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PERTURBATION TO THE SYMMETRIES AND ADIABATIC INVARIANTS OF HOLONOMIC VARIABLE MASS SYSTEMS*

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The perturbation problem of symmetries for the holonomic variable mass systems under small excitation is discussed. The concept of high-order adiabatic invariant is presented, and the form of adiabatic invariants and the conditions for their existence are given. Then the corresponding inverse problem is studied. Finally an example is presented to illustrate these results.

Keywords: analytical mechanics, variable mass, symmetry, perturbation, adiabatic invariant
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I. INTRODUCTION

More and more attention has been paid to the research of symmetries and conserved quantities of mechanical systems and many results have sprung up during recent years. These results turn the knowledge of people to the essence of mechanical system.[1,2] In 1917, Burgers first proposed adiabatic invariants which pointed to a special kind of Hamilton system.[3] The adiabatic invariants mean that they are almost not changed when the parameter varies very slowly and they play a very important role in the research on quasi-integrability of a mechanical system. The rigorous mathematical meaning of the adiabatic invariants is a subtle work, so far, it has not been solved yet.[4] In fact, that the parameter varies very slowly, is equal to the action of a small perturbation, so we would make researches by the latter model in this paper.

Due to the development of the space and other technologies, the study on the dynamics of variable mass systems becomes more and more important. The differential equations of motion,[5−7] the Hamilton's principle,[7,8] the Noether's theory,[9] the Lie symmetries and conserved quantities of the variable mass systems,[10] etc., have been obtained. But the problem of adiabatic invariants of the systems has not been dealt with.

The study of adiabatic invariants has become a popular subject in mechanics,[11] atomic and molecular physics,[12,13] etc. In Refs.[14, 15] adiabatic invariants of general mechanical system were studied and some important results obtained. In this paper, we discuss the adiabatic invariants of the holonomic variable mass systems from the viewpoint of symmetries and conserved quantities. Firstly, we propose the concept of higher order adiabatic invariants. Then, we prove the conditions for their existence and give their forms. Finally we present the relationship between the symmetries of the mechanical systems and the adiabatic invariants.

II. EQUATIONS OF MOTION OF THE SYSTEMS

Let us consider a mechanical system of N particles. At time t, the mass of the i-th particle is $m_i (i = 1, ..., N)$; at time $t + \mathrm{d}t$, the mass of the corpuscle abandoned from (or merging into) the particle is $\mathrm{d}m_i$. Let the position of the system be determined by the n generalized coordinates $q_s (s = 1, ..., n)$ and let[5]

$$m_i = m_i(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{1}$$

the equations of motion can be written in the form[6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s \quad (s = 1, ..., n), \tag{2}$$

where L is Lagrangian of the system, Q_s are generalized nonpotential forces, P_s are generalized reverse thrusts, and we have[5,6]

$$P_s = \sum_{i=1}^{N} \Big\{ (\boldsymbol{R}_i + \dot{m}_i \dot{\boldsymbol{r}}_i) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \frac{\partial m_i}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \frac{\partial m_i}{\partial \dot{q}_s}\Big) \Big\}, \tag{3}$$

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where $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are respectively the position vector and the velocity vector of the i-th particle and

$$\boldsymbol{R}_i = \frac{\mathrm{d}m_i}{\mathrm{d}t}\boldsymbol{u}_i, \tag{4}$$

where $\boldsymbol{u}_i$ is the corpuscle's velocity relative to the i-th particle.

Ⅲ. INFINITESIMAL TRANSFORMATIONS AND CONSERVED QUANTITIES

Introducing the infinitesimal transformations

$$t^* = t + \Delta t, \quad q_s^* = q_s + \Delta q_s \quad (s = 1, ..., n) \tag{5}$$

or their expanded formulas

$$\begin{aligned} t^* =& t + \varepsilon\tau^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad q_s^* = q_s + \varepsilon\xi_s^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \\ & (s = 1, ..., n) \end{aligned} \tag{6}$$

and taking the generator vector of the infinitesimal transformations

$$X^{(0)} = \tau^0\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s^0\frac{\partial}{\partial q_s}, \tag{7}$$

and its first extended generator

$$X^{(1)} = X^{(0)} + \sum_{s=1}^{n}(\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0)\frac{\partial}{\partial \dot{q}_s}, \tag{8}$$

we have:

Proposition 1. If there exists a gauge function $G^0 = G^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$\begin{aligned} L\dot{\tau}^0 + X^{(1)}(L) + \sum_{s=1}^{n}(Q_s + P_s)(\xi_s^0 - \dot{q}_s\tau^0) \\ + \dot{G}^0 = 0, \end{aligned} \tag{9}$$

then the holonomic variable mass system has the following conserved quantity:

$$I = L\tau^0 + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\xi_s^0 - \dot{q}_s\tau^0) + G^0 = \text{const.} \tag{10}$$

Proof

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} =& \dot{L}\tau^0 + L\dot{\tau}^0 + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s^0 - \ddot{q}_s\tau^0 - \dot{q}_s\dot{\tau}^0) \\ & + \sum_{s=1}^{n}(\xi_s^0 - \dot{q}_s\tau^0)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} \\ & - L\dot{\tau}^0 - X^{(1)}(L) - \sum_{s=1}^{n}(Q_s + P_s)(\xi_s^0 - \dot{q}_s\tau^0) \\ =& \sum_{s=1}^{n}(\xi_s^0 - \dot{q}_s\tau^0)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s - P_s\Big) = 0. \end{aligned}$$

Ⅳ. PERTURBATION TO THE SYMMETRIES AND ADIABATIC INVARIANTS

Firstly we give the concept of high-order adiabatic invariants.

Definition If $I_z(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a z-th order adiabatic invariant of the mechanical system.

Suppose the holonomic variable mass system corresponding to Eq.(2) is perturbed by a small quantity εW_s, then the equations of motion of the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s + \varepsilon W_s \quad (s = 1, ..., n), \tag{11}$$

due to the action of εW_s, the primary symmetries and conserved quantities of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformation of the system without perturbation, and $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ express the generators of time and space respectively after being perturbed, then

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \quad \xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots, \tag{12}$$

and they satisfy

$$\begin{aligned} L\dot{\tau} + X^{(1)'}(L) + \sum_{s=1}^{n}(Q_s + P_s)(\xi_s - \dot{q}_s\tau) \\ + \varepsilon\sum_{s=1}^{n}W_s(\xi_s - \dot{q}_s\tau) + \dot{G} = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} X^{(1)'} =& X^{(0)'} + \sum_{s=1}^{n}(\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s} \\ =& \tau\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s\frac{\partial}{\partial q_s} + \sum_{s=1}^{n}(\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s}, \end{aligned}$$

with G in Eq.(13) being a gauge function. Let

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \tag{14}$$

$$X^{(1)^*} = \tau^k\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s^k\frac{\partial}{\partial q_s} + \sum_{s=1}^{n}(\dot{\xi}_s^k - \dot{q}_s\dot{\tau}^k)\frac{\partial}{\partial \dot{q}_s}. \tag{15}$$

Substituting Eqs.(12) and (14) into Eq.(13), we obtain

$$\begin{aligned} L\dot{\tau}^k + X^{(1)^*}(L) + \sum_{s=1}^{n}(Q_s + P_s)(\xi_s^k - \dot{q}_s\tau^k) \\ + \sum_{s=1}^{n}W_s(\xi_s^{k-1} - \dot{q}_s\tau^{k-1}) + \dot{G}^k = 0 \\ (k = 0, 1, 2, \cdots). \end{aligned} \tag{16}$$

When $k=0$, the condition W_s=0 holds, and we may prove that, for the differential equation (11) of the mechanical system, the following quantity:

$$I_z=\sum_{k=0}^{z}\varepsilon^k\left[L\tau^k+\sum_{s=1}^{n}\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s^k-\dot{q}_s\tau^k)+G^k\right] \quad (17)$$

is a z-th order adiabatic invariant of the mechanical system, i.e., $\dfrac{\mathrm{d}I_z}{\mathrm{d}t}$ is in direct proportion to ε^{z+1}. In fact

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\sum_{k=0}^{z}\varepsilon^k\Big[\dot{L}\tau^k+L\dot{\tau}^k+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\dot{\xi}_s^k-\ddot{q}_s\tau^k-\dot{q}_s\dot{\tau}^k)\\&+\sum_{s=1}^{n}(\xi_s^k-\dot{q}_s\tau^k)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-L\dot{\tau}^k-X^{(1)^*}(L)\\&-\sum_{s=1}^{n}(Q_s+P_s)(\xi_s^k-\dot{q}_s\tau^k)\\&-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Big]\\=&\sum_{k=0}^{z}\varepsilon^k\Big[\sum_{s=1}^{n}(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q_s-P_s)(\xi_s^k-\dot{q}_s\tau^k)\\&-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Big].\end{aligned}$$

Using Eq.(11), we have

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\sum_{k=0}^{z}\varepsilon^k\Big[\sum_{s=1}^{n}\varepsilon W_s(\xi_s^k-\dot{q}_s\tau^k)\\&-\sum_{s=1}^{n}W_s(\xi_s^{k-1}-\dot{q}_s\dot{\tau}^{k-1})\Big].\end{aligned}$$

Expanding the above formula and making summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^{z+1}\sum_{s=1}^{n}W_s(\xi_s^z-\dot{q}_s\tau^z). \quad (18)$$

So we have

Proposition 2. For a holonomic variable mass system perturbed by a small quantity εW_s, if the generators $\tau^k(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\xi_s^k(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ under infinitesimal transformations satisfy Eq.(16), and $G^k(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the gauge function, then

$$I_z=\sum_{k=0}^{z}\varepsilon^k\left[L\tau^k+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s^k-\dot{q}_s\tau^k)+G^k\right]$$

is a z-th order adiabatic invariant of the mechanical system.

V. INVERSE PROBLEM

Suppose that the system has a first order adiabatic invariant as follows:

$$I_1=\lambda_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\lambda_1(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \quad (19)$$

Differentiating this with respect to t, we have

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}t}=&\frac{\partial\lambda_0}{\partial t}+\sum_{s=1}^{n}\Big(\frac{\partial\lambda_0}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_0}{\partial\dot{q}_s}\ddot{q}_s\Big)\\&+\varepsilon\Big[\frac{\partial\lambda_1}{\partial t}+\sum_{s=1}^{n}\Big(\frac{\partial\lambda_1}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_1}{\partial\dot{q}_s}\ddot{q}_s\Big)\Big]. \quad (20)\end{aligned}$$

Multiplying Eq.(11) by $\xi_s-\dot{q}_s\tau$, and taking summation for s, we have

$$\sum_{s=1}^{n}(\xi_s-\dot{q}_s\tau)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q_s-P_s-\varepsilon W_s\Big)=0. \quad (21)$$

From formula (3) we know that P_s are linear with respect to $\ddot{\boldsymbol{q}}$ in general case

$$P_s=\sum_{r=1}^{n}V_{sr}(t,\boldsymbol{q},\dot{\boldsymbol{q}})\ddot{q}_r+V_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad (22)$$

where

$$\begin{aligned}V_{sr}=&\sum_{i=1}^{n}\Big\{\frac{\partial m_i}{\partial\dot{q}_r}(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i)\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_s}\\&+\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial^2 m_i}{\partial\dot{q}_s\partial\dot{q}_r}+\frac{\partial m_i}{\partial\dot{q}_s}\dot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_r}\Big\}.\end{aligned}$$

If

$$\frac{\partial m_i}{\partial\dot{q}_s}=0\quad(i=1,...,N;\quad s=1,...,n),$$

we have

$$V_{sr}=0.$$

By using Eq.(18), and adding Eqs.(20) and (21), we have

$$\begin{aligned}&\frac{\partial\lambda_0}{\partial t}+\sum_{s=1}^{n}\Big(\frac{\partial\lambda_0}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_0}{\partial\dot{q}_s}\ddot{q}_s\Big)\\&+\varepsilon\Big[\frac{\partial\lambda_1}{\partial t}+\sum_{s=1}^{n}\Big(\frac{\partial\lambda_1}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_1}{\partial\dot{q}_s}\ddot{q}_s\Big)\Big]\\&+\sum_{s=1}^{n}(\xi_s-\dot{q}_s\tau)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}-Q_s-P_s-\varepsilon W_s\Big)\\=&\varepsilon^2\sum_{s=1}^{n}W_s(\xi_s^1-\dot{q}_s\tau^1), \quad (23)\end{aligned}$$

where the expressions of τ and ξ_s are Eqs.(12).

Firstly, we seek the generators τ^0 and ξ_s^0 of the infinitesimal transformations without perturbation. Separating the terms not containing ε in Eq.(23), then

separating the terms containing $\ddot{q}_r$ and taking their coefficients as zeros, we have

$$\frac{\partial\lambda_0}{\partial\dot{q}_r}+\sum_{s=1}^{n}\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_r}-V_{sr}\right)(\xi_s^0-\dot{q}_s\tau^0)=0. \tag{24}$$

Let

$$\det(\omega_{sr})=\det\left(V_{sr}-\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_r}\right)\neq 0,$$

from Eq.(24) we can obtain

$$\xi_s^0-\dot{q}_s\tau^0=\sum_{r=1}^{n}\tilde{\omega}_{sr}\frac{\partial\lambda_0}{\partial\dot{q}_r}, \tag{25}$$

where

$$\tilde{\omega}_{sk}\omega_{kr}=\delta_{sr}.$$

Further let

$$\lambda_0=L\tau^0+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s^0-\dot{q}_s\tau^0)+G^0. \tag{26}$$

Thus we can seek the generators τ^0 and ξ_s^0 of the infinitesimal transformations from Eqs.(25) and (26) when the gauge function G^0 is given.

Further, separating the terms containing ε in Eq.(23), then separating the terms containing $\ddot{q}_r$ and taking their coefficients as zeros, we have

$$\xi_s^1-\dot{q}_s\tau^1=\sum_{r=1}^{n}\tilde{\omega}_{sr}\frac{\partial\lambda_1}{\partial\dot{q}_r}, \tag{27}$$

$$\lambda_1=L\tau^1+\sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s^1-\dot{q}_s\tau^1)+G^1. \tag{28}$$

Thus we can seek the generators τ^1 and ξ_s^1 of the infinitesimal transformations from Eqs.(27) and (28) when the gauge function G^1 is given.

VI. AN ILLUSTRATIVE EXAMPLE

Consider a variable mass particle whose mass is

$$m=m_0\mathrm{e}^{-\alpha t}\quad(m_0=\text{const},\alpha=\text{const},\alpha>0). \tag{29}$$

Its Lagrangian is

$$L=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2), \tag{30}$$

the generalized nonpotential forces are

$$Q_1=Q_1(t,q_1,\dot{q}_1,\dot{q}_2),\quad Q_2=C_2\dot{q}_2+C_1q_1\dot{q}_1, \tag{31}$$

where C_1 and C_2 are constants. The corpuscle's absolute velocity is zero, i.e.,

$$\boldsymbol{u}=-\dot{\boldsymbol{r}}=-\dot{q}_1\boldsymbol{i}-\dot{q}_2\boldsymbol{j}, \tag{32}$$

and the system is perturbed by the small quantities as follows:

$$W_1=-\frac{D_1}{1+b^2t^2},\quad W_2=-\frac{D_2bt}{1+b^2t^2}, \tag{33}$$

where D_1, D_2 and b are constants. Try to study the adiabatic invariants of the system.

From Eqs.(3), (4) and (32) we know

$$P_1=P_2=0. \tag{34}$$

Equations(11) give

$$\begin{aligned}(m\dot{q}_1)^{\cdot}&=Q_1-\frac{D_1\varepsilon}{1+b^2t^2},\\(m\dot{q}_2)^{\cdot}&=C_2\dot{q}_2+C_1q_1\dot{q}_1-\frac{D_2\varepsilon bt}{1+b^2t^2}.\end{aligned} \tag{35}$$

Firstly, we seek the 0-th order adiabatic invariant, i.e., the exact invariant. The structure equation (16) gives

$$\begin{aligned}L\dot{\tau}^0+\tau^0\frac{\partial L}{\partial t}+\sum_{s=1}^{2}\xi_s^0\frac{\partial L}{\partial q_s}+\sum_{s=1}^{2}(\dot{\xi}_s^0-\dot{q}_s\dot{\tau}^0)\frac{\partial L}{\partial\dot{q}_s}\\+\sum_{s=1}^{2}Q_s(\xi_s^0-\dot{q}_s\tau^0)+\dot{G}^0=0.\end{aligned} \tag{36}$$

Choosing the infinitesimal generators as

$$\tau^0=0,\quad \xi_1^0=0,\quad \xi_2^0=1, \tag{37}$$

then Eq.(36) becomes

$$Q_2+\dot{G}^0=0, \tag{38}$$

so we obtain

$$G^0=-C_2q_2-\frac{1}{2}C_1q_1^2. \tag{39}$$

From proposition 1, we can obtain the 0-th order adiabatic invariant

$$I_0=m(t)\dot{q}_2-C_2q_2-\frac{1}{2}C_1q_1^2=\text{const}. \tag{40}$$

Secondly, we seek the first order adiabatic invariant. The structure equation (16) gives

$$\begin{aligned}L\dot{\tau}^1+\tau^1\frac{\partial L}{\partial t}+\sum_{s=1}^{2}\xi_s^1\frac{\partial L}{\partial q_s}+\sum_{s=1}^{2}(\dot{\xi}_s^1-\dot{q}_s\dot{\tau}^1)\frac{\partial L}{\partial\dot{q}_s}\\+\sum_{s=1}^{2}Q_s(\xi_s^1-\dot{q}_s\tau^1)+\sum_{s=1}^{2}W_s(\xi_s^0+\dot{q}_s\tau^0)+\dot{G}^1=0.\end{aligned} \tag{41}$$

Choosing the infinitesimal generators as

$$\tau^1 = 0, \quad \xi_1^1 = 0, \quad \xi_2^1 = 1, \tag{42}$$

then Eq.(41) becomes

$$Q_2 + W_2 + \dot{G}^1 = 0, \tag{43}$$

so we obtain

$$G^1 = -C_2 q_2 - \frac{1}{2} C_1 q_1^2 + \frac{D_2}{2b} \ln(1 + b^2 t^2). \tag{44}$$

From proposition 2, we can obtain the first order adiabatic invariant

$$I_1 = m(t)\dot{q}_2 - C_2 q_2 - \frac{1}{2} C_1 q_1^2 + \varepsilon \Big[m(t)\dot{q}_2 - C_2 q_2 - \frac{1}{2} C_1 q_1^2 + \frac{D_2}{2b} \ln(1 + b^2 t^2) \Big] = \text{const.} \tag{45}$$

Further we can obtain higher order adiabatic invariants.

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