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Generalized synchronization of two different chaotic systems*

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In this paper, generalized synchronization of two different chaotic dynamical systems is investigated. An active control is adopted to construct a response system which synchronizes with a given drive system for a function relation. Based on rigorous analysis, the error system is asymptotically stable at the equilibrium. Numerical simulations illustrate the effectiveness of the proposed theory.

Keywords: chaos, generalized synchronization, active control
PACC: 0545

1. Introduction

Synchronization in chaotic systems has been extensively investigated both theoretically and experimentally since 1990.[1] Many control methods[1−8] have been developed to achieve chaos synchronization. Originally, chaos synchronization refers to complete synchronization (CS), where two chaotic systems possess the same state trajectories for $t \to \infty$. However, CS is a rather restrictive concept and it is often difficult to achieve CS except under ideal conditions. Recently, a generalization of synchronization, called generalized synchronization (GS) was proposed, where two systems are said to synchronize if there is a functional relation between the states of both systems. This kind of synchronization in a generalized sense leads to richer behaviour than CS. Although Kocarev and Parlitz[8] have developed a general theorem on the necessary and sufficient conditions of the presence of functional relation between the states of drive and response systems, it is still a hard task to know how they are related to each other even if we know the existence of functional relation. To the best of our knowledge, there are few theoretical results for detecting the presence of this kind of functional relation though numerous examples and different control methods for GS have been introduced in Refs.[9,10].

In this paper, we study GS between two different chaotic systems and design the appropriate controller to implement GS with the given drive chaotic system via the active control.

The rest of this paper is organized as follows. In Section 2, the definition of GS between two dynamical systems is restated. An approach for constructing a response system via an active control to synchronize with the drive system is proposed in Section 3. Finally, numerical simulations are given to verify the effectiveness of the method.

2. Definition of GS

In general, a standard definition of GS is as follows: given a drive system $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$ and a response system $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}} \in R^m$, two dynamical systems x and y exhibit GS if there exists a function φ such that $y = \varphi(x)$.

Definition 1 Given two dynamical systems

$$\begin{aligned} \dot{x} &= f(x,t) \quad \leftarrow \text{drive system}, \\ \dot{y} &= h(x,y,t) \quad \leftarrow \text{response system}, \end{aligned} \tag{1}$$

the two systems in Eq.(1) are said to be in a state of GS if there exists a differentiable transformation $\varphi : R^n \to R^m$, a manifold $M = \{(x,y)|y = \varphi(x)\}$, and a set $B = B_x \times B_y \subset R^n \times R^m$ with $M \subset B$ such that all trajectories of Eq.(1) with initial conditions in basin B approach M as time goes to infinity, i.e.

$$\lim_{t\to\infty} \|y - \varphi(x)\| = 0\,. \tag{2}$$

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Remark 1 One can predict the state of one of the systems through the function $\varphi(x)$ provided that the state of the other system is known.

The purpose of this paper is to construct a response system in the systems (1) to satisfy Eq.(2) by using active control method. Hence we rewrite the response system of Eq.(1) in the form

$$\dot{y} = h(x, y, t) = g(y) + u(y, x), \tag{3}$$

where $u(y, x) = u(u_1, u_2, \ldots, u_m)^{\mathrm{T}} \in R^m$ is a controller to be designed later.

3. Simulation results

In this section we give some simulation results to realize GS by using the active control method.

3.1. GS between Lorenz system and Rossler hyperchaotic system

Take the well-known Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= \gamma x_1 - x_1 x_3 - x_2, \\ \dot{x}_3 &= x_1 x_2 - \xi x_3, \end{aligned} \tag{4}$$

as the drive system and Rossler hyperchaotic system

$$\begin{aligned} \dot{y}_1 &= b y_2 + c y_1 + u_1(t), \\ \dot{y}_2 &= 3 + y_2 y_3 + u_2(t), \\ \dot{y}_3 &= -y_2 - y_4 + u_3(t), \\ \dot{y}_4 &= y_1 + y_3 + a y_4 + u_4(t), \end{aligned} \tag{5}$$

as the response system. Our goal is to design the appropriate active controllers $u_i(t)$, $i = 1, 2, 3, 4$ such that the trajectory of the salve system (5) asymptotically approaches the master system (4) for a given functional relation and finally implements GS, in the sense that $\lim\limits_{t\to\infty} \|y - \varphi(x)\| = 0$. Define the error differences as

$$\begin{aligned} e_1 &= y_1 - \varphi_1, \quad e_2 = y_2 - \varphi_2, \\ e_3 &= y_3 - \varphi_3, \quad e_4 = y_4 - \varphi_4. \end{aligned} \tag{6}$$

For simplicity, consider a special GS–linear GS in which $\varphi(x) = (x_2, x_1 - 2x_3, 3x_1 - x_2, 3x_3)^{\mathrm{T}}$. Using the notation (6), we obtain the error dynamical system

$$\dot{e} = \dot{y} - \dot{\varphi} = \dot{y} - D\varphi(x) f(x), \tag{7}$$

where $D\varphi(x)$ is the Jacobian matrix of the map $\varphi(x)$, i.e.

$$D\varphi(x) = \begin{pmatrix} \dfrac{\partial\varphi_1(x)}{\partial x_1} & \dfrac{\partial\varphi_1(x)}{\partial x_2} & \dfrac{\partial\varphi_1(x)}{\partial x_3} & \dfrac{\partial\varphi_1(x)}{\partial x_4} \\ \dfrac{\partial\varphi_2(x)}{\partial x_1} & \dfrac{\partial\varphi_2(x)}{\partial x_2} & \dfrac{\partial\varphi_2(x)}{\partial x_3} & \dfrac{\partial\varphi_2(x)}{\partial x_4} \\ \dfrac{\partial\varphi_3(x)}{\partial x_1} & \dfrac{\partial\varphi_3(x)}{\partial x_2} & \dfrac{\partial\varphi_3(x)}{\partial x_3} & \dfrac{\partial\varphi_3(x)}{\partial x_4} \\ \dfrac{\partial\varphi_4(x)}{\partial x_1} & \dfrac{\partial\varphi_4(x)}{\partial x_2} & \dfrac{\partial\varphi_4(x)}{\partial x_3} & \dfrac{\partial\varphi_4(x)}{\partial x_4} \end{pmatrix}.$$

In the following, an active control, proposed by Bai and Lonngren,[7] is adopted to design the controllers. Define the active controller as

$$u(t) = v(t) + D\varphi(x) f(x), \tag{8}$$

where $v(t) = (v_1(t), v_2(t), v_3(t), v_4(t))^{\mathrm{T}}$. Hence

$$\begin{aligned} \dot{e}_1 &= c e_1 + b e_2 + b(x_1 - 2x_3) + c x_2 + v_1(t), \\ \dot{e}_2 &= 3 + y_2 y_3 + v_2(t), \\ \dot{e}_3 &= -e_2 - e_4 - x_1 - x_3 + v_3(t), \\ \dot{e}_4 &= e_1 + e_3 + a e_4 + 3x_1 + 3a x_3 + v_4(t). \end{aligned} \tag{9}$$

We re-define the control functions to eliminate all items which cannot be shown in the form of e_1, e_2, e_3 and e_4 as follows:

$$\begin{aligned} v_1 &= -b(x_1 - 2x_3) - c x_2 + w_1(t), \\ v_2 &= -3 - y_2 y_3 + w_2(t), \\ v_3 &= x_1 + x_3 + w_3(t), \\ v_4 &= -3x_1 - 3a x_3 + w_4(t). \end{aligned} \tag{10}$$

Substituting Eq.(10) in Eq.(9) we get

$$\begin{aligned} \dot{e}_1 &= c e_1 + b e_2 + w_1(t), \\ \dot{e}_2 &= w_2(t), \\ \dot{e}_3 &= -e_2 - e_4 + w_3(t), \\ \dot{e}_4 &= e_1 + e_3 + a e_4 + w_4(t). \end{aligned} \tag{11}$$

The error system (11) to be controlled is a linear system with a control input $w_1(t)$, $w_2(t)$, $w_3(t)$ and $w_4(t)$ as functions of e_1, e_2, e_3 and e_4. As long as these feedbacks stabilize the system, e_1, e_2, e_3 and e_4 will converge to zero as time tends to infinity. This implies that systems (4) and (5) are synchronized in a generalized differentiable transformation $y = \varphi(x)$. Using the active control method,[7] we choose

$$\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

where A is a constant matrix. There are many possible choices for the elements of the matrix A provided that all eigenvalues of A have a negative real part. As a result, the system (11) will be stable. For simplicity, we choose

$$A = \begin{pmatrix} -1-c & -b & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & -a-1 \end{pmatrix}, \quad (12)$$

so the eigenvalues of the system (11) are found to be -1, -1 and -1.

3.2. Numerical results

We select the parameters of Lorenz system as $\sigma = 10$, $\gamma = 28$, $\xi = 8/3$ and the parameters of Rossler hyperchaotic system as $a = 0.25$, $b = -0.5$, $c = 0.05$, so that both systems exhibit chaotic behaviour without the controller. The initial values of the drive and response systems are arbitrarily given by $(-5.5,\ 2.8,\ 3.0)$ and $(10.0,\ 7.9,\ -10.0,\ -6.0)$ respectively. The simulation results of the two systems are shown in Fig.1, where the control is activated after $t = 2$. As shown in Fig.1, after $t = 2$, the controller is then activated and we find that the signals (Fig.1) decay to zero after the transient time $t = 13$ as expected, which implies that the GS can be realized.

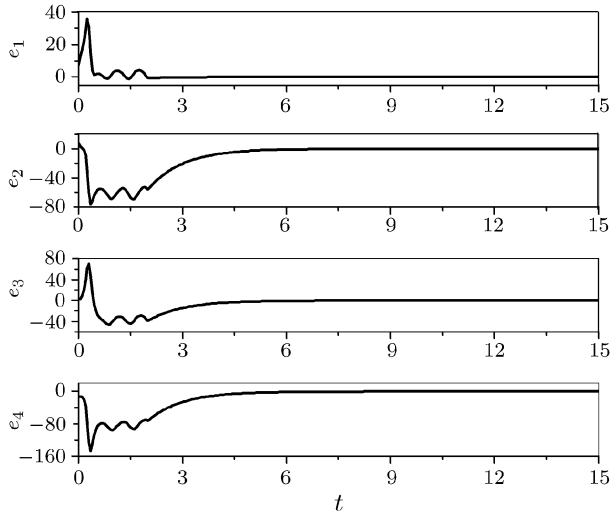


Fig.1. Difference signals: e_1, e_2, e_3 and e_4 for Lorenz system and Rossler hyperchaotic system with the active controller activated when $t = 2$.

3.3. GS between Lorenz system and Chen system

In 2002, Lü *et al*[11] presented a new unified chaotic system which can bridge the gap between the Lorenz and the Chen systems.[12] As another example, we consider the so-called unified chaotic system,[11] which is described by

$$\begin{aligned} \dot{y}_1 &= (25a+10)(y_2 - y_1) + u_1, \\ \dot{y}_2 &= (28-35a)y_1 - y_1 y_3 + (29a-1)y_2 + u_2, \\ \dot{y}_3 &= y_1 y_2 - \frac{a+8}{3} y_3 + u_3, \end{aligned} \quad (13)$$

where a is a parameter. When $0 < a < 0.8$, the system (13) belongs to the generalized Lorenz system; when $0.8 < a \leq 1.0$, the system (13) is the generalized Chen system;[12] if $a = 0.8$, system (13) is the generalized Lü one.[13]

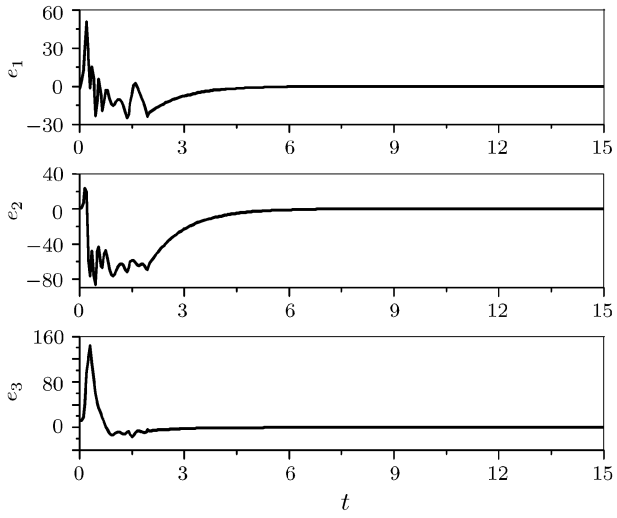


Fig.2. Synchronization error: e_1, e_2 and e_3 for Lorenz system and unified chaotic system with the active controller activated when $t = 2$, $a = 0.95$.

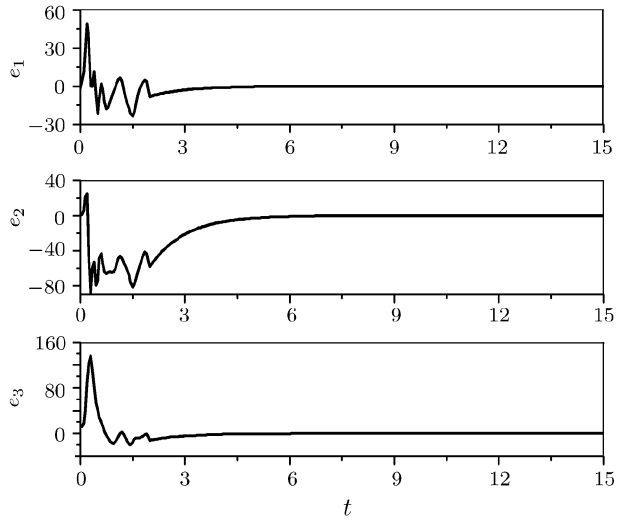


Fig.3. Synchronization error: e_1, e_2 and e_3 for Lorenz system and unified chaotic system with the active controller activated when $t = 2$, $a = 0.8$.

In order to get GS, the drive system (4) and response systems (13) are considered respectively. Now we give some numerical examples to illustrate our results. We choose $a = 0.95$ (which corresponds to the generalized Chen system) and $a = 0.8$ (which corresponds to the generalized Lü one) respectively. The initial values for the response system (13) are arbitrarily chosen by (1.0, 0.9, −1.0). The simulation results are shown in Figs.2 and 3 respectively, where the control is activated after $t = 2$. As shown in Figs.2 and 3, the error vector $\boldsymbol{e}(t)$ becomes zero as time tends to infinity. So the GS is realized.

4. Conclusion

In this paper, GS between two non-identical chaotic systems has been achieved. The central point of this paper is to construct a response system to synchronize with the drive system for a given functional relation by using active control. From the simulations, we can draw a conclusion that GS is achieved provided that the error system is asymptotically stable at the equilibrium $e = 0$, irrespective of whether they have multiple merged basins of attraction or not. The proposed scheme is efficient in achieving GS in our example and can be applied to other chaotic systems.

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