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An exact solution to paraxial propagation of laser beams in longitudinal inhomogeneous plasmas*

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An exact, general solution for laser beams propagating in longitudinally inhomogeneous plasmas is obtained in the form of the diffraction integral. The Gaussian beam and the Hermite–Gaussian beam are taken for example. In the case of an increasing plasma density along the propagation distance, natural diffraction of the Gaussian beam is retarded. This retardance has a less effect on the central part of the Hermite–Gaussian beam while a considerable rise of the power in bucket (PIB) occurs in the surrounding part of the beam.

Keywords: inhomogeneous plasmas, laser beam, paraxial propagation, exact solution
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1. Introduction

Plasma density plays a key role in the interaction of intense laser beams with plasmas, which can be found in a large number of high power laser applications, such as x-ray lasers,[1−3] harmonic generation,[4,5] laser-driven plasma accelerators,[6−8] and inertial confinement fusion.[9] The plasma density affects not only the propagation properties of the laser field,[10,11] but also the laser energy deposition and the hot electron production from parametric instabilities such as stimulated Raman scattering[12] and filamentation.[13] Therefore, the plasma density is one of the most important parameters measured in the plasma diagnostics.[14−17] Usually, such as in the x-ray interferometry method,[14−17] the refractive index of the plasma medium offers a convenient way to measure the plasma density, since the refractive index is directly related to the plasma density and determines the beam propagation properties (e.g. the variation of the beam parameters). However, the formula of the beam phase difference, commonly used to evaluate the plasma density, is derived in the Wentzel–Kramers–Brillouin (WKB) approximation,[14−16] where the plasma is assumed to be uniform along the propagation distance. Although this formula is modified with higher-order corrections by Fu and Guo,[17] to our knowledge, an exact solution of the beam field, with the longitudinal plasma-density inhomogeneity taken into consideration, is still absent.

In this paper, we consider the propagation of laser beams in longitudinally inhomogeneous plasmas. A modified paraxial wave equation is derived and the corresponding exact solution is calculated by using the Fourier transform method. The Gaussian beam as well as the Hermite–Gaussian beam are taken for example.

2. Paraxial wave equation and exact solution

Making use of the Maxwell's equations and the cold electron fluid equations, the wave equation for an laser beam propagating in a plasma is given by[10]

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r},z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}(\boldsymbol{r},z,t) + \nabla[\boldsymbol{E}(\boldsymbol{r},z,t)\cdot\nabla\ln\epsilon] = k_{\rm p}^2\boldsymbol{E}(\boldsymbol{r},z,t), \quad (1)$$

where $\boldsymbol{E}(\boldsymbol{r},z,t)$ is the laser electric field; c is the speed of light in vacuum; $k_{\rm p}=\sqrt{4\pi ne^2/m_0c^2}$ is the plasma wave number; n is the plasma (electron) density; e and m_0 are the charge and the rest mass of electron respectively; $\epsilon = 1 - n/n_{\rm c}$ is the dielectric function of the plasma; $n_{\rm c}$ is the critical plasma density; $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$,

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with $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ being unit vectors along the x- and the y- axes, respectively.

For simplicity, we assume that the plasma is inhomogeneous only along the z-direction, i.e. $n = n(z)$. Now, we consider a laser beam linearly polarized in the x-direction and propagating along the z-direction. Then, the laser electric field can be written as

$$\boldsymbol{E}(\boldsymbol{r},z,t) = \frac{E}{2}\exp\Big[\mathrm{i}\Big(\int_0^z k(z')\mathrm{d}z' - \omega_0 t\Big)\Big]\hat{\boldsymbol{x}} + \text{c.c.}, \quad (2)$$

where $E = E(\boldsymbol{r},z,t)$ is assumed to be a slowly-varying complex distribution profile of the laser electric field; ω_0 and $k(z) = \eta(z)\omega_0/c$ are the laser centre frequency and wave number respectively; $\eta(z) = \sqrt{\epsilon(z)}$ is the refractive index of the plasma; c.c. denotes the complex conjugate. Then, the group speed of laser light in a plasma is given by $v_{\mathrm{g}}(z) = \eta(z)c$. Substituting Eq.(2) into Eq.(1), introducing the co-moving coordinate system ($z = z, \tau = t - \int_0^z v_{\mathrm{g}}^{-1}(z')\mathrm{d}z'$), and using the paraxially and slowly-varying envelope approximation, i.e. $|\partial E/\partial z| \ll |kE|$ and $|\partial E/\partial\tau| \ll |\omega_0 E|$, Eq.(1) reduces to

$$\nabla_\perp^2 E(\boldsymbol{r},z) + 2\mathrm{i}k\frac{\partial}{\partial z}E(\boldsymbol{r},z) + \mathrm{i}\frac{\partial k}{\partial z}E(\boldsymbol{r},z) = 0. \quad (3)$$

By comparing with the standard paraxial wave equation,[17] we see that there is an additional term (the last term) in Eq.(3). This comes from the longitudinal inhomogeneity of the plasma density.

The Fourier transform of Eq.(3) with respect to $\boldsymbol{r}$ gives

$$2\mathrm{i}k\frac{\partial}{\partial z}\tilde{E}(\boldsymbol{k}_\perp,z) = -\mathrm{i}\frac{\partial k}{\partial z}\tilde{E}(\boldsymbol{k}_\perp,z) + k_\perp^2\tilde{E}(\boldsymbol{k}_\perp,z), \quad (4)$$

where

$$\tilde{E}(\boldsymbol{k}_\perp,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} E(\boldsymbol{r},z)\exp(-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{r})\mathrm{d}^2\boldsymbol{r} \quad (5)$$

is the Fourier transform of $E(\boldsymbol{r},z)$ and $\boldsymbol{k}_\perp = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$. It is straightforward to integrate Eq.(4) with respect to z, which yields

$$\tilde{E}(\boldsymbol{k}_\perp,z) = \tilde{E}(\boldsymbol{k}_\perp,0)\sqrt{\frac{\eta_0}{\eta(z)}}\exp\Big[-\mathrm{i}\frac{\zeta(z)}{2k_0}k_\perp^2\Big], \quad (6)$$

where

$$\zeta(z) = \int_0^z \frac{\eta_0}{\eta(z')}\mathrm{d}z', \quad (7)$$

$\tilde{E}(\boldsymbol{k}_\perp,0)$ is the Fourier transform of the initial distribution profile $E(\boldsymbol{r},0)$, $\eta_0 = \eta(0)$, and $k_0 = k(0)$. Therefore, the inverse Fourier transform of Eq.(6) is given by

$$E(\boldsymbol{r},z) = \frac{k_0}{2\mathrm{i}\pi\zeta(z)}\sqrt{\frac{\eta_0}{\eta(z)}}\int_{-\infty}^{\infty}\mathrm{d}^2\boldsymbol{r}'E(\boldsymbol{r}',0) \times \exp\Big[\frac{\mathrm{i}k_0}{2\zeta(z)}|\boldsymbol{r}'-\boldsymbol{r}|^2\Big]. \quad (8)$$

Equation (8) is an exact diffraction-integral-like solution of the modified paraxial wave equation (3). That is to say, an analytic solution of the beam field can be obtained from Eq.(8) as long as the initial distribution profile $E(\boldsymbol{r},0)$ is given. It is also indicated that both the amplitude and the phase of the laser field are affected by the longitudinal plasma density inhomogeneity, which results in the change of the deflection and distribution of beam intensities.

3. Examples

3.1. Gaussian beam

We first consider the simplest Gaussian beam, i.e.

$$E(r,0) = E_0\exp\Big[-\frac{r^2}{w_0^2}\Big], \quad (9)$$

as an example, where E_0 is the initial amplitude and w_0 is the waist width of the Gaussian amplitude distribution. Substituting Eq.(9) into Eq.(8) yields

$$E(r,z) = \frac{E_0\sqrt{\eta_0/\eta}}{1+\mathrm{i}\frac{\zeta(z)}{Z_{\mathrm{R}}}}\exp\Big[-\frac{r^2}{w_0^2\Big(1+\mathrm{i}\frac{\zeta(z)}{Z_{\mathrm{R}}}\Big)}\Big], \quad (10)$$

where $Z_{\mathrm{R}} = k_0 w_0^2$ is the Rayleigh length. Equation (10) can be written in the form of the standard Gaussian beam as follows:

$$E(r,z) = E_r(z)\exp\Big[-\frac{r^2}{w^2(z)} - \mathrm{i}\varphi(z) + \mathrm{i}\frac{k_0 r^2}{2R(z)}\Big], \quad (11)$$

where

$$E_r(z) = \frac{E_0 w_0}{w(z)}\sqrt{\frac{\eta_0}{\eta(z)}}, \quad (12)$$

$$w(z) = w_0\sqrt{1+\Big(\frac{\zeta(z)}{Z_{\mathrm{R}}}\Big)^2}, \quad (13)$$

$$\varphi(z) = \arctan\Big(\frac{\zeta(z)}{Z_{\mathrm{R}}}\Big), \quad (14)$$

and

$$R(z) = \zeta(z) + \frac{Z_{\mathrm{R}}^2}{\zeta(z)} \quad (15)$$

are the amplitude, spot size, additional phase factor, and wave front radius of curvature, respectively.

It is implied in Eq.(11) that the beam diffraction is retarded if the plasma density increases along the propagation distance. This is plotted in Fig.1(a), in

which a linear rise of plasma density with z is assumed, i.e. $n(z) = n_0(1 + z/L)$, where $n_0 = n_c/N$ is the plasma density at $z = 0$ and L is the scale length. With the increase of the plasma density gradient (i.e. a smaller value of L and/or N), retardance of the beam diffraction is enhanced as shown by the dashed and the dash-dotted curves. And it is also shown in Fig.1(b) that the increase of the beam intensity mainly occurs in the central part of the beam.

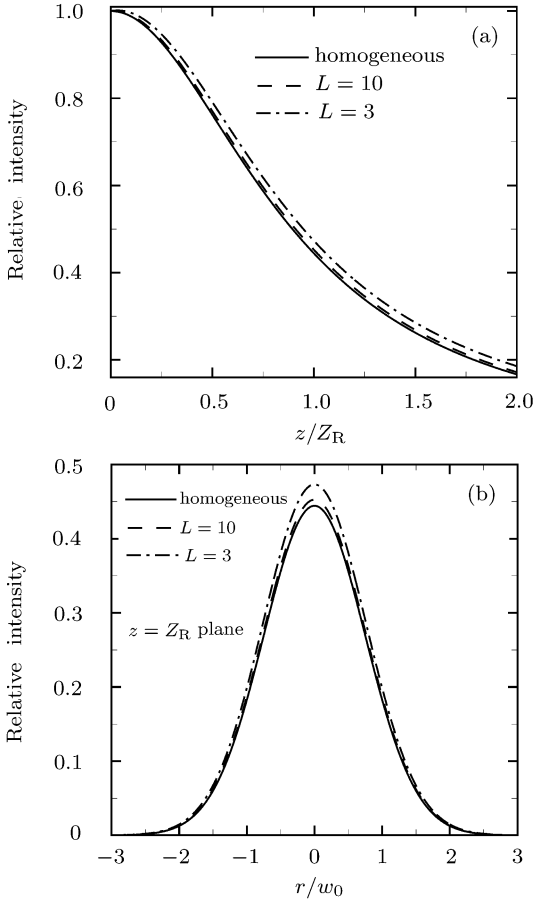


Fig.1. (a) Relative intensity versus propagation distance in a homogeneous plasma (solid curve) and inhomogeneous plasmas with linear rise of the density ($L = 10$, dashed curve; $L = 3$, dash-dotted curve) for the parameter $N = 5$, and (b) intensity distribution in $z = Z_R$ plane corresponding to Fig.1(a).

3.2. Hermite–Gaussian beam

It is more general to consider a Hermite–Gaussian beam[18] as follows:

$$E(x, y, 0) = E_0 H_m\Big[\frac{x}{w_0}\Big] H_l\Big[\frac{y}{w_0}\Big] \exp\Big[-\frac{r^2}{w_0^2}\Big], \quad (16)$$

where m and l are the order indices associated with the Hermite polynomials. Substituting Eq.(16) into Eq.(8), one has

$$E(x, y, z) = E_0 \sqrt{\frac{\eta_0}{\eta(z)}} \Big(\frac{w_0}{w_i}\Big)^{2+m+l} H_m\Big[\frac{x}{w_i}\Big] \times H_l\Big[\frac{y}{w_i}\Big] \exp\Big[-\frac{r^2}{w_i^2}\Big], \quad (17)$$

where $w_i = w_0\sqrt{1 + \mathrm{i}\zeta(z)/Z_R}$.

Equation (17) can be used to analyse the propagation properties of the Hermite–Gaussian beam in longitudinally inhomogeneous as well as homogeneous plasmas. However, we restrict ourselves to only discussion on the variation of the beam power. One can readily obtain the total power of the beam from the definition

$$P(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |E(x, y, z)|^2 \mathrm{d}x\mathrm{d}y. \quad (18)$$

Moreover, another useful parameter for characterizing the beam quality in practice is the power in bucket (PIB), which is defined as the fractional power within the size of a given bucket,[18] i.e.

$$\mathrm{PIB}(r, z) = \frac{1}{P_0}\int_0^{2\pi}\int_0^{r} |E(r', \theta, z)|^2 r' \mathrm{d}r'\mathrm{d}\theta, \quad (19)$$

where the coordinate ($r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$) has been used. And in order to show the effect of plasma density inhomogeneity on the beam power evolution, the PIB is defined with the initial total power $P_0 = P(0)$ rather than the varying total power $P(z)$ (in the case of a homogeneous plasma, the beam power does not vary with the propagation distance). Equation (19) shows the transverse distribution of the beam power at the propagation distance z in the case of a longitudinally inhomogeneous plasma.

To see the effect of plasma density inhomogeneity on the beam power evolution in detail, Fig.2 plots the relative total power (scaled by P_0) and the PIB according to Eqs.(18) and (19) with the order indices $m = l = 2$. The dashed curves correspond to a linearly increasing plasma density with z, i.e. $n(z) = n_0(1 + z/L)$. In the dash-dotted curves, the plasma density is assumed to increase quadratically with z, i.e. $n(z) = n_0(1 + z^2/L^2)$. The parameters used in Fig.2 are $N = n_c/n_0 = 5$ and $L = 3$. The total power increases along the propagation distance in the case of longitudinally inhomogeneous plasmas as shown in Fig.2(a). However, this longitudinal inhomogeneity has a less effect on the central part of the beam; the increase of the beam power mainly comes from the surrounding part of the beam, as shown in Fig.2(b) (the dashed and the dash-dotted curves), in which the distribution of the beam power (PIB) in the $z = 0$ and $z = 1.5Z_R$ planes are plotted.

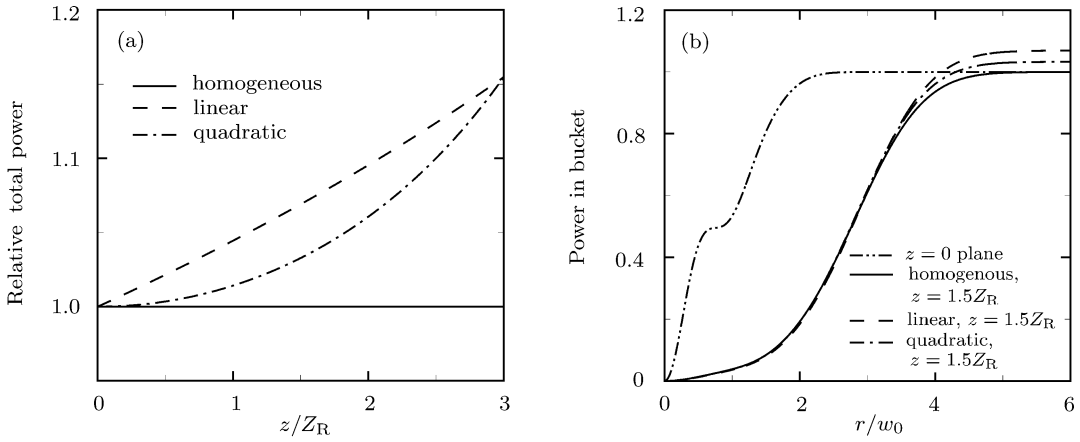


Fig.2. (a) Relative total power versus propagation distance in a homogeneous plasma (solid curve) and inhomogeneous plasmas with linear (dashed curve) as well as quadratic (dash-dotted curve) rise of the density for the parameters $N = 5$ and $L = 3$, and (b) PIB in $z = 0$ and $z = 1.5Z_R$ planes corresponding to Fig.2(a).

4. Conclusion

In conclusion, we obtain an exact, general solution for the propagation of laser beams in longitudinally inhomogeneous plasmas in the form of the diffraction integral by using the Fourier transform method. This solution can be used to analyse the propagation properties of laser beams with any profile in longitudinally inhomogeneous as well as homogeneous plasmas. The Gaussian beam and the Hermite–Gaussian beam are taken for examples to show the effect of longitudinal plasma density inhomogeneity on the laser beam propagation properties. As expected, the increase of plasma density leads to the retardance of the diffraction of the Gaussian beam. And this retardance is enhanced with longitudinal plasma density gradient increasing. Increasing plasma density also results in the rise of beam power. In the case of the Hermite–Gaussian beam, the rise of beam power mainly comes from the surrounding part of the beam. These can be used in the analysis of the measurement results of plasma density.

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