0}^{\infty}\frac{(2m+1)!(A^2/4)^m(2m)!(A^{+2}/4)^{m+1}}{(2m)!(2m+1)!m!(m+1)!} \\ &= \pi\sum_{n=0}^{\infty}\frac{(A^2/4)^m(A^{+2}/4)^{m+1}}{m!(m+1)!} \\ &= \frac{\pi A^{+2}}{4}\sum_{n=0}^{\infty}\frac{\left[\frac{1}{4}(A^+A/2)^2\right]^m}{m!(m+1)!} \\ &= \pi\frac{A^+}{A}\frac{1}{2}\left(\frac{A^+A}{2}\right)\sum_{n=0}^{\infty}\frac{\left[\frac{1}{4}\left(\frac{A^+A}{2}\right)^2\right]^m}{m!(m+1)!} \\ &= \pi\frac{A^+}{A}\mathrm{I}_1\left(\frac{A^+A}{2}\right). \end{aligned} \tag{B.2}$$

APPENDIX C

In this appendix we prove Eq.(12).

$$\begin{aligned} W &\equiv \int_0^{2\pi}\mathrm{d}\varphi\exp(\mathrm{i}\varphi+\omega^2)\omega\mathrm{erf}\omega \\ &= \int_0^{2\pi}\mathrm{d}\varphi\exp(\mathrm{i}\varphi+\omega^2)\omega\frac{2}{\sqrt{\pi}}\exp(-\omega^2)\sum_{n=0}^{\infty}\frac{2^n\omega^{2n+1}}{(n+1)!!} \\ &= \frac{2}{\sqrt{\pi}}\int_0^{2\pi}\mathrm{d}\varphi\mathrm{e}^{\mathrm{i}\varphi}\sum_{n=0}^{\infty}\frac{2^n(A\mathrm{e}^{\mathrm{i}\varphi}+A^+\mathrm{e}^{-\mathrm{i}\varphi})^{2n+2}}{2^{2n+2}(n+1)!!} \\ &= \frac{1}{\sqrt{\pi}}\oint_{|z|=1}\frac{\mathrm{d}z}{\mathrm{i}}\sum_{n=0}^{\infty}\frac{(Az^2+A^+)^{2n+2}}{2^{2n+2}(n+1)!!z^{2n+2}}. \end{aligned} \tag{C.1}$$

At z=0, only when $2m = 2n+1$ and m, n are integers, we have $W \neq 0$. Because $2m = 2n+1$ is inconsistent with m and n being integers. We can immediately obtain

$$W \equiv \int_0^{2\pi}\mathrm{d}\varphi\exp(\mathrm{i}\varphi+\omega^2)\omega\mathrm{erf}\omega = 0. \tag{C.2}$$

Similarly Eq.(11) can also be proved easily.

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