Vol 16 No 9, September 2007
1009-1963/2007/16(09)/2665-06
Chinese Physics
© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Lie symmetry algebra of one-dimensional nonconservative dynamical systems*

Liu Cui-Mei(刘翠梅)[b], Wu Run-Heng(吴润衡)[c], and Fu Jing-Li(傅景礼)[a)d)†]

[a] *Institute of Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*
[b] *Department of Physics, Shangqiu Teachers College, Shangqiu* 476000, *China*
[c] *Department of Mathematics, Beijing Industrial University, Beijing* 100080, *China*
[d] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai* 200072, *China*

(Received 13 January 2007; revised manuscript received 12 March 2007)

Lie symmetry algebra of linear nonconservative dynamical systems is studied in this paper. By using 1–1 mapping, the Lie point and Lie contact symmetry algebras are obtained from two independent solutions of the one-dimensional linear equations of motion.

Keywords: Lie algebra, symmetry, infinitesimal transformation, nonconserved dynamical system
PACC: 1130L, 0420M

1. Introduction

Research on Lie symmetry in differential equations and other relevant topics has now aroused great interest not only for searching conserved quantities but also for obtaining the number of symmetries of systems. It is known that the Lie symmetry algebra for one-dimensional harmonic oscillator has been extensively discussed.[1−18] It has shown there to be an eight-parameter symmetry group. Schwarz[19] extended this study to Lie contact symmetry and pointed out there existed a much large group of contact transformations leaving harmonic oscillator equation invariant. Martini and Kersten[20] studied the complete Lie symmetry algebra of general linear second-order ordinary differential equations by using 1–1 mapping. Minorn Omote[21] given an infinite-dimensional symmetry algebra of the Davey–Stewartson equations, showing that this is a gauge generalization of the symmetry transformation for the equation. Mahomed and Leach[22] discussed the Lie symmetry with scalar second-order ordinary differential equations, and pointed out how second-ordinary differential equations can be classified by investigating the realization of real low-dimensional Lie algebra in terms of vector fields in two coordinates. Recently, Kaushal and Mishra[23] have generalized the dynamical algebraic approach, which is known already, from one-dimensional time-dependent systems to two-dimensional time-dependent systems.

In this paper we extend the study of Lie symmetry algebra to linear nonconservative dynamical systems. It is shown that this study can be further extended to nonholonomic dynamical systems.

2. Lie symmetry algebra of nonconservative dynamical systems

We can prove that the Lie symmetry algebra of a nonconservative mechanical system can be determined if two independent solutions are obtained. A one-degree-of-freedom mechanical system with Lagrangian $L(t, q, \dot{q})$, subjected to a non-potential generalized force $Q'(t, q, \dot{q})$, has the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q. \tag{1}$$

Equation (1) may be rewritten as the following linear second-order ordinary differential equation:

$$\ddot{q} + a_1(t)\dot{q} + a_2(t)q = a_3(t). \tag{2}$$

Correspondingly, its homogeneous general linear second-order ordinary differential equation is

$$\ddot{q} + a_1(t)\dot{q} + a_2(t)q = 0, \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant No 10672143) and the Natural Science Foundation of Henan Province, China (Grant Nos 0511022200 and 072300440220).
†Corresponding author. E-mail: sqfujingli@163.com

which is described by Arnold.[24] There exists a 1–1 mapping which transforms a linear differential equation (3) into

$$Y_{XX} = 0 \tag{4}$$

i.e. by rectifying the solutions of Eq.(3).

The Lie point and Lie contact symmetry algebras of Eq.(4) are[25]

$$\begin{aligned} &U_1 = \partial_Y,\ U_2 = X\partial_Y,\ U_3 = Y\partial_Y,\ U_4 = Y_X\partial_Y,\\ &U_5 = XY_X\partial_Y,\ U_6 = YY_X\partial_Y,\\ &U_7 = X(Y - XY_X)\partial_Y,\\ &U_8 = Y(Y - XY_X)\partial_Y, \end{aligned} \tag{5}$$

and[3]

$$U_9 = (YG(Y-XY_X, Y_X)+H(Y-XY_X, Y_X))\partial_Y, \tag{6}$$

where U_i are group generators, G and H are arbitrary functions of their variables, and the notation $\partial_Y = \dfrac{\partial}{\partial Y}$ is used.

Now, we start to discuss Eq.(2). There exists a 1–1 mapping which transforms a linear differential equation (3) into

$$P_{XX} = 0, \tag{7}$$

which is a differential equation defined on $J^2(X, P)$. Let $u_1(t)$ and $u_2(t)$ be two solutions of the homogeneous equation (3), and $f(t)$ be an extra solution of the non-homogeneous equation (2). We take two independent solutions of the non-homogeneous equation (2) such that

$$q_1(t) = u_1(t) + f(t), \quad \text{and} \quad q_2(t) = u_2(t) + f(t), \tag{8}$$

satisfying

$$q_1(0) = 0, \quad \text{and} \quad q_2(0) \neq 0. \tag{9}$$

We now define the mapping $\phi : \{t, u\} \to \{X, P\}$ as

$$\bar{X} = \frac{q_1(t)}{q_2(t)}, \quad \text{and} \quad \bar{P} = \frac{q(t)}{q_2(t)}; \tag{10}$$

since

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = \frac{q_1'(t)q_2(t) - q_1(t)q_2'(t)}{q_2(t)^2}\,|_{t=0} \neq 0,$$

ϕ is an invertible mapping, and

$$\phi^{-1} : \{(X, P)\} \to \{(t, q)\}$$

is defined as

$$\bar{t} = [q_1/q_2]^{-1}(X), \quad \text{and} \quad \bar{q} = Pq_2(\bar{t}). \tag{11}$$

The prolongation mapping of ϕ, leaving the contact invariant,[26]

$$p^1\phi : \{(t, q, \dot{q})\} \to \{(X, P, P_X)\},$$

is defined as

$$\bar{X} = \frac{q_1(t)}{q_2(t)}, \quad \bar{P} = \frac{q(t)}{q_2(t)},$$

and

$$\bar{P}_X = \frac{q_2(t)q'(t) - q_2'(t)q(t)}{q_2(t)^2}\left[\frac{\mathrm{d}\bar{X}}{\mathrm{d}t}\right]^{-1}. \tag{12}$$

After somewhat tedious calculation, we have $p^2\phi : J^2(t, q) \to J^2(X, P)$ which is defined as expressions (12) and

$$\begin{aligned} \bar{P}_{XX} \;=\; & q_2(t)(q_1'(t)q_2(t) - q_1(t)q_2'(t))^{-1}\\ & \times(\ddot{q} + a_1(t)\dot{q} + a_2(t)q - a_3(t)), \end{aligned} \tag{13}$$

from which it is clearly seen that ϕ transforms solutions of Eq.(2) into Eq.(7) in a 1–1 manner.

We now derive the general formula for the transformation of vector fields. We consider only vertical vector fields, i.e.

$$V = F(X, P, P_X)\partial_P + \cdots, \tag{14}$$

where the dots denote the prolongation terms, since each vector field

$$V^1 = K(X, P, P_X)\partial_X + H(X, P, P_X)\partial_P$$

is equivalent to[27]

$$V^2 = (H(X, P, P_X) - P_X K(X, P, P_X))\partial_P + \cdots;$$

since the prolonged field can be obtained from $F(X, P, P_X)$ by applying the total derivative operator D_X to it, we suppress the prolongation in the sequel. The mapping $(\phi^{-1})_* : T\{(X, P)\} \to T\{(t, q)\}$ is defined as

$$\begin{aligned} ((\phi^{-1})_*V)f(t, q) \;=\; & V(f(\bar{t}, \bar{q}))\,|_{t,q}\\ =\; & F(\bar{X}, \bar{P}, \bar{P}_X)q_2(t)\partial f(t, q)/\partial q, \end{aligned} \tag{15}$$

from which we obtain a general formula

$$(\phi^{-1})_*\{F(X, P, P_X)\partial_P\} = F(\bar{X}, \bar{P}, \bar{P}_X)q_2(t)\partial_q. \tag{16}$$

So, by using Eq.(16) infinitesimal symmetries of Eq.(7) are transformed into symmetries of Eq.(2) in a 1–1 manner.

Equation (16) is a general formula which is the same as a homogeneous general linear second-order ordinary differential equation of the dynamical system (3).

3. Illustrative examples

Example 1 Assume there exists a 6 inch long spring whose end a 20 pound object is hung on. Suppose the spring is subjected to an additional damping force of βv pounds, in which v is instantaneous speed.

We discuss Lie symmetry algebra of the system in the two cases of an additional damping force: (a) without additional damping force, i.e. $\beta = 0$, (b) with to additional damping forces being $\beta = 8$, $\beta = 10$, and $\beta = 12.5$.

The motion of the object under the action of additional damping force $\beta v = \beta\dot{q}$ is described by[28]

$$\frac{20}{32}\ddot{q} = -40q - \beta\dot{q} \tag{17}$$

or

$$\ddot{q} + \frac{8\beta}{5}\dot{q} + 64q = 0. \tag{18}$$

(a) If $\beta = 0$, motion equation of the object is

$$\ddot{q} + 64q = 0, \tag{19}$$

which has two independent solutions as follows:

$$q_1(t) = \sin(8t), \quad \text{and} \quad q_2(t) = \cos(8t). \tag{20}$$

The mapping $p^{\mathrm{I}}\phi : J^{\mathrm{I}}(t, y) \to J^{\mathrm{I}}(X, Y)$ is defined as

$$\begin{aligned} &\bar{X} = \tan(8t), \quad \bar{Y} = \frac{q}{\cos(8t)}, \\ &\bar{Y}_X = \dot{q}\cos(8t) + 8q\sin(8t), \end{aligned} \tag{21}$$

while $\phi^{-1} : \{(X, Y)\} \to \{(t, y)\}$ is given as

$$\bar{t} = \frac{1}{8}\tan^{-1} X, \quad \bar{q} = Y(1 + X)^{-1/2}. \tag{22}$$

Applying formula (16) to Lie point and Lie contact symmetries of Eq.(4), i.e. Eqs.(5) and (6), yields

$$\begin{aligned} U_1 &= \cos(8t)\frac{\partial}{\partial q}, \quad U_2 = \sin(8t)\frac{\partial}{\partial q}, \\ U_3 &= q\frac{\partial}{\partial q}, \quad U_4 = (\dot{q}\cos^2(8t) + 4q\sin(16t))\frac{\partial}{\partial q}, \\ U_5 &= \left(\frac{1}{2}q_t\sin(2t) + 8q\sin^2(8t)\right)\frac{\partial}{\partial q}, \\ U_6 &= (q\dot{q}\cos(8t) + 8q^2\sin(8t))\frac{\partial}{\partial q}, \\ U_7 &= \sin(8t)\left(\frac{q}{\cos(8t)} - \tan(8t)(\dot{q}\cos(8t) + 8q\sin(8t))\right)\frac{\partial}{\partial q}, \\ U_8 &= \left(\frac{q^2}{\cos(8t)} - (q\dot{q}\sin(8t) + 8q^2\tan(8t)\sin(8t))\right)\frac{\partial}{\partial q}, \\ U_9 &= \frac{q}{\cos(8t)}G(v, u) + \cos(8t)H(v, u)\frac{\partial}{\partial q}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} v &= \frac{q}{\cos(8t)} - \tan(8t)(\dot{q}\cos(8t) + 8q\sin(8t)) \\ &= \frac{q}{\cos(8t)} - \dot{q}\sin(8t) - 8q\frac{\sin^2(8t)}{\cos(8t)}, \\ u &= \dot{q}\cos(8t) + 8q\sin(8t). \end{aligned} \tag{24}$$

(b) if $\beta = 8$, Eq.(18) has the form

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + 12.8\frac{\mathrm{d}q}{\mathrm{d}t} + 64q = 0, \tag{25}$$

which has two independent solutions as follows:

$$q_1(t) = \mathrm{e}^{-6.4t}\cos(4.8t),$$

and

$$q_2(t) = \mathrm{e}^{-6.4t}\sin(4.8t), \tag{26}$$

by using expression (10), $p^{\mathrm{I}}\phi : J^{\mathrm{I}}(t, y) \to J^{\mathrm{I}}(X, Y)$ is defined as

$$\bar{X} = \tan(4.8t), \quad \bar{P} = \frac{q\mathrm{e}^{6.4t}}{\cos(4.8t)}, \quad \text{and} \quad \bar{P}_X = \frac{(\dot{q}+6.4)\mathrm{e}^{6.4t}\cos(4.8t)}{4.8} + q\mathrm{e}^{6.4t}\sin(4.8t), \tag{27}$$

where $y_t = \mathrm{d}y/\mathrm{d}t$, and $\phi^{-\mathrm{I}} : \{(X,Y)\} \to \{(t,y)\}$ is defined as

$$\bar{t} = \frac{1}{4.8}\tan^{-1}X, \quad \text{and} \quad \bar{q} = q_2 P = \mathrm{e}^{-\frac{4}{3}\tan^{-1}X}(1+X)P. \tag{28}$$

The Lie point and Lie contact symmetry algebras of Eq.(25) with $\beta = 8$ are computed from expressions (5), (6), (16) and (27) as follows:

$$\begin{aligned}
&U_1 = \mathrm{e}^{-6.4t}\cos(4.8t)\frac{\partial}{\partial q}, \quad U_2 = \mathrm{e}^{-6.4t}\tan(4.8t)\sin(4.8t)\frac{\partial}{\partial q}, \quad U_3 = q\tan(4.8t)\frac{\partial}{\partial q},\\
&U_4 = \left(\frac{(\dot{q}+6.4)\cos^2(4.8t)}{4.8} + \frac{1}{2}q\sin(9.6t)\right)\frac{\partial}{\partial q},\\
&U_5 = \sin(4.8t)\left(\frac{(\dot{q}+6.4)\sin(9.6t)}{9.6} + q\sin^2(4.8t)\right)\frac{\partial}{\partial q},\\
&U_6 = \left(\frac{(q\dot{q}+6.4)\mathrm{e}^{6.4t}\cos(4.8t)}{4.8} + q^2\mathrm{e}^{6.4t}\sin(4.8t)\right)\frac{\partial}{\partial q},\\
&U_7 = \left(\frac{q\sin(4.8t)}{\cos(4.8t)} - \tan(4.8t)\left(\frac{(\dot{q}+6.4)\cos 9.6t}{9.6} + q\sin^2(4.8t)\right)\right)\frac{\partial}{\partial q},\\
&U_8 = \left(\frac{q^2\mathrm{e}^{6.4t}}{\cos(4.8t)} - \tan(4.8t)\left(\frac{(\dot{q}+6.4)q\mathrm{e}^{6.4t}\cos(4.8t)}{4.8} + q^2\mathrm{e}^{6.4t}\sin(4.8t)\right)\right)\frac{\partial}{\partial q},\\
&U_9 = \left(\frac{q}{\cos(4.8t)}G(v,u) + \mathrm{e}^{-6.4t}\cos(4.8t)H(v,u)\right)\frac{\partial}{\partial q},
\end{aligned} \tag{29}$$

in which G and H are arbitrary functions of v and u, namely

$$\begin{aligned}
v &= \frac{q\mathrm{e}^{6.4t}}{\cos(4.8t)} - \left(\frac{(\dot{q}+6.4)\mathrm{e}^{6.4t}\sin(4.8t)}{4.8} + q\mathrm{e}^{6.4t}\frac{\sin^2(4.8t)}{\cos(4.8t)}\right),\\
u &= \frac{(\dot{q}+6.4)\mathrm{e}^{6.4t}\cos(4.8t)}{4.8} + q\mathrm{e}^{6.4t}\sin(4.8t).
\end{aligned}$$

If $\beta = 10$, Eq.(18) becomes

$$\ddot{q} + 16\dot{q} + 64q = 0. \tag{30}$$

Equation (30) has two independent solutions:

$$q_1 = t\mathrm{e}^{-4t}, \quad \text{and} \quad q_2 = \mathrm{e}^{-4t}, \tag{31}$$

by using expression (10), $p^{\mathrm{I}}\phi : J^{\mathrm{I}}(t,y) \to J^{\mathrm{I}}(X,Y)$ is defined as

$$\bar{X} = t, \quad \bar{P} = q\mathrm{e}^{4t}, \quad \text{and} \quad \bar{P}_X = (\dot{q}+4q)\mathrm{e}^{4t}. \tag{32}$$

While $\phi^{-\mathrm{I}} : \{(X,Y)\} \to \{(t,y)\}$ is given by

$$\bar{t} = X, \quad \text{and} \quad \bar{q} = P\mathrm{e}^{-4X}.$$

The Lie point and Lie contact symmetry algebras of Eq.(30) with $\beta = 10$ are computed from expressions (5), (6), (16) and (30) as follows:

$$\begin{aligned}
&U_1 = \mathrm{e}^{-4t}\frac{\partial}{\partial q}, \quad U_2 = t\mathrm{e}^{-4t}\frac{\partial}{\partial q}, \quad U_3 = q\frac{\partial}{\partial q}, \quad U_4 = (\dot{q}+4q)\frac{\partial}{\partial q}, \quad U_5 = t(\dot{q}+4q)\frac{\partial}{\partial q},\\
&U_6 = q\mathrm{e}^{4t}(\dot{q}+4q)\frac{\partial}{\partial q}, \quad U_7 = (tq - t^2(y_t\dot{q}+4q))\frac{\partial}{\partial q},\\
&U_8 = (q^2\mathrm{e}^{4t} - t(\dot{q}q+4q^2)\mathrm{e}^{4t})\frac{\partial}{\partial q}, \quad \text{and} \quad U_9 = (qG(v,u) + \mathrm{e}^{-4t}H(v,u))\frac{\partial}{\partial q},
\end{aligned} \tag{33}$$

in which G and H are arbitrary functions of v and u, namely

$$v = q\mathrm{e}^{4t} - t(\dot{q}+4q)\mathrm{e}^{4t}, \quad \text{and} \quad u = (\dot{q}+4q)\mathrm{e}^{4t}.$$

If $\beta = 12.5$, Eq.(18) leads to

$$\ddot{q} + 20\dot{q} + 64q = 0, \tag{34}$$

which has two independent solutions in the following forms:

$$q_1 = \mathrm{e}^{-4t}, \quad \text{and} \quad q_2 = \mathrm{e}^{-16t}. \tag{35}$$

By using expression (9) $p^{\mathrm{I}}\phi : J^{\mathrm{I}}(t,y) \to J^{\mathrm{I}}(X,Y)$ is defined as

$$\bar{X} = \mathrm{e}^{12t}, \quad \bar{P} = y\mathrm{e}^{16t}, \quad \text{and} \quad \bar{P}_X = (y_t + 16y)\mathrm{e}^{16t} \tag{36}$$

while $\phi^{-\mathrm{I}} : \{(X,Y)\} \to \{(t,y)\}$ is given by

$$\bar{t} = \frac{1}{12}\ln X, \quad \text{and} \quad \bar{q} = \mathrm{e}^{-\frac{4}{3}\ln X}P = X^{-\frac{4}{3}}P.$$

The Lie point and Lie contact symmetry algebras of Eq.(25), with $\beta = 8$ are computed from expressions (5), (6), (16) and (34) as follows:

$$\begin{aligned}
&U_1 = \mathrm{e}^{-16t}\frac{\partial}{\partial q}, \quad U_2 = \mathrm{e}^{-4t}\frac{\partial}{\partial q},\\
&U_3 = q\frac{\partial}{\partial q}, \quad U_4 = (\dot{q}+16q)\frac{\partial}{\partial q},\\
&U_5 = \mathrm{e}^{12t}(\dot{q}+16q)\frac{\partial}{\partial q}, \quad U_6 = q\mathrm{e}^{16t}(\dot{q}+16q)\frac{\partial}{\partial q},\\
&U_7 = \mathrm{e}^{12t}(q - \mathrm{e}^{12t}(\dot{q}+16q))\frac{\partial}{\partial q},\\
&U_8 = q\mathrm{e}^{16t}(q - \mathrm{e}^{12t}(\dot{q}+16q))\frac{\partial}{\partial q},\\
&U_9 = (qG(v,u) + \mathrm{e}^{-16t}H(v,u))\frac{\partial}{\partial q},
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
&v = q\mathrm{e}^{16t} - \mathrm{e}^{12t}(\dot{q}+16q)\mathrm{e}^{16t},\\
&\text{and} \quad u = (\dot{q}+16q)\mathrm{e}^{16t}.
\end{aligned} \tag{38}$$

Example 2 Consider a dynamical system with Lagrangian $L = \dot{q}^2/2 - \frac{1}{2}q^2$, subjected to a nonconservative force $Q' = 2\mathrm{e}^t - 1$, its non-homogeneous second-order ordinary differential equation is

$$\ddot{q} + q = 2\mathrm{e}^t - 1, \tag{39}$$

with a corresponding homogeneous second-order ordinary differential equation being

$$\ddot{q} + q = 0,$$

which has the following two solutions:

$$u_1(t) = \sin t, \quad \text{and} \quad u_2(t) = \cos t. \tag{40}$$

Moreover, Eq.(39) has an extra solution

$$f(t) = \mathrm{e}^t - 1, \tag{41}$$

then we take two independent solutions as

$$\begin{aligned}
&q_1(t) = \sin t + \mathrm{e}^t - 1,\\
&\text{and} \quad q_2(t) = \cos t + \mathrm{e}^t - 1,
\end{aligned} \tag{42}$$

which satisfy

$$y_1(0) = 0, \quad \text{and} \quad y_2(0) \neq 0.$$

By using expression (10) $p^{\mathrm{I}}\phi : J^{\mathrm{I}}(t,y) \to J^{\mathrm{I}}(X,Y)$ is defined as

$$\begin{aligned}
&\bar{X} = \frac{\sin t + \mathrm{e}^t - 1}{\cos t + \mathrm{e}^t - 1}, \quad \bar{P} = \frac{q}{\cos t + \mathrm{e}^t - 1},\\
&\text{and} \quad \bar{P}_X = \frac{\dot{q}(\cos t + \mathrm{e}^t - 1) - q(\mathrm{e}^t - \sin t)}{(\cos t + \mathrm{e}^t - 1)^2},
\end{aligned} \tag{43}$$

while $\phi^{-\mathrm{I}} : \{(X,Y)\} \to \{(t,y)\}$ is given by expression (43). Applying Eq.(15) to the Lie point and Lie contact symmetries of Eq.(2), i.e. Eqs.(5) and (6), yields

$$\begin{aligned}
&U_1 = (\cos t + \mathrm{e}^t - 1)\frac{\partial}{\partial q}, \quad U_2 = (\sin t + \mathrm{e}^t - 1)\frac{\partial}{\partial q}, \quad U_3 = q\frac{\partial}{\partial q},\\
&U_4 = \frac{\dot{q}(\cos t + \mathrm{e}^t - 1) - q(\mathrm{e}^t - \sin t)}{(\cos t + \mathrm{e}^t - 1)}\frac{\partial}{\partial q},\\
&U_5 = \frac{(\sin t + \mathrm{e}^t - 1)(\dot{q}(\cos t + \mathrm{e}^t - 1) - q(\mathrm{e}^t - \sin t))}{(\cos t + \mathrm{e}^t - 1)^2}\frac{\partial}{\partial q},\\
&U_6 = \frac{q\dot{q}(\cos t + \mathrm{e}^t - 1) - q^2(\mathrm{e}^t - \sin t)}{(\cos t + \mathrm{e}^t - 1)^2}\frac{\partial}{\partial q}, \quad U_7 = (\sin t + \mathrm{e}^t - 1)v\frac{\partial}{\partial q},\\
&U_8 = qv\frac{\partial}{\partial y}, \quad U_9 = (yG(v,u) + (\cos t + \mathrm{e}^t - 1)H(v,u))\frac{\partial}{\partial q},
\end{aligned} \tag{44}$$

where

$$v = \frac{q}{\cos t + \mathrm{e}^t - 1} - \frac{\sin t + \mathrm{e}^t - 1}{\cos t + \mathrm{e}^t - 1}\frac{\dot{q}(\cos t + \mathrm{e}^t - 1) - q(\mathrm{e}^t - \sin t)}{(\cos t + \mathrm{e}^t - 1)^2},$$
$$u = \frac{\dot{q}(\cos t + \mathrm{e}^t - 1) - q(\mathrm{e}^t - \sin t)}{(\cos t + \mathrm{e}^t - 1)^2}. \tag{45}$$

4. Conclusions

The present investigation is a generalization of the results for second-order ordinary differential equations to linear nonconservative dynamical systems. By using 1–1 mapping, which transforms differential equations into $P_{XX} = 0$, we can construct the Lie contact symmetry algebra for two and more dimensional nonconservative dynamical systems. It should be pointed out that the present research can be generalized to nonholonomic dynamical systems.

References

[1] Anderson R L and Davison S M 1974 *J. Math. Anal. Appl.* **48** 301
[2] Wulfman C E and Wybourne B G 1976 *J. Phys.* A: *Math. Gen.* **9** 507
[3] Lutzky M 1978 *J. Phys.* A: *Math. Gen.* **11** 249
[4] Aquirre M and Krause J 1984 *J. Math. Phys.* **25**2103
[5] Aguirre M and Krause J 1987 *J. Phys.* A **20** 3553
[6] Govinder K S and Leach P G L 1998 *SIAM Rev.* **40** 945
[7] Leach P G L 1980 *J. Aust. Math. Soc. Ser.* B **22** 12
[8] Leach P G L 1978 *SIAM J. Appl. Math.* **34** 496
[9] Leach P G L 1977 *J. Aust. Math. Soc. Ser.* **20** 97
[10] Guo Y X, Yu Y and Huang H J 2001 *Chin. Phys.* **10** 1
[11] Mei F X and Xu X J 2005 *Chin. Phys.* **14** 443
[12] Mei F X 1999 *Applied of Lie Groups and Lie Algebraic to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[13] Fang J H, Liao Y P, Ding N and Wang P 2006 *Chin. Phys.* **15** 470
[14] Chen X W, Liu C M and Li Y M 2006 *Chin. Phys.* **15** 470
[15] Zhang Y, Shang M and Mei F X 2000 *Chin. Phys.* **9** 401
[16] Zhang H B, Chen L Q, Liu R W and Gu S L 2005 *Acta Phys. Sin.* **54** 2489 (in Chinese)
[17] Qiao Y F, Li R J and Zhao S H 2005 *Chin. Phys.* **14** 882
[18] Lou Z M 2006 *Chin. Phys.* **15** 891
[19] Schwarz F 1983 *J. Phys.* A: *Math. Gen.* **16** L133
[20] Martini R and Kersten P H M 1983 *J. Phys.* A: *Math. Gen.* **16** L455
[21] Minoru Omote 1988 *J. Math. Phys.* **29** 2599
[22] Mahomed F M and Leach P G L 1989 *J. Math. Phys.* **30** 2770
[23] Kaushal R S and Mishra B C 1993 *J. Math. Phys.* **24** 5543
[24] Arnold V 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* chap.1 (Grundlehren der Mathematischen Wissenschaften 250) (New York: Springer)
[25] Lie S 1893 *Theorie der Transformationgruppen* (Leipzig: Teubner)
[26] Pirani F A E, Robinson D C and Shadwick W F 1979 *Local Jet Bundle Formulation of Bäcklund Transformations, Mathematical Physics* Studies 1 (Dordrecht: Reidel)
[27] Kumei S and Bluman G W 1982 *SIAM Appl. Math.* **42** 1157
[28] Spiegel M R 1978 *Theories and Exercises of Advanced Mathematics* (Shanghai: Shanghai Science and Technology Press) p102 [Xie G R *et al* translator] (in Chinese)