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Space-charge limiting currents for magnetically focused intense relativistic electron beams*

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The self-consistent differential equations, which describe a laminar-flow equilibrium state in a magnetically focused intense relativistic electron beam propagating inside a conducting waveguide, are presented. The canonical angular momentum, p_θ, defined under the conditions at the source, uniquely determines the possible solutions of these equations. By numerically solving these equations, the space-charge limited current and the externally applied magnetic field are obtained in a solid beam and a hollow beam in two cases of $p_\theta = 0$ (magnetically shielded source) and $p_\theta = $ const. (immersed source) separately. It is shown that the hollow beam is more beneficial to the propagation of the intense relativistic beam through a drift tube than the solid beam.

Keywords: space-charge limiting current, intense relativistic beam, laminar-flow equilibrium state
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1. Introduction

Intense relativistic electron beam transport has important applications in a variety of areas, including collective ion acceleration, x-ray production, microwave generation,[1,2] pasteurization of food, and plasma heating for achieving magnetically confined thermonuclear fusion. One of the most important issues in intense relativistic charged particle beam is the space-charge limiting currents. As the charged particle beam current in a drift tube increases to a limiting value, an electrostatic potential depression builds up due to the beam's self-space charge field and, as a result, the beam cannot propagate any more.

Bogdankevich and Rukhadze[3] have derived the limiting current to annular beams and extended to solid beams in the case of an 'infinite' axial magnetic field. Olson and Poukey[4] have derived a limiting current expression which does not include radial variation in γ, but it is used as an upper current limit to the transport of electron beams in a gas-filled conducting tube. Thompson and Sloan[5] have derived a limiting current estimate that is determined by maximizing an expression for the current as a function of the equilibrium energy on axis. Uhm[6] has derived a limiting current for a solid electron beam with a strong, externally applied magnetic field. Reiser[7] has studied the limiting currents of laminar-flow equilibria in magnetically focused relativistic electron beams propagating inside a conducting tube that confines the electric field and the magnetic self-fields. He obtained analytical solutions of the limiting currents and the externally applied magnetic fields of the solid beams in the case of magnetically shielded source ($p_\theta = 0$).

We present a general theoretical analysis of the laminar-flow equilibria of magnetically focused relativistic electron beams, including the solid beams and hollow beams. This theory is described by a set of self-consistent differential equations. By numerically solving the equations, the limiting currents and the externally applied magnetic fields for solid beams and hollow beams are obtained in two cases of $p_\theta = 0$ and $p_\theta = $ const. separately.

2. Differential equations for relativistic laminar flow equilibrium

The laminar-flow equilibrium implies that all particles in an equilibrium region move along the helical paths with a constant radius r. The following assumptions are needed.

(1) The effect of transverse temperature or emittance is negligible compared with the effect of self-field

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force.

(2) Exact radial force balance exists for each particle within the beam such that the sum of centrifugal, static electric, and magnetic forces at every radius is zero.

(3) $\partial/\partial t$=0, $\upsilon_r = 0$, $\partial/\partial z = 0$ and $\partial/\partial\theta = 0$ due to axial symmetry; thus all quantities are functions of radius r only.

(4) The externally applied magnetic field B_0 is constant in the entire drift tube.

With these assumptions, the following two equations can be obtained:[7]

$$\frac{\gamma(r)m_0\upsilon_\theta^2(r)}{r} - eE_r(r) - e\upsilon_\theta(r)B_z(r) + e\upsilon_z(r)B_\theta(r) = 0, \tag{1}$$

$$\gamma(r)m_0 r\upsilon_\theta(r) - erA_\theta(r) = \gamma(r)m_0 r\upsilon_\theta(r) - \mathrm{e}\int_0^r B_z(r)r\mathrm{d}\,r = p_\theta(r). \tag{2}$$

The first equation represents the balance of all radial forces, and the second one is the conservation law for the canonical angular momentum which is neglected in a macroscopic fluid approach. A_θ is the magnetic vector potential. p_θ is the canonical angular momentum. m_0 is the rest electron mass, $-e$ is the electron charge.

The axial speed of the particle, υ_z, may generally be a function of radius r, and, since $\upsilon_r = 0$, it is related to υ_θ and γ by the relation $\upsilon_z^2+\upsilon_\theta^2 = c^2(1-\gamma^{-2})$. The potential is defined such that $(\gamma-1)m_0c^2 = e\phi$, and a normalized potential is defined as $\Phi = (m_0c^2/e)\phi = \gamma-1$, which represents the diode voltage. The Poisson equation in our case takes the form

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\,r}\Big(r\frac{\mathrm{d}\,\phi}{\mathrm{d}\,r}\Big) = \frac{m_0c^2}{e}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\,r}\Big(r\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,r}\Big) = \frac{en}{\varepsilon_0}, \tag{3}$$

where $n = n(r)$ is the electron density.

In Eq.(1), E_r represents the radial static electric self-field due to the space charge. From the definition of the scalar potential, we have $E_r = -\mathrm{d}\,\phi/\mathrm{d}\,r = -\phi' = -(m_0c^2/e)\gamma'$. B_θ is the magnetic self-field component due to the axial current $J_z = -en\upsilon_z$, and they are related by the form $(1/r)(rB_\theta)' = -\mu_0en\upsilon_z$. B_z is the sum of the externally applied magnetic field B_0 and the magnetic self-field component $B_{z,\mathrm{s}}$. The angular current $J_\theta = -en\upsilon_\theta$ yields $B_z' = \mu_0en\upsilon_\theta$ in form. Two differential equations for the velocity components can be obtained[7] as follows:

$$\upsilon_z'' + \Big(2\frac{\gamma'}{\gamma}+\frac{1}{r}\Big)\upsilon_z' + \frac{1}{\gamma m_0}\Big(\frac{\upsilon_\theta p_\theta'}{\upsilon_z}\Big)' = 0, \tag{4}$$

$$\upsilon_\theta'' + \Big(2\frac{\gamma'}{\gamma}+\frac{1}{r}\Big)\upsilon_\theta' - \frac{\upsilon_\theta}{r^2} - \frac{1}{\gamma m_0}\Big(\frac{p_\theta'}{r}\Big)' = 0, \tag{5}$$

where $\gamma' = (\gamma^3/c^2)(\upsilon_\theta\upsilon_\theta'+\upsilon_z\upsilon_z')$. The canonical angular momentum, p_θ, which depends on the geometry and axial magnetic field at the cathode, determines the characteristics of the laminar flow in the equilibrium region downstream from the resource. The self-consistent set of Eqs.(4) and (5) leads to two independent functional relations involving the azimuthal and axial velocity components in which the canonical angular momentum imposes a constraint on the allowed solutions. Thus there appear three cases of practical interest, namely, (a) $p_\theta = 0$, (b) p_θ = const., and (c) $p_\theta \propto r^2$.

3. Simulation results and discussion

By numerically solving the self-consistent equations (4) and (5) with initial conditions at the inner surface of the beam, the values of υ_z, υ_θ and γ at an arbitrary radius r can be obtained. The initial conditions are determined as follows. The normalized potential on the inner surface of beams is denoted by Φ_0, which describes the extent of space-charge potential depression. Because υ_θ in the inner surface is zero ($\upsilon_{\theta0} = 0$), υ_z in the inner surface is $\upsilon_{z0} = c(1-\gamma^{-2})^{1/2}$. Due to $\gamma_0' = 0$, we have $\upsilon_{z0}' = 0$. The last boundary condition is $\upsilon_{\theta0}'$, which is an exclusive value when the potential of the tube wall is set in the case of laminar-flow equilibrium state. So we can scan the value of $\upsilon_{\theta0}'$, and use the initial condition ($\upsilon_{\theta0}$, υ_{z0}, υ_{z0}' and $\upsilon_{\theta0}'$) to solve the two differential equations (4) and (5). When the calculated potential of tube wall is equal to the set value Φ_b, the value of $\upsilon_{\theta0}'$ is the one that we search for. At this time, a beam current is calculated, but it may not be a maximal value (the limiting current). Then by scanning the value of Φ_0, the maximal value of the beam current can be obtained.

In the cases of $p_\theta = 0$ and p_θ = const., the last terms on the left-hand sides in both Eqs.(4) and (5) vanish, which leads to υ_z = const. In Fig.1, we plotted I/I_0 versus the potential ratio Φ_0/Φ_a at several values of the potential Φ_a, where $I_0 = 4\pi\varepsilon_0m_0c^3/e \approx$ 17000 A and Φ_a is the potential on the outer surface

of the solid electron beam. In the case of $r_a = r_b$ (r_a is the radius of the outer surface of the electron beam and r_b is the radius of the drift tube), Φ_a is the diode voltage. This figure is the same as Fig.2 in Ref.[7], and it is obtained from the following analytical expression:

$$\frac{I}{I_0} = \frac{1}{2}\Big[\Phi_a^2\Big(\frac{1}{\Phi_a}+\frac{\Phi_0}{\Phi_a}\Big)^2 - 1\Big]^{1/2}\Big[\Big(\frac{1}{\Phi_a}+\frac{\Phi_0}{\Phi_a}\Big)^{-2} - 1\Big]. \quad (6)$$

We can see that the beam current has a maximum at small values of the potential ratio, and it is zero valued at $\Phi_0/\Phi_a = 0$ and $\Phi_0/\Phi_a = 1$ separately. In the case of $r_b > r_a$, we know the value of Φ_b, not Φ_a. By Gauss's law, the relation between Φ_b and Φ_a is expressed as

$$\Phi_b = \Phi_a + \frac{e^2}{\varepsilon_0 m_0 c^2}\ln\Big(\frac{r_b}{r_a}\Big)\int_{r_c}^{r_a} n(r) r \mathrm{d}r, \quad (7)$$

where r_c is the radius of inner surface of hollow electron beam; for a solid electron beam, $r_c = 0$. The electron density profile $n(r)$ can be obtained from Eq.(3). Plotted in Fig.2 is the beam current I/I_0 versus the potential ratio Φ_0/Φ_b at several values of the potential Φ_b in the case of $r_b/r_a = 1.2$ for a solid beam. It is seen from Eqs.(4) and (5), that the beam current is the same as those in two cases of $p_\theta = $ const. and $p_\theta = 0$.

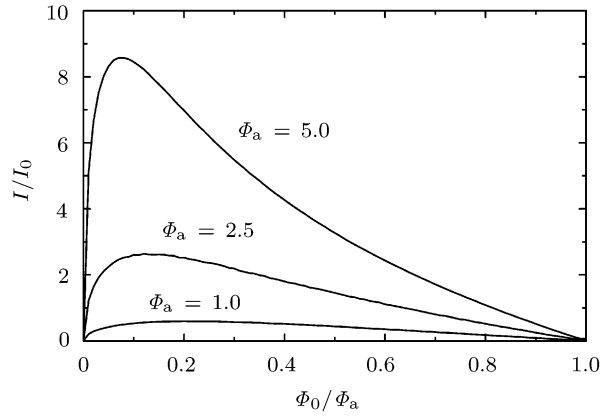


Fig.1. Beam current I/I_0 versus potential ratio Φ_0/Φ_a at different values of Φ_a for a solid beam with $p_\theta = $ const.

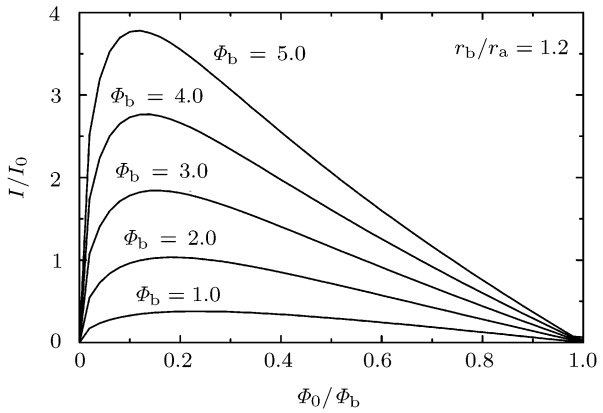


Fig.2. Beam current versus potential ratio $\Phi_0\Phi_b$ at different values of Φ_b for a solid beam with $r_b/r_a = 1.2$ and $p_\theta = $ const.

The axial component of the magnetic self-field can be written as

$$B_{z,s}(r) = \begin{cases} -\mu_0 e 2\pi \displaystyle\int_{r_c}^{r_a} n(r')\upsilon_\theta(r') r' \mathrm{d}r' + B_c, & (\text{for } 0 < r \le r_c), \\ -\mu_0 e 2\pi \displaystyle\int_{r}^{r_a} n(r')\upsilon_\theta(r') r' \mathrm{d}r' + B_c, & (\text{for } r_c < r \le r_a), \\ B_c, & (\text{for } r_a < r \le r_b), \end{cases} \quad (8)$$

where B_c is the uniform axial field generated by the azimuthal induced current in the tube wall, and from the flux conservation law, one obtains

$$B_c = \frac{4\pi\mu_0 e}{r_b^2}\Big[\frac{r_a^2}{2}\int_{r_c}^{r_a} n(r')\upsilon_\theta(r') r' \mathrm{d}r' - \int_{r_c}^{r_a} r \mathrm{d}r \int_{r_c}^{r} n(r')\upsilon_\theta(r') r' \mathrm{d}r'\Big]. \quad (9)$$

Substituting $B_z(r) = B_{z,s}(r) + B_0$ into Eq.(2) with $r = r_a$ yields the following expression for the applied magnetic field:

$$B_0 = 2\frac{m_0\gamma_a\upsilon_\theta(r_a)}{e r_a} + B_c\frac{r_b^2 - r_a^2}{r_a^2} - \frac{2p_\theta(r_a)}{e r_a^2}, \quad (10)$$

where $\upsilon_\theta(r_a)$ and $p_\theta(r_a)$ are the values of $\upsilon_\theta(r)$ and $p_\theta(r)$ when $r = r_a$, respectively. Figure 3 shows the required applied magnetic field versus potential ratio Φ_0/Φ_b at different values of Φ_b for a solid beam with

$r_b/r_a = 1.2$ and p_θ=0. The required applied magnetic field can also been easily obtained from Eq.(10) in the case of p_θ = const. From Fig.4, we can see the effect of beam-to-tube radius ratio r_b/r_a on the beam current in a solid beam, that is, as r_b/r_a increases, the limiting current decreases on the assumption that the tube voltage is constant. When the entire drift tube is filled with electron beam ($r_b/r_a = 1.0$), the limiting current is biggest.

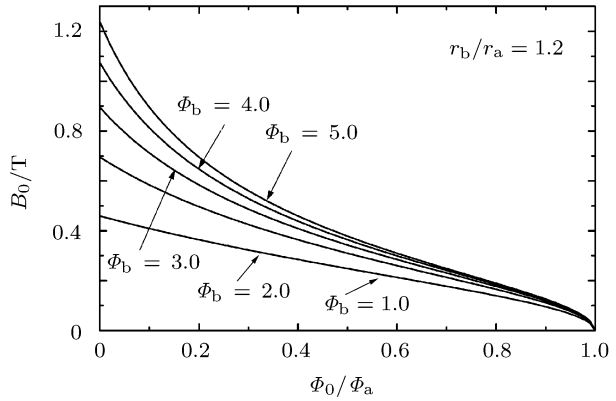


Fig.3. The required applied magnetic field B_0 versus potential ratio Φ_0/Φ_b at different values of Φ_b for a solid beam with $r_b/r_a = 1.2$ and $p_\theta = 0$.

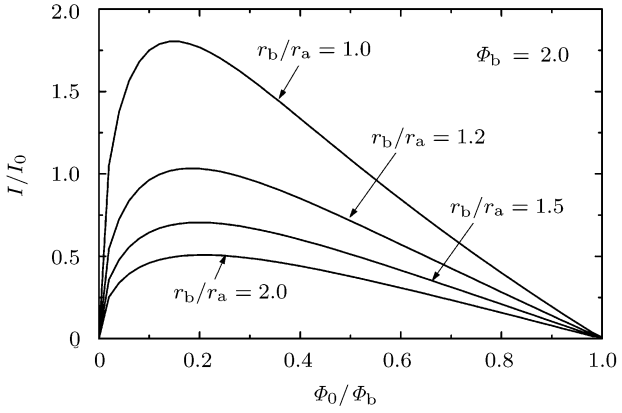


Fig.4. The effect of beam-to-tube radius ratio r_b/r_a on the beam current for a solid beam in the case of $\Phi_b = 2.0$ and $p_\theta = 0$.

For a hollow beam, r_c is the radius of the inner surface of electron beam. We can see the effect of the radius of the inner surface of electron beam on the limiting current and the externally applied magnetic field in Figs.5 and 6. As the ratio r_c/r_a increases, the limiting current increases but the required applied magnetic field decreases on the assumption that the tube voltage is constant and the radius ratio is a constant, i.e. $r_b/r_a = 1.2$. For an 'infinite' thin beam ($r_c/r_a = 1.0$), the limiting current is biggest and the required applied magnetic field approaches zero. For a solid beam with $r_c/r_a = 0.0$, the limiting current is smallest but the required applied magnetic field is biggest. So the hollow beam is beneficial to the propagation of the intense relativistic beam through a drift tube.

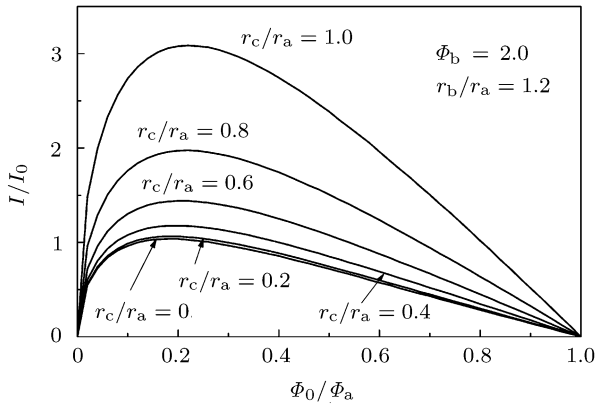


Fig.5. The effect of r_c/r_a on beam current for a hollow beam with $\Phi_b = 2.0$, $r_b/r_a = 1.2$ and p_θ = const.

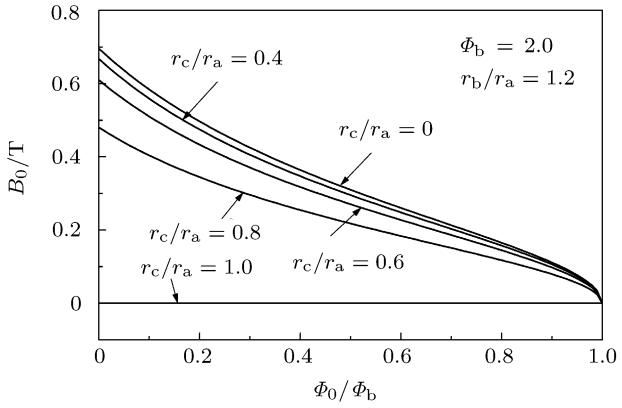


Fig.6. The effect of r_c/r_a on the required applied magnetic field B_0 for a hollow beam with $\Phi_b = 2.0$, $r_b/r_a = 1.2$ and $p_\theta = 0$.

4. Conclusion

In summary, we have obtained the self-consistent differential equations that describe a laminar-flow equilibrium state in an intense, magnetized, relativistic electron beam propagating inside a conducting waveguide, are presented. By numerically solving these equations, the azimuthal speed v_θ, the axial speed v_z, the electron beam density n as functions of radius r are calculated, and the space-charge limiting current and the externally applied magnetic field are obtained in a solid beam and a hollow beam in

two cases of $p_\theta = 0$ (magnetically shielded source) and $p_\theta = \text{const.}$ (immersed source) separately. It can be seen that the limiting current is larger and the externally applied magnetic field is smaller in a hollow beam than in a solid beam. So the hollow beam is more beneficial to the propagation of an intense relativistic electron beam through a drift tube than a solid beam.

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