Vol 16 No 9, September 2007
1009-1963/2007/16(09)/2612-04
Chinese Physics
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Projective synchronization in coupled fractional order chaotic Rossler system and its control*

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(Received 27 December 2006; revised manuscript received 24 January 2007)

This paper proposes a method to achieve projective synchronization of the fractional order chaotic Rossler system. First, construct the fractional order Rossler system's corresponding approximate integer order system, then a control method based on a partially linear decomposition and negative feedback of state errors is utilized on the new integer order system. Mathematic analyses prove the feasibility and the numerical simulations show the effectiveness of the proposed method.

Keywords: fractional order, chaos synchronization, projective synchronization, Rossler system
PACC: 0545

1. Introduction

It is well known that the fractional calculus is a classical mathematical notion, with a history as long as calculus itself. But its applications to physics and engineering are just a recent subject of interest. As an important branch of the applications of fractional calculus, the fractional order chaotic system has attracted much attention. Many general integer chaotic systems have proved that its corresponding fractional order system can be found by numerical experiment.[1−7] These studies told us how to represent a general integer order chaotic system by fractional order system, and the characters of the new fractional order system were also analysed.

Since Pecora and Carroll first studied the synchronization of chaotic systems in 1990,[8,9] chaos synchronization has become a subject of great interest.[10−14] In 1999, Mainieri and Rehacek[15] observed a new phenomenon in partially linear chaotic system, which they called projective synchronization (PS), PS is the dynamical behaviour in which the responses of two identical systems synchronize up to a constant scaling factor. This phenomenon was observed and studied widely in the coupled integer-order chaotic system.[16−20]

For the fractional order system, chaos synchronization was studied[1−7] and the PS of the fractional order Chen's system was proved which can be obtained by numerical experiment.[21] Unfortunately, to our knowledge, almost all the current fractional order systems were based on the numerical experiment; no profound theoretical analysis was done for the control mechanism of PS of fractional order chaotic systems.

In this paper, we present a nonlinear control approach to achieve PS of the coupled fractional order chaotic Rossler system. First, we utilize an integer order system instead of the fractional one approximately. Then, the common method on the general integer order system can be utilized. In other words, we study the corresponding integer order system instead of the fractional order system by the integer order system's method. The stability is proved mathematically and numerical simulation is introduced to verify the result.

2. Construction of the coupled fractional order Rossler system

2.1. Fractional derivatives and its approximation

There are several definitions of fractional derivatives. In this paper, we use the best known Riemann–Liouville definition in Ref.[22] to solve the fractional derivatives, i.e.

$$\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}^{\alpha} t} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, \qquad (1)$$

where $\Gamma(g)$ is the gamma function and $n-1 \leq \alpha < n$.

Considering the initial conditions at $t = 0$, one

*Project supported by the Key Youth Project of Southwest University for Nationalities of China and the Natural Science Foundation of the State Nationalities Affairs Commission of China (Grant Nos 05XN07 and 07XN05).
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then has the Laplace transform of Eq.(1),

$$L\Big\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\Big\} = s^\alpha L\{f(t)\}. \tag{2}$$

So, the fractional integral operator of order 'α' can be represented by the transfer function $F(s) = 1/s^\alpha$; a general method to solve the fractional differential is to approximate the fractional operators by using the standard integer order operators;[23] in Table 1 of Ref.[24], the authors introduced the approximations for $1/s^\alpha$ with $\alpha = 0.1 - 0.9$ in steps of 0.1 with errors of approximately 2dB; in this paper, $\alpha = 0.9$ will be studied in the following. When $\alpha = 0.9$,

$$\frac{1}{s^{0.9}} = \frac{\mathrm{d}\,s^2 + es + f}{s^3 + gs^2 + hs + i}, \tag{3}$$

where $d = 1.766$, $e = 38.27$, $f = 4.914$, $g = 36.15$, $h = 7.789$, $i = 0.01$.[24]

2.2. Two coupled fractional order Rossler system

From the above, one can get the following fractional-order Rossler system easily.

Consider a fractional order Rossler system mentioned in Ref.[5],

$$\begin{aligned}
&\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = -(y - z), \qquad \frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} = x + ay,\\
&\frac{\mathrm{d}^\alpha z}{\mathrm{d}t^\alpha} = b + z(x - c),
\end{aligned} \tag{4}$$

where $a = 0.6$, $b = 0.2$, $c = 10$ in our paper.
Then, the coupled fractional order Rossler system can be constructed as

$$\begin{aligned}
&\frac{\mathrm{d}^\alpha x_\mathrm{m}}{\mathrm{d}t^\alpha} = -(y_\mathrm{m} - z),\\
&\frac{\mathrm{d}^\alpha y_\mathrm{m}}{\mathrm{d}t^\alpha} = x_\mathrm{m} + ay_\mathrm{m},\\
&\frac{\mathrm{d}^\alpha z}{\mathrm{d}t^\alpha} = b + z(x_\mathrm{m} - c),\\
&\frac{\mathrm{d}^\alpha x_\mathrm{s}}{\mathrm{d}t^\alpha} = -(y_\mathrm{s} - z),\\
&\frac{\mathrm{d}^\alpha y_\mathrm{s}}{\mathrm{d}t^\alpha} = x_\mathrm{s} + ay_\mathrm{s},
\end{aligned} \tag{5}$$

where m and s mean master system and slave system respectively. When $\alpha = 0.9$, doing the Laplace transform of Eq.(5), one can obtain

$$\begin{aligned}
&X_\mathrm{m}(s) = -\frac{1}{s^{0.9}}L(y_\mathrm{m} - z),\\
&Y_\mathrm{m}(s) = \frac{1}{s^{0.9}}L(x_\mathrm{m} + ay_\mathrm{m}),\\
&Z(s) = \frac{1}{s^{0.9}}L(b + z(x_\mathrm{m} - c)),\\
&X_\mathrm{s}(s) = -\frac{1}{s^{0.9}}L(y_\mathrm{s} - z),\\
&Y_\mathrm{s}(s) = \frac{1}{s^{0.9}}L(x_\mathrm{s} + ay_\mathrm{s}).
\end{aligned} \tag{6}$$

Consider Eq.(3), define

$$\begin{aligned}
&x_\mathrm{m} = x_{1\mathrm{m}}, \qquad \dot{x}_\mathrm{m} = \dot{x}_{1\mathrm{m}} = x_{2\mathrm{m}},\\
&\ddot{x}_\mathrm{m} = \dot{x}_{2\mathrm{m}} = x_{3\mathrm{m}}, \qquad y_\mathrm{m} = x_{4\mathrm{m}},\\
&\dot{y}_\mathrm{m} = \dot{x}_{4\mathrm{m}} = x_{5\mathrm{m}}, \qquad \ddot{y}_\mathrm{m} = \dot{x}_{5\mathrm{m}} = x_{6\mathrm{m}},\\
&z = x_7, \quad \dot{z} = \dot{x}_7 = x_8, \quad \ddot{z} = \dot{x}_8 = x_9,\\
&x_\mathrm{s} = x_{1\mathrm{s}}, \qquad \dot{x}_\mathrm{s} = \dot{x}_{1\mathrm{s}} = x_{2\mathrm{s}},\\
&\ddot{x}_\mathrm{s} = \dot{x}_{2\mathrm{s}} = x_{3\mathrm{s}}, \quad y_\mathrm{s} = x_{4\mathrm{s}},\\
&\dot{y}_\mathrm{s} = \dot{x}_{4\mathrm{s}} = x_{5\mathrm{s}}, \quad \ddot{y}_\mathrm{s} = \dot{x}_{5\mathrm{s}} = x_{6\mathrm{s}},
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\boldsymbol{x}_\mathrm{m} = \mathrm{col}(x_{1\mathrm{m}}, x_{2\mathrm{m}}, x_{3\mathrm{m}}, x_{4\mathrm{m}}, x_{5\mathrm{m}}, x_{6\mathrm{m}}),\\
&\boldsymbol{z} = \mathrm{col}(x_7, x_8, x_9),\\
&\boldsymbol{x}_\mathrm{s} = \mathrm{col}(x_{1\mathrm{s}}, x_{2\mathrm{s}}, x_{3\mathrm{s}}, x_{4\mathrm{s}}, x_{5\mathrm{s}}, x_{6\mathrm{s}}).
\end{aligned} \tag{8}$$

From Eqs.(3), (5), (6), (7) and (8),one can get the time domain formula of Eq.(5),

$$\dot{\boldsymbol{x}}_\mathrm{m} = A\boldsymbol{x}_\mathrm{m} + B\boldsymbol{z}, \quad \dot{\boldsymbol{z}} = C\boldsymbol{z}, \quad \dot{\boldsymbol{x}}_\mathrm{s} = A\boldsymbol{x}_\mathrm{s} + B\boldsymbol{z}, \tag{9}$$

where $\boldsymbol{x}_\mathrm{m} \in R^{6\times1}$, $\boldsymbol{z} \in R^{3\times1}$, $\boldsymbol{x}_\mathrm{s} \in R^{6\times1}$, and

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
-i & -h & -g & -f & -e & -d\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
f & e & d & af - i & ae - h & ad - g
\end{bmatrix}, \qquad B = \begin{bmatrix}
0 & 0 & 0\\
0 & 0 & 0\\
-f & -e & -d\\
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0
\end{bmatrix},$$

$$C = \begin{bmatrix}
0 & 1 & 0\\
0 & 0 & 1\\
dx_{3\mathrm{m}} + ex_{2\mathrm{m}} + f(x_{1\mathrm{m}} - c) - i & 2dx_{2\mathrm{m}} + ex_{1\mathrm{m}} - h - ce & dx_{1\mathrm{m}} - cd - g
\end{bmatrix}. \tag{10}$$

From the above, we know that Eq.(9) is approximately equivalent to Eq.(5), Then, studying the PS of Eq.(9) instead of that of Eq.(5), one can use the integer order method to study the corresponding fractional order system.

3. PS of two coupled fractional order Rossler system

To make the two coupled system achieve PS, we add control $u(t)$ to the slave system of Eq.(9), then one can obtain

$$\begin{aligned}\dot{\boldsymbol{x}}_{\rm m} &= A\boldsymbol{x}_{\rm m} + B\boldsymbol{z}\\ \dot{\boldsymbol{z}} &= C\boldsymbol{z}\\ \dot{\boldsymbol{x}}_{\rm s} &= A\boldsymbol{x}_{\rm s} + B\boldsymbol{z} + u.\end{aligned} \tag{11}$$

Now, the problem becomes, what kind of u can make system (11) achieve PS?

Theorem 1 If the control input in Eq.(11) takes the form of

$$u = (1-\beta)B\boldsymbol{z} - K(\beta\boldsymbol{x}_{\rm m} - \boldsymbol{x}_{\rm s}), \tag{12}$$

where β is the assigned scaling factor and

$$K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_{\rm n}\end{bmatrix}. \tag{13}$$

satisfies the condition

$$A - K < 0\,, \tag{14}$$

the PS can be achieved between the master and the slave systems in the integer order coupled system (5) and its corresponding fractional order coupled system (8).

Proof Define the error vector of PS of Eq.(9) as

$$\boldsymbol{e} = \beta\boldsymbol{x}_{\rm m} - \boldsymbol{x}_{\rm s}. \tag{15}$$

Construct the Lyapunov function $v = \frac{1}{2}\boldsymbol{e}^{\rm T}\boldsymbol{e}$, then

$$\begin{aligned}\dot{v} &= \boldsymbol{e}^{\rm T}\dot{\boldsymbol{e}} = \boldsymbol{e}^{\rm T}(\beta\dot{\boldsymbol{x}}_{\rm m} - \dot{\boldsymbol{x}}_{\rm s})\\ &= \boldsymbol{e}^{\rm T}(\beta A\boldsymbol{x}_{\rm m} + \beta B\boldsymbol{z} - A\boldsymbol{x}_{\rm s} - B\boldsymbol{z}\\ &\quad + (1-\beta)B\boldsymbol{z} - K(\beta\boldsymbol{x}_{\rm m} - \boldsymbol{x}_{\rm s}))\\ &= \boldsymbol{e}^{\rm T}(A-k)\boldsymbol{e}.\end{aligned} \tag{16}$$

Consider Eq.(14); obviously, $\dot{v} < 0$. So, $\lim\limits_{t\to\infty}\boldsymbol{e} = 0$, this means $\lim\limits_{t\to\infty}\boldsymbol{x}_{\rm s} = \beta\boldsymbol{x}_{\rm m}$, PS of Eq.(9) is achieved. And because Eq.(9) is the approximation of the fractional system (5), the PS will also be achieved in system (5).

The proof is complete.

4. Numerical simulations

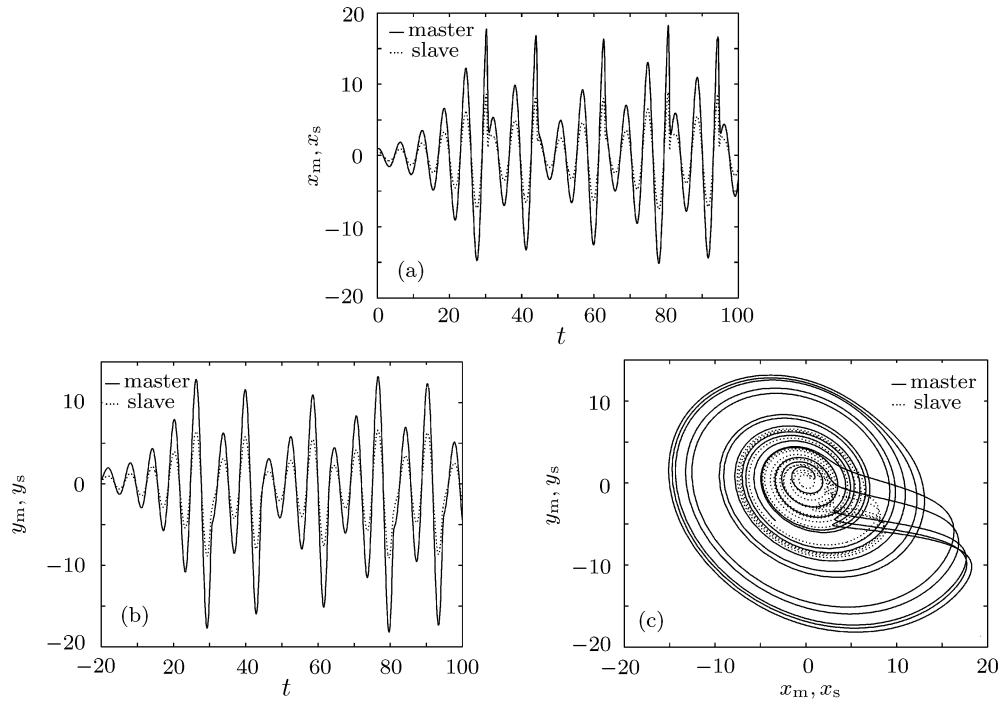


Fig.1. The numerical simulation result ($\beta = 0.5$). (a) Time evolution of the state $x_{\rm m}$, $x_{\rm s}$. (b) Time evolution of the state $y_{\rm m}$, $y_{\rm s}$. (c) The projection of the master and slave system onto the $x - y$ plane.

Because A is a constant matrix, one can find a matrix k which satisfies Eq.(14) easily. In this paper, we only show the 0.9 order system; the other fractional order system can be obtained by the same method. When $\beta = 0.5$, and $\beta = -0.5$, the simulation is shown in Figs.1 and 2 respectively.

The simulation results above demonstrate that PS can be achieved in the coupled fractional order chaotic Rossler system by the method presented in this paper.

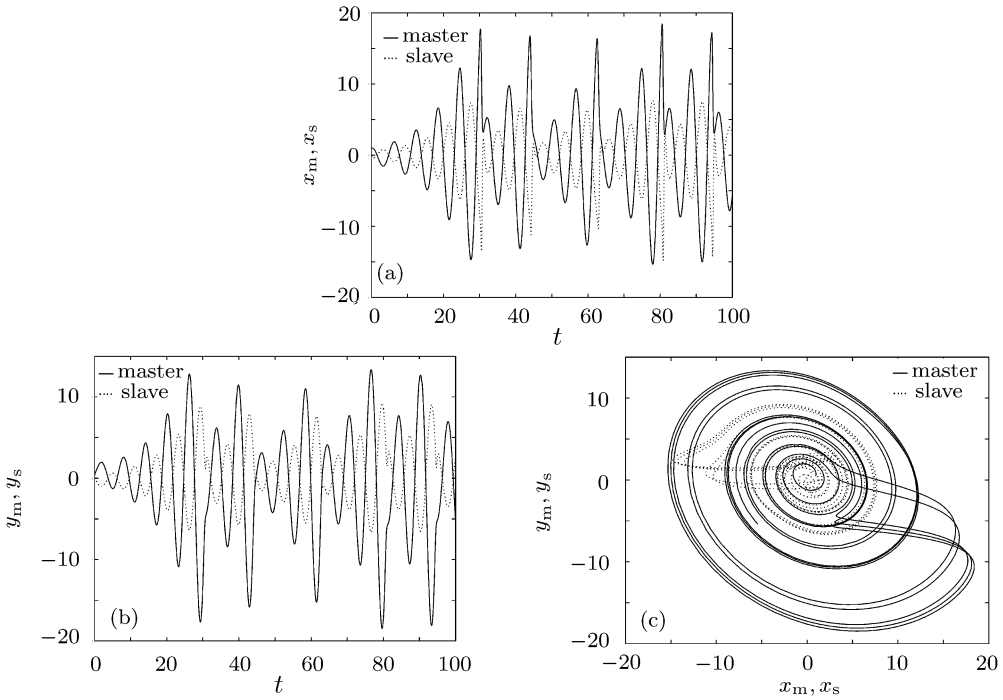


Fig.2. The numerical simulation result ($\beta = -0.5$). (a) Time evolution of the state x_m, x_s. (b) Time evolution of the state y_m, y_s. (c) The projection of the master and slave system onto the $x - y$ plane.

5. Conclusions

This paper has proposed a method to achieve the PS of fractional order chaotic Rossler system. Analyse the fractional order system according to the common method used in general integer order system by transforming the fractional order system into integer order system. First, construct the corresponding integer order system of fractional order Rossler system, then a control method based on a partial linear decomposition and negative feedback of state errors is utilized on the new integer order system. Numerical simulations show the effectiveness and feasibility of the proposed approach. The method can also be used in other fractional chaotic systems.

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