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Canards in a rheodynamic model of cardiac pressure pulsations*

Xie Feng (谢 峰)[a)c)] and Chen Xian-Feng (陈贤峰)[b)c)]

[a)] *Department of Applied Mathematics, Donghua University, Shanghai 201620, China*
[b)] *Department of Mathematics, Shanghai Jiaotong University, Shanghai* 200240, *China*
[c)] *Division of Computational Science, E-Institute of Shanghai Universities, at SJTU, Shanghai* 200030, *China*

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This paper reports on the canard phenomenon occurring in a rheodynamic model of cardiac pressure pulsations. By singular perturbation techniques the corresponding parameter value at which canards exist is obtained. The physiological significance of canards in this model is given.

Keywords: canards, rheodynamic model, cardiac self-oscillations, singular perturbations
PACC: 0547, 0200

1. Introduction

To describe the mechanical properties of cardiac muscle, some mathematical models have been proposed (see Refs.[1–4], for instance). Recently, in Ref.[5], by introducing additional viscous and inertial elements in a typical three-element model,[4] Petrov and Nikolov deduced a rheodynamical model of cardiac pressure pulsations, which was applied to parametrically analyse the oscillations of the left ventricular pressure, moved cardiac blood volume and aortic blood flow.

The model in Ref.[5] takes the form:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = \xi - e, \tag{1}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = [a - b(\xi - c - e)^2](\xi - c - e) - g\zeta, \tag{2}$$

where the variables ζ, ξ and τ present dimensionless blood volume momentum, left ventricular pressure and time, respectively, and the dimensionless constant coefficients a, b, c, g and e are defined as follows:

$$a = \frac{(r_1 - r_0)T}{m}, \quad b = \frac{\chi A_{\mathrm{C}}^2 \bar{P}^2 T}{mk^2},$$
$$c = \frac{SP_{\mathrm{AV}}}{\bar{P}A_{\mathrm{C}}}, \quad e = \frac{P_{\mathrm{A}}}{\bar{P}}, \tag{3}$$
$$g = \frac{kT^2}{m}, \quad \bar{P} = \frac{1}{T}\int_0^T (P - P_{\mathrm{A}})\mathrm{d}t. \tag{4}$$

All the parameters taken in Eqs.(3) and (4) have physical meanings, which are given in Table 1; for details see Ref.[5].

Table 1. Physical meanings of the parameters in Eqs.(3) and (4).

r_0	positive friction coefficient
r_1	absolute value of constant negative friction
m	mass of myocard muscle
T	expecting period of oscillation of the model
χ	phenomenological coefficient of variable positive friction
P	left ventricular pressure
S	cross-section of mitral valve entry
k	new effective coefficient of elasticity
A_{C}	cross-section of aortic artery
P_{A}	average arterial pressure
P_{AV}	left ventricular diastolic pressure

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In Ref.[5], the authors gave a simple equilibrium analysis and numerical analysis of the rheodynamical model for $a = 80, b = 70, e = 1.2$, $g = 216$ or 255, and some values of c lying in $(0, 1)$. Very recently, a rigorous qualitative analysis and complete parameter analysis for this model were given in Ref.[6].

In the present article, we report on the canard phenomenon occurring in this model. A canard is a trajectory which follows both the attracting and repelling parts of a slow manifold in a slow–fast vector field. Canard limit cycles are associated with a dramatic change in properties, such as the amplitude and period of a periodic orbit within an exponentially small interval of a control parameter. In addition to their intrinsic interest, it has recently been argued that canards provide a mechanism for the synchronization of neurons.[7] Canards are first found in a study of the van der Pol system using techniques from nonstandard analysis.[8] Later, Eckhaus[9] and Mishchenko *et al*[10] applied classical asymptotic analysis to the study of canards. Recently, some geometric methods (see for instance, Refs.[11–13]) have also been used to analyse a variety of canards. In addition to the van der Pol system, canards have been found and analysed to varying degree for a variety of chemical, biological and other systems; see for instance Refs.[14–16] and references therein.

The article is organized as follows. In Section 2 some qualitative results are represented. In Section 3, we first write Eqs.(1)–(2) as a slow–fast system by introducing a singular perturbation parameter,[17−24] then give an asymptotic analysis of the system obtained, and by smoothly connecting associated attracting and repelling parts of slow manifolds in an asymptotic limit, the canard parameter value (namely, the parameter value at which canards exist) is predicted. This prediction matches very well with the numerically obtained value. In the end, we represent the physiological significance of canards in this model. Concluding remarks are given Section 4.

2. Qualitative aspect for the model

In this section, we represent some qualitative results of Eqs.(1)–(2). Obviously, system (1)–(2) has a unique equilibrium $\left(\dfrac{c(bc^2-a)}{g}, e\right)$. The following theorem concerns the stability of equilibrium and the existence of limit cycle.

Theorem[6] For system (1)–(2), the following assertions hold.

(I) If $a \le 3bc^2$, $\left(\dfrac{c(bc^2-a)}{g}, e\right)$ is asymptotic stable, and the system has no limit cycle.

(II) If $a > 3bc^2$, $\left(\dfrac{c(bc^2-a)}{g}, e\right)$ is unstable, and the system has a unique and asymptotic stable limit cycle.

System (1)–(2) undergoes a Hopf bifurcation at $c = \sqrt{a/3b}$, that is, when c is varied from $c > \sqrt{a/3b}$ to $c < \sqrt{a/3b}$, a stable limit cycle is born via Hopf bifurcation. Figure 1 shows the periodic orbits obtained numerically for $a = 80, b = 70,\ g = 216,\ e = 1.2$ and different values of c. We observe that the periodic orbit changes its properties very dramatically in a very narrow interval of the bifurcation parameter c. In fact, for the appropriate value of parameter c the periodic orbit follows both the left branch and middle branch of the ξ-nullcline, that is, the attracting and repelling parts of a slow manifold, which implies the occurrence of canards.

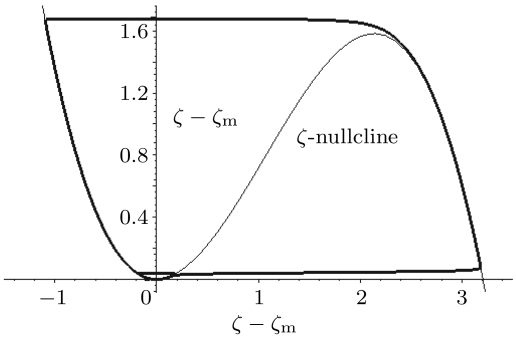


Fig.1. For $a = 80, b = 70, g = 216$ and $e = 1.2$, the "small" periodic orbit corresponding to $c = 0.6167080063$ and the "big" periodic orbit corresponding to $c = 0.6166478998$.

In order to convert Eqs.(1)–(2) into the canonical form, we need the following proposition, which can be easily verified.

Proposition For positive parameters a, b, c, g, e, the ξ-nullcline is S-shaped and has a local minimum at $(\xi_{\rm m}, \zeta_{\rm m})$, where

$$\xi_{\rm m} = e + c - \sqrt{\frac{a}{3b}},$$

$$\zeta_{\rm m} = \frac{c(bc^2-a)}{g} + \frac{1}{g}\left[a - 3bc^2 + 3bc\left(c - \sqrt{\frac{a}{3b}}\right) - b\left(c - \sqrt{\frac{a}{3b}}\right)^2\right]\left(c - \sqrt{\frac{a}{3b}}\right).$$

For $a = 80, b = 70, c = 0.4, g = 216$ and $e = 1.2$, the nullclines of system (1)–(2) are depicted in Fig.2.

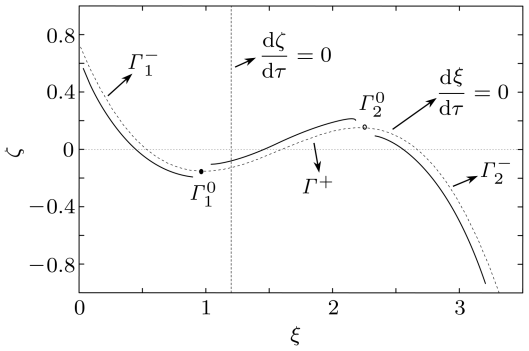


Fig.2. For $a = 80, b = 70, c = 0.4, g = 216$ and $e = 1.2$, the nullclines of system (1)–(2).

3. Analysis of canards for the model

Note that, for the values of interest of the parameters a, b, c, g, e, the quantities $1/a, 1/b$ and $1/g$ are all small. We introduce a singular perturbation parameter ε by letting $\varepsilon = 1/g$, and let $\alpha = a/g$ and $\beta = b/g$. Thus, system (1)–(2) becomes

$$\frac{d\zeta}{d\tau} = u(\zeta, \xi), \tag{5}$$

$$\varepsilon\frac{d\xi}{d\tau} = v(\zeta, \xi), \tag{6}$$

where

$$u(\zeta, \xi) = \xi - e,$$
$$v(\zeta, \xi) = [\alpha - (\xi - c - e)^2](\xi - c - e) - \zeta.$$

For ε small system (5)–(6) has two time scales: the slow scale and the fast scale. By transforming from the slow time scale τ to the fast time scale $t = \tau/\varepsilon$, one obtains the following equivalent fast system

$$\frac{d\zeta}{dt} = \varepsilon u(\zeta, \xi), \tag{7}$$

$$\frac{d\xi}{dt} = v(\zeta, \xi). \tag{8}$$

For $\varepsilon = 0$ the set $\Gamma \equiv \{(\zeta, \xi) | v(\zeta, \xi) = 0\}$ corresponds to the equilibria of system (7)–(8). We call Γ the critical manifold. For typical parameter values, for instance, $a = 80, b = 70, c = 0.4, g = 216$, and $e = 1.2$, Fig.2 depicts the shape of Γ in the domain of interest, which consists of five nonintersecting parts $\Gamma_1^-, \Gamma_1^0, \Gamma^+, \Gamma_2^0, \Gamma_2^-$. Obviously, Γ_1^- and Γ_2^- are attracting, Γ^+ repelling, and Γ_1^0, Γ_2^0, which are bifurcation points of the critical manifold Γ, are non-hyperbolic. Let $\bar{\Gamma}_1^-$, $\bar{\Gamma}_2^-$ and $\bar{\Gamma}^+$ are any compact submanifolds of Γ_1^-, Γ_2^- and Γ^+ respectively. It follows from the geometric singular perturbation theory (see Ref.[25], for example) that for sufficiently small ε, $\bar{\Gamma}_1^-$, $\bar{\Gamma}_2^-$ and $\bar{\Gamma}^+$ persist as nearby invariant slow manifolds $\bar{\Gamma}_{1,\varepsilon}^-$, $\bar{\Gamma}_{2,\varepsilon}^-$ and $\bar{\Gamma}_\varepsilon^+$ of system (5)–(6) with $\bar{\Gamma}_{1,0}^- = \bar{\Gamma}_1^-$, $\bar{\Gamma}_{2,0}^- = \bar{\Gamma}_2^-$, $\bar{\Gamma}_0^+ = \bar{\Gamma}^+$ respectively. An important issue is to study the dynamics of system (5)–(6), and in particular, to clarify the behaviour of the slow manifold $\bar{\Gamma}_{1,\varepsilon}^-$, $\bar{\Gamma}_{2,\varepsilon}^-$ and $\bar{\Gamma}_\varepsilon^+$ in the vicinity of the points Γ_1^0 and Γ_2^0.

In system (5)–(6) we treat c as a control parameter. Therefore, the slow manifolds change as c is varied. The manifolds $\bar{\Gamma}_{1,\varepsilon}^-$ and $\bar{\Gamma}_{2,\varepsilon}^-$ can be extended beyond the point Γ_1^0 according to the flow. In what follows, we will seek the value $c^*(\varepsilon)$ of the parameter c for which the manifolds $\bar{\Gamma}_{1,\varepsilon}^-$ and $\bar{\Gamma}_{2,\varepsilon}^-$ merge in the neighbourhood of the point Γ_1^0. The manifold obtained for the parameter value $c^*(\varepsilon)$ is called a canard manifold. In this situation, nearby trajectories follow the canard manifold both along its attracting and repelling parts.

In order to use asymptotic expansions to approximate the value for c at which canards exist, it is convenient to move the point Γ_1^0 to the origin. Therefore, we make the transformation

$$\xi = \xi_m + x, \quad \zeta = \zeta_m + y.$$

Thus, system (7)–(8) is transformed into

$$\frac{dx}{dt} = 3\sqrt{\alpha\beta}x^2 - \beta x^3 - y, \tag{9}$$

$$\frac{dy}{dt} = \varepsilon\left(x + c - \sqrt{\frac{\alpha}{3\beta}}\right). \tag{10}$$

Let us begin by noting that trajectories of system (9)–(10) must satisfy

$$(3\sqrt{\alpha\beta}x^2 - \beta x^3 - y)\frac{dy}{dx} = \varepsilon\left(x + c - \sqrt{\frac{\alpha}{3\beta}}\right). \tag{11}$$

Let

$$c = c_0 + c_1\varepsilon + c_2\varepsilon^2 + \cdots. \tag{12}$$

We seek solutions of Eq.(11) of the form

$$y(x) = \sum_{i=0}^{\infty} y_i(x)\varepsilon^i. \tag{13}$$

Substituting Eqs.(12) and (13) into Eq.(11) and equating the coefficients in like powers of ε, we get a recurrent sequence of algebraic equations for the functions

$y_i(x)$. The constants c_i can be found from the condition of boundedness of the functions $y_i(x)$ as x tends to zero.

At the first step of the algorithm, we easily get

$$y_0 = 3\sqrt{\alpha\beta}x^2 - \beta x^3.$$

To determine c_0, at the second step of the algorithm we obtain

$$y_1(x) = -\frac{x + c_0 - \sqrt{\dfrac{\alpha}{3\beta}}}{y_0'},$$

where the prime denotes the differentiation with respect to x. For the coefficient $y_1(x)$ to be a smooth function in the neighbourhood of the point $x = 0$, taking $y_0'(0) = 0$ into account, we find that it is sufficient to choose $c_0 = \sqrt{\alpha/3\beta}$. Thus,

$$y_1(x) = \frac{1}{3\beta x - 6\sqrt{\alpha\beta}}.$$

To determine c_1, consider the order of $\mathcal{O}(\varepsilon^2)$, and then we have

$$y_2(x) = -\frac{y_1(x)y_1'(x) + c_1}{y_0'(x)}.$$

Therefore, for $y_2(x)$ to be a smooth function in the neighbourhood of the point $x = 0$, we choose c_1 to satisfy

$$y_1(0)y_1'(0) + c_1 = 0,$$

that is,

$$c_1 = -y_1(0)y_1'(0) = -\frac{1}{72\alpha\sqrt{\alpha\beta}}.$$

Thus, for $a = 80, b = 70$ and $g = 216$ the canard parameter value c^* is

$$c^* = c_0 + c_1\varepsilon + \cdots \approx 0.616712285,$$

which matches quite well with the canard parameter values in Fig.1 obtained numerically. According to Ref.[5], the parameter c presents a dimensionless analogue P_{AV}. For $0 < c < 0.6172$ system (1)–(2) has a stable limit cycle, presenting self-oscillations. When c arrives at 0.618 (corresponding to $P_{\mathrm{AV}} = 26.5\,\mathrm{mm\ Hg}$) the self-oscillations rapidly disappear. Physiologically, this state corresponds to lack of cardiac pulsations of the heart muscle.[5] Therefore, physiologically, canards correspond an extremely transitory transition state, along which the normal heart function may rapidly cease.

In view of Eq.(12), it follows that for g large enough, g does have little influence on canard parameter values c^*; see Fig.3 for a numerical portrait.

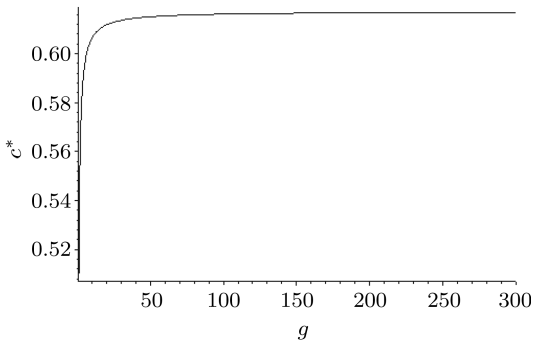


Fig.3. For $a = 80, b = 70$ and $e = 1.2$, the canard parameter value c^* as a function of g.

4. Conclusions

In this paper, we have reported on the canard phenomenon occurring in a rheodynamic model of cardiac pressure pulsations. Canards in this model have physiological significance, that is, canards correspond an extremely transitory transition state, along which the normal heart function may rapidly cease. We also find that for g large enough, g does have little influence on canard parameter values c^*.

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