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ABUNDANT DROMION-LIKE STRUCTURES TO THE (2+1) DIMENSIONAL KdV EQUATION*

ZHANG JIE-FANG(张解放)

Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, China;

Research Center of Engineering Science, Zhejiang University of Technology, Hangzhou 310032, China

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The abundant generalized dromion structures for the (2+1)-dimensional KdV equation are obtained using the homogeneous balance method. We give not only the general curve soliton which is finite on a curved line and localized apart from the curve, find but also the dromion solutions which can be driven by two perpendicular line soliton and by two non-perpendicular line soliton and by one line soliton and one curve line soliton. Various types of multi-dromion solutions can be constituted by selecting different arbitrary functions of y. The $(1+N)$ dromion obtained by Radha *et al.*[3] is only a very special case of our results.

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The study of the exponentially localized soliton solutions, which was called dromions in (2+1) dimensions, has attracted much attention of physicists and mathematicians. For instance, for some (2+1) dimensional integrable models such as the Devey-Stewartson (DS) equation[1], the Kadomtsev-Petviashvilli (KP) equation[2], the Nizhnik-Novikov-Veselov (NNV) equation[3], the breaking soliton equation[4] and the long dispersive wave equation[5], the dromion-like structures are found by using different approaches. Generally, a dromion solution occurs at the cross point of two nonparallel line solitons. A line soliton is finite on a infinitely long straight line while it exponentially decays in other directions. For the DS equation and the NNV equations, the dromion solution can be obtained from two perpendicular line ghost solitons[1,3]; for the KP equation, the dromion solution can be driven by non-perpendicular line soliton[2]; for the breaking soliton equation and the long dispersive wave equation, the dromion solution can be gotten from perpendicular line soliton and line ghost soliton.[4,5] In this paper, we would like to study the dromion structures of more generalized form for the (2+1) dimesional KdV equation

$$u_t + u_{xxx} = 3(u\partial_y^{-1}u_x)_x\,, \tag{1}$$

which reduces to the KdV equation if $x = y$. This nonlocal equation has been studied first by Boiti *et al.*[6] and the Cauchy problem has been linearized. Radha and Lakshmanan[3] carried out its singularity structure analysis and constructed its bilinear form and used it to generate multidromion solutions.

Now we use the homogeneous balance method[7] to construct its dromion-like structures of more generalized form. In order that nonlinear term and higher order partial derivative term in Eq.(1) can be partially balanced, we consider the following conjecture solution form

$$u(x,y,t) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f[w(x,y,t)] = f''w_xw_y + f'w_{xy}\,, \tag{2}$$

where prime means derivative with respect to w, $f(w)$ is a function of one argument w only, $w(x,y,t)$ and $f(w)$ are differentiable functions. From Eq.(2) it is easy to deduce that

$$u_t = f^{(3)}w_xw_yw_t + f''w_{xt}w_y + f''w_xw_{yt} + f''w_tw_{xy} + f'w_{xyt}\,, \tag{3}$$

$$\begin{aligned} u_{xxx} =& f^{(5)}w_x^4w_y + 4f^{(4)}w_x^3w_{xy} + 6f^{(4)}w_x^2w_yw_{xx} + 12f^{(3)}w_xw_{xx}w_{xy} + 6f^{(3)}w_x^2w_{xxy} + 4f^{(3)}w_xw_yw_{xxx} \\ &+ 4f''w_{xy}w_{xxx} + 4f''w_xw_{xxxy} + 3f^{(3)}w_yw_{xx}^2 + 6f''w_{xx}w_{xxy} + f''w_yw_{xxxx} + f'w_{xxxxy}\,, \end{aligned} \tag{4}$$

$$\begin{aligned} 3(u\partial_y^{-1}u_x)_x =& 6f''f^{(3)}w_x^4w_y + 3f'f^{(3)}(w_x^3w_{xy} + w_x^2w_{xx}w_y) + 6f''^2(w_x^3w_{xy} + 2w_x^2w_{xx}w_y) \\ &+ 3f'f''(5w_xw_{xx}w_{xy} + w_xw_{xxx}w_y + w_x^2w_{xxy} + w_{xx}^2w_y) + 3f'^2(w_{xxx}w_{xy} + w_{xx}w_{xxy})\,. \end{aligned} \tag{5}$$

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Substituting Eqs.(3)—(5) into Eq.(1) and collecting all homogeneous terms in partial derivatives of $w(x,y,t)$ we have

$$(f^{(5)}-6f''f^{(3)})w_x^4w_y+f^{(4)}(4w_x^3+6w_x^2w_{xx}w_y)+f^{(3)}(w_xw_yw_t+12w_xw_{xy}w_{xx}+6w_x^2w_{xxy}+4w_xw_yw_{xx}$$
$$+3w_{xx}^2w_y)+f''(w_{xt}w_y+w_{yt}w_x+w_{xy}w_t+4w_{xy}w_{xxx}+4w_xw_{xxxy}+6_{xx}w_{xy}+w_yw_{xxx})$$
$$-3f'f^{(3)}(w_x^3w_{xy}+w_x^2w_{xx}w_y)-3f''^2(4w_x^2w_{xx}w_y+2w_x^3w_{xy})-3f'f''(5w_xw_{xx}w_{xy}+w_xw_{xxx}w_y$$
$$+w_x^2w_{xxy}+w_{xx}^2w_y)-3f'^2(w_{xxx}w_{xy}+w_{xx}w_{xxy})+f'(w_{xyt}+w_{xxxxy})=0\,. \tag{6}$$

Setting the coefficients of $w_x^4w_y$ in Eq.(6) to zero yields an ordinary differential equation for $f[w(x,y,t)]$

$$f^{(5)}-6f''f^{(3)}=0\,, \tag{7}$$

which indicates that the nonlinear term and the higher order derivative term appearing in Eq.(1) have been partially balanced. Solving Eq.(7), we have

$$f=-2\ln w\,, \tag{8}$$

which yields

$$f''^2=\frac{1}{3}f^{(4)},\quad f'f^{(3)}=\frac{2}{3}f^{(4)},$$
$$f'f''=f^{(3)},\quad f'^2=2f''\,. \tag{9}$$

By using Eqs.(7) and (9), Eq.(6) becomes

$$f^{(3)}(w_xw_yw_t-3w_xw_{xx}w_{xy}+w_xw_{xxx}w_y+3w_x^2w_{xxy})$$
$$+f''(w_{xt}w_y+w_xw_{yt}+w_{xy}w_t-2w_{xy}w_{xxx}+w_yw_{xxxx}$$
$$+4w_xw_{xxxy})+f'(w_{xyt}+w_{xxxxy})=0\,. \tag{10}$$

Setting the coefficients of $f^{(3)}$, f'' and f' in Eq.(10) to zero, we obtain a set of equations

$$w_y(w_{xt}+w_{xxxx})+w_x(w_{yt}+w_{xxxy})+w_{xy}(w_t+w_{xxx})$$
$$+3(w_xw_{xxxy}-w_{xy}w_{xxx})=0\,, \tag{11}$$
$$w_xw_y(w_t+w_{xxx})+3w_x(w_xw_{xxy}-w_{xx}w_{xy})=0, \tag{12}$$
$$w_{xyt}+w_{xxxxy}=0\,. \tag{13}$$

Obviously, taking

$$w_t+w_{xxx}=0 \tag{14}$$

and

$$w_xw_{xxy}-w_{xx}w_{xy}=0\,, \tag{15}$$

Eqs.(11)—(13) are satisfied automatically. From Eqs.(14) and (15) we can get a special solution

$$w(x,y,t)=1+\sum_{i=1}^{N}\exp[k_ix+\alpha(y)-k_i^3t]\,, \tag{16}$$

where k_i are arbitrary real constants and $\alpha(y)$ is a differentiable function of variable y.

Substituting solution (16) into Eq.(2) the generalized dromion-like solutions of Eq.(2) are easily gotten

$$u(x,y,t)=-2\frac{\sum_{i=1}^{N}k_i\alpha'(y)\exp\xi_i}{\left(1+\sum_{i=1}^{N}\exp\xi_i\right)^2}\,, \tag{17}$$

where

$$\xi_i=k_ix-\alpha(y)-k_i^3t\,. \tag{18}$$

To understand the meaning of solutions (17), we can find some special cases from solution (17) with (18) for Eq.(1). For instance:

To obtain "single" soliton solution, we select $N=1$ in solutions (17) so that we have

$$u(x,y,t)=-2\frac{k_1\alpha'(y)\exp\xi_1}{(1+\exp\xi_1)^2}$$
$$=-\frac{1}{2}k_1\alpha'(y)\mathrm{sech}^2\left(\frac{1}{2}\xi_1\right), \tag{19}$$

obviously, when $\alpha(y)=ly$, solution (19) is the bell-like soliton solution.

If $\alpha'(y)$ is fixed as a single soliton which is parallel to x aixs, then solution (19) indicates the original straight line soliton and curved line soliton $\mathrm{sech}^2\left(\frac{1}{2}\xi_1\right)$ disappear which become ghosts and only a single peak which is localized in all directions (named dromion) survives. And the dromion is located at the intersection of the lines soliton and the curve soliton. Considering $\alpha(y)$ is an arbitrary function of y, so the single dromion still possesses rich structures. Here are three simple examples:

$$u_1(x,y,t) = k_1 \mathrm{sech}^n(y-y_0)\mathrm{sech}^2\left[\frac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(y'-y_0)\mathrm{d}y'\right)\right]$$

$$\equiv k_1 h_1(y)\mathrm{sech}^2\left[\frac{1}{2}(k_1 x - k_1^3 t - h_1(y))\right], \tag{20}$$

$$u_2(x,y,t) = k_1 \mathrm{sech}^n(\cosh(y-y_0)-1)\mathrm{sech}^2\left[\frac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(\cosh(y'-y_0)-1)\mathrm{d}y'\right)\right]$$

$$\equiv k_1 h_2(y)\mathrm{sech}^2\left[\frac{1}{2}(k_1 x - k_1^3 t - h_2(y))\right], \tag{21}$$

$$u_3(x,y,t) = k_1 \frac{1}{(y-y_0)^{2n}+1}\mathrm{sech}^2\left[\frac{1}{2}\left(k_1 x - k_1^3 t - \int^y \frac{1}{(y'-y_0)^{2n}+1}\mathrm{d}y'\right)\right]$$

$$\equiv k_1 h_3(y)\mathrm{sech}\left[\frac{1}{2}(k_1 x - k_1^3 t - h_3(y))\right]. \tag{22}$$

The first type of dromion solution, $u_1(x,y,t)$, decays exponentially in all directions. The second type of dromion solution, $u_2(x,y,t)$, decays tremendously quicker than the first one in the y direction. The third type of dromion solution, $u_3(x,y,t)$, decays much slower than the first one in y direction. These results have been obtained using a different method in Ref.[8].

If $\alpha'(y)$ is selected as N parallel line solitions (parallel to x axis), them we get an N-dromion bound state

$$u = \left(\sum_{j=1}^N f_j(y)\right)\mathrm{sech}^2\frac{1}{2}\left(k_1 x - k_1^3 t - \int^y \sum_{j=1}^N f_j(y')\mathrm{d}y'\right), \tag{23}$$

driven by N line ghost solitons and one curved line ghost soliton. In Eq.(23), $f_j(y)$ can be selected quite freely, say $h_1(y)$, $h_2(y)$, $h_3(y)$ shown in Eqs.(20)—(22). Because all N parallel line solitons are static in y direction, the N dromions can only move with the same speed in the x direction as the curved line ghost soliton does and they cannot pass through each other. In other words, the behavior of these dromions looks like a bound state.

We put $N=2$ in solutions (17) to give

$$u_1(x,y,t) = -2\alpha'(y)\frac{k_1\exp\xi_1 + k_2\exp\xi_2}{(1+\exp\xi_1+\exp\xi_2)^2}. \tag{24}$$

This type of multi-dromion solution possesses much more aboundant structure than the case of $N=1$.

If we take some further restrictions on Eq.(17), we would get the dromion solutions studied by Radha and Laskshmanan.[3] For example, taking

$$N=3, \quad \xi_1=\chi_1=k_1x-k_1^3t, \quad \xi_2=\chi_2=l_1y, \quad \xi_3=\chi_1+\chi_3=k_1x-k_1^3t+l_1y \tag{25}$$

in Eq.(17), the RL's (1,1) dromion solution[3] follows immediately

$$u = -2\frac{k_1 l_1\exp(\chi_1+\chi_2)}{[1+\exp\chi_1+\exp\chi_2+\exp(\chi_1+\chi_2)]^2}. \tag{26}$$

Similarly, inserting the special form

$$N=5, \quad \xi_1=\chi_1=k_1x-k_1^3t, \quad \xi_2=\chi_2=l_1y, \quad \xi_3=\chi_3=k_2x-k_x^3t,$$
$$\xi_4=\chi_1+\chi_2=k_1x-k_1^3t+l_1y, \quad \xi_5=\chi_2+\chi_3=k_2x-k_2^3t+l_1y \tag{27}$$

into Eq.(17) leads to the RL's (2,1) dromion.[3]

$$u = -\frac{k_1 l_1\exp(\chi_1+\chi_2)+k_2 l_1\exp(\chi_2+\chi_3)}{[1+\exp\chi_1+\exp\chi_2+\exp(\chi_1+\chi_2)+\exp(\chi_2+\chi_3)]^2}. \tag{28}$$

In summary, we have given many types of new dromion solutions for the (2+1) dimensional KdV equation (1) by making use of the homogeneous balance method. The method described here, which is different from the bilinear method, Backlund transformations and the inverse scattering method, is concise

and direct.

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