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Stochastic resonance in linear system driven by multiplicative noise and additive quadratic noise*

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In this paper the stochastic resonance (SR) is studied in an overdamped linear system driven by multiplicative noise and additive quadratic noise. The exact expressions are obtained for the first two moments and the correlation function by using linear response and the properties of the dichotomous noise. SR phenomenon exhibits in the linear system. There are three different forms of SR: the bona fide SR, the conventional SR and SR in the broad sense. Moreover, the effect of the asymmetry of the multiplicative noise on the signal-to-noise ratio (SNR) is different from that of the additive noise and the effect of multiplicative noise and additive noise on SNR is different.

Keywords: stochastic resonance, signal-to-noise ratio, dichotomous noise, additive quadratic noise
PACC: 0540, 0250

1. Introduction

Since Benzi *et al*,[1] and Nicolis and Nicolis[2] discovered the phenomenon which they termed stochastic resonance (SR), the phenomenon of SR has attracted enormous interest due to both its potential technological applications for optimizing the response to weak external signals in nonlinear dynamical systems and its connection with some biological mechanisms. A large number of applications are shown in science and technology, ranging from electronic circuits and lasers to chemical systems and the connection with some situations of biological interest.[3–6]

In the initial stage of investigation of SR, the nonlinearity system, noise, and periodic signal were thought of three essential ingredients for the onset of SR. Although an external periodic force can be replaced by some internal source of a periodic nature, nonlinearity seems to be the necessary ingredient for SR. There were many theoretical developments of SR in bistable systems.[7–17] In 1996, Berdichevsky and Gitterman[18] found that SR can occur in the linear system when the multiplicative noise is dichotomous, then SR in linear system is studied in Ref.[19–22].

In this paper, we consider a linear system driven by multiplicative noise and additive quadratic noise and the exact expressions are obtained for the first moments and the correlation function. We found three different forms of SR: the bona fide SR, the conventional SR and SR in the broad sense.

2. The SNR for a linear system

Consider an overdamped linear system described by the stochastic differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(a+\xi(t))x + A\cos(\Omega t) + \eta^2(t), \qquad (1)$$

where A and Ω denote the amplitude and the frequency, $\xi(t)$ and $\eta(t)$ are Gaussian noise with a vanishing mean and correlation functions described as

$$\begin{aligned} &\langle \xi(t)\xi(s)\rangle = \sigma_1 \exp[-\lambda|t-s|], \\ &\langle \eta(t)\eta(s)\rangle = \sigma_2 \exp[-\lambda|t-s|], \\ &\langle \xi(t)\eta(s)\rangle = \langle \eta(t)\xi(s)\rangle = 0. \end{aligned} \qquad (2)$$

In the following, we consider nonsymmetric dichotomous noise (random telegraphic process), which is allowed to take on the values A_1 and $-B_1$ for $\xi(t)$, and A_2 and $-B_2$ for $\eta(t)$, where $A_1, A_2, B_1, B_2 > 0$. The rate for the transition from A_1 to $-B_1$ will be denoted by α_1 and the reverse rate will be denoted by α_2. The rate for the transition from A_2 to $-B_2$ will be denoted by β_1 and the reverse rate will be denoted by β_2. We consider the special form of exponentially

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correlated noise (2), which has one of the two values, and its parameters are

$$\xi(t) \in \{A_1, -B_1\}, \quad \sigma_1 = A_1 B_1, \quad \lambda = \alpha_1 + \alpha_2,$$
$$\Lambda_1 = A_1 - B_1, \quad \eta(t) \in \{A_2, -B_2\}, \quad \sigma_2 = A_2 B_2,$$
$$\lambda = \beta_1 + \beta_2, \quad \Lambda_2 = A_2 - B_2. \tag{3}$$

In order to find the first two moments, let us average Eq.(1) and that multiplied by $2x$, which gives

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = -a\langle x\rangle - \langle \xi(t)x\rangle + A\cos(\Omega t) + \sigma_2, \tag{4}$$
$$\frac{\mathrm{d}\langle x^2\rangle}{\mathrm{d}t} = -2a\langle x^2\rangle - 2\langle \xi(t)x^2\rangle + 2A\cos(\Omega t)\langle x\rangle + 2\langle \eta^2(t)x\rangle. \tag{5}$$

Equation (4) contains the new correlator $\langle \xi(t)x\rangle$. Using the Shapiro–Loginov formula for differentiation of exponentially correlated random functions,[23] we obtain

$$\frac{\mathrm{d}\langle \xi(t)x\rangle}{\mathrm{d}t} = \left\langle \xi(t)\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle - \lambda\langle \xi(t)x\rangle. \tag{6}$$

Multiplying Eq.(1) by $\xi(t)$, averaging and using Eq.(6), we have

$$\frac{\mathrm{d}\langle \xi(t)x\rangle}{\mathrm{d}t} = -(a+\lambda)\langle \xi(t)x\rangle - \langle \xi^2(t)x\rangle. \tag{7}$$

Equation (7) contains the higher-order correlator $\langle \xi^2(t)x\rangle$. Using the property (3) of dichotomous noise, we can denote the higher-order correlator $\langle \xi^2(t)x\rangle$ by

$$\langle \xi^2(t)x\rangle = \sigma_1\langle x\rangle + \Lambda_1\langle \xi(t)x\rangle. \tag{8}$$

Inserting Eq.(8) into Eq.(7) we transform the latter equation together with Eq.(4) into the closed system of two equations for two unknown functions $\langle x\rangle$ and $\langle \xi(t)x\rangle$. We obtain the asymptotic value at $t \to \infty$ of the first moment, which is

$$\langle x\rangle = A\frac{f_1\cos(\Omega t) + f_2\sin(\Omega t)}{f_3} + f_4, \tag{9}$$

where

$$f_1 = a\Omega^2 + (a + \Lambda_1 + \lambda)b_1 b_2, \tag{10}$$
$$f_2 = \Omega(\Omega^2 + (a + \Lambda_1 + \lambda)^2 + \sigma_1), \tag{11}$$
$$f_3 = (\Omega^2 + b_1^2)(\Omega^2 + b_2^2), \tag{12}$$
$$f_4 = \frac{\sigma_2}{b_1 b_2}(a + \Lambda_1 + \lambda), \tag{13}$$
$$b_{1,2} = a + \varepsilon_{1,2} = a + \frac{\lambda + \Lambda_1}{2} \pm \sqrt{\frac{(\lambda + \Lambda_1)^2}{4} + \sigma_1}. \tag{14}$$

We can obtain four differential equations about $\langle x^2\rangle$, $\langle \xi(t)x^2\rangle$, $\langle \eta(t)x\rangle$ and $\langle \xi(t)\eta(t)x\rangle$ in a similar manner. Solving the differential equations and omitting the tedious solution of these equations, we have the stationary $t \to \infty$ value of the second moment

$$\langle x^2\rangle_{\mathrm{st}} = \Big\{\frac{\sigma_2\Lambda_2^2}{(a+\lambda)(a+\Lambda_1+2\lambda) - \sigma_1} \times [2(a+\Lambda_1+\lambda)^2 + (a+\Lambda_1)\lambda + 2\sigma_1] + \frac{A^2}{2f_3}[f_1(2a+2\Lambda_1+\lambda) - 2\sigma_1(\Omega^2 - b_1 b_2)] + \sigma_2\Big[f_4(2a+2\Lambda_1+\lambda) + \frac{2\sigma_1\sigma_2}{b_1 b_2}\Big]\Big\} \times [a(2a+2\Lambda_1+\lambda) - 2\sigma_1]^{-1}. \tag{15}$$

In order to get the stationary correlation function $\langle x(t+\tau)x(t)\rangle$, we solve Eq.(1) and obtain

$$x(t+\tau) = x(t)g(\tau)\exp(-a\tau) + A\int_0^\tau \exp(-av)g(v)\cos[\Omega(t+\tau-v)]\mathrm{d}v + \int_0^\tau \exp(-av)h(v)\mathrm{d}v, \tag{16}$$

where

$$g(v) = \Big\langle \exp\Big(-\int_0^v \xi(u)\mathrm{d}u\Big)\Big\rangle,$$
$$h(t-v) = \Big\langle \eta^2(v)\exp\Big(-\int_v^t \xi(u)\mathrm{d}u\Big)\Big\rangle. \tag{17}$$

We expand $g(v)$ and $h(v)$ into exponential in series and calculate, then we get[24]

$$g(v) = \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\exp(-\varepsilon_2 v) - \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\exp(-\varepsilon_1 v),$$
$$h(t-v) = \sigma_2 g(t-v). \tag{18}$$

Multiplying Eq.(16) by $x(t)$ and averaging, we obtain the asymptotic value of the correlation function

$$\langle x(t+\tau)x(t)\rangle = \langle x^2\rangle_{\mathrm{st}}g(\tau)\exp(-a\tau) + \frac{\langle x\rangle A}{\varepsilon_1 - \varepsilon_2}[f_5\sin(\Omega t) + f_6\cos(\Omega t)] + \langle x\rangle\int_0^\tau \exp(-av)h(v)\mathrm{d}v, \tag{19}$$

where

$$f_5 = \frac{\varepsilon_2 b_1\sin(\Omega\tau) - \varepsilon_2\Omega f_7}{b_1^2 + \Omega^2} + \frac{\varepsilon_1\Omega f_8 - \varepsilon_1 b_2\sin(\Omega\tau)}{b_2^2 + \Omega^2}, \tag{20}$$
$$f_6 = -\frac{\varepsilon_2\Omega\sin(\Omega\tau) + \varepsilon_2 b_1 f_7}{b_1^2 + \Omega^2}$$

$$+\frac{\varepsilon_1 b_2 f_8+\varepsilon_1\Omega\sin(\Omega\tau)}{b_2^2+\Omega^2}, \tag{21}$$

$$f_{7,8}=\cos(\Omega\tau)-\exp(-b_{1,2}\tau). \tag{22}$$

Averaging Eq.(19) over the period of the external signal $2\pi/\Omega$, we get

$$\begin{aligned}\langle x(t+\tau)x(t)\rangle_{\rm st}&=\langle x^2\rangle_{\rm st}g(\tau)\exp(-a\tau)\\&\quad+\frac{A^2}{\varepsilon_1-\varepsilon_2}\frac{f_1f_6+f_2f_5}{2f_3}\\&\quad+f_4\int_0^\tau\exp(-av)h(v)\mathrm{d}v.\end{aligned} \tag{23}$$

Translate Eq.(23) into the power spectrum $S(w)$ by Fourier transform

$$\begin{aligned}S(w)&=\int_{-\infty}^{+\infty}\langle x(t+\tau)x(t)\rangle_{\rm st}\exp(-\mathrm{i}w\tau)\mathrm{d}\tau\\&=S_0\delta(w)+S_1(w)+S_2(w)\delta(w-\Omega),\end{aligned} \tag{24}$$

and

$$S_0=\frac{2\pi f_4\sigma_2}{(\varepsilon_1-\varepsilon_2)b_1b_2}(\varepsilon_1b_1-\varepsilon_2b_2), \tag{25}$$

$$\begin{aligned}S_1(w)&=\frac{2\langle x^2\rangle_{\rm st}}{\varepsilon_1-\varepsilon_2}(b_2l_1-b_1l_2)+\frac{2f_4\sigma_2}{\varepsilon_1-\varepsilon_2}(l_2-l_1)\\&\quad+\frac{A^2}{f_3(\varepsilon_1-\varepsilon_2)}\Big[\frac{b_1l_2u_1}{b_1^2+w^2}-\frac{b_2l_1u_2}{b_2^2+w^2}\Big],\end{aligned} \tag{26}$$

$$S_2(w)=\frac{\pi A^2}{2f_3(\varepsilon_1-\varepsilon_2)}(u_2l_1-u_1l_2), \tag{27}$$

$$l_{1,2}=(b_{1,2}-a)/(b_{2,1}^2+w^2),\quad u_{1,2}=f_2w+f_1b_{1,2}, \tag{28}$$

where S_0 is power spectrum at zero frequency, $S_1(w)$ and $S_2(w)$ are the output power spectrum of noise and signal, respectively. Then, the signal-to-noise ratio (SNR) is

$$R=\frac{\int_0^{+\infty}S_2(w)\delta(w-\Omega)\mathrm{d}w}{S_1(w=\Omega)}=\frac{\pi A^2(u_2l_1-u_1l_2)}{4\langle x^2\rangle_{\rm st}f_3(b_2l_1-b_1l_2)+4\sigma_2f_4(l_2-l_1)+2A^2\Big(\dfrac{b_1l_2u_1}{b_1^2+w^2}-\dfrac{b_2l_1u_2}{b_2^2+w^2}\Big)}. \tag{29}$$

3. Discussion and conclusions

According to Eq.(29), we will discuss the dependence of the SNR upon frequency, amplitude and different parameters of noise.

The curve of the SNR versus the external frequency w for different values of the amplitude A is given in Fig.1. As shown in the figure, there is a single-peak in the curve of SNR with increasing w and the SR phenomenon appears, which is a bona fide resonance phenomenon. The SNR increases with increasing A.

In Fig.2, the SNR is shown as a function of the external frequency w for different values of the asymmetry of the multiplicative noise Λ_1. As shown in Fig.2, the SNR increases with increasing w, then decreases with increasing w. The curve of SNR appears a single-peak. SR exists for this case and it is a bona fide SR. At the same time, the SNR increases with increasing the asymmetry of the multiplicative noise Λ_1.

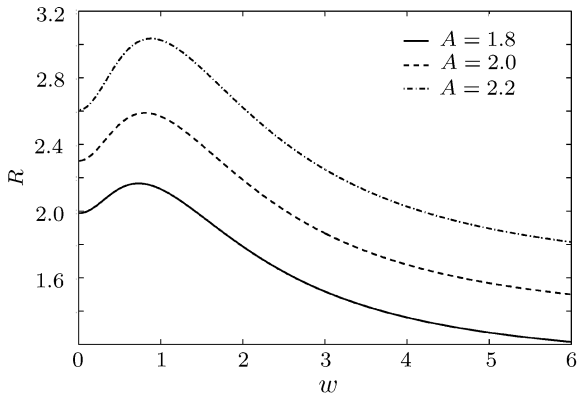


Fig.1. SNR as a function of the frequency w for $a=1$, $\lambda=1$, $\sigma_1=\sigma_2=1$, $\Lambda_1=\Lambda_2=1$ with varied A.

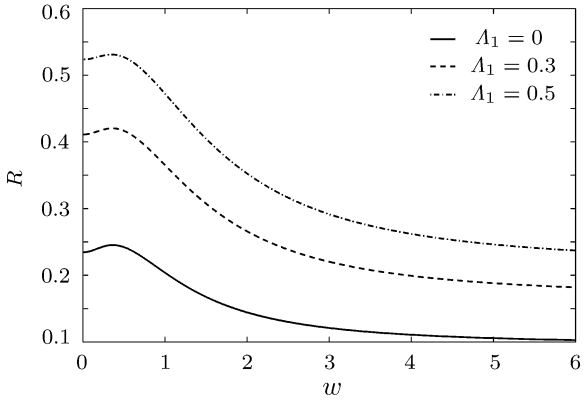


Fig.2. SNR as a function of the frequency w for $a=1$, $\lambda=1$, $\sigma_1=\sigma_2=1$, $A=\Lambda_2=1$ with varied Λ_1.

In Fig.3, we plot the curve of SNR R as a function of the external frequency w for different values of the asymmetry of the additive noise Λ_2. As shown in the figure, the SNR also behaves non-monotonically as the frequency w varies. The SNR decreases as increasing the asymmetry of the additive noise Λ_2, whereas the SNR increases as increasing the asymmetry of the multiplicative noise Λ_1, as shown in Fig.2. In this case, the bona fide SR appears as shown Figs.1 and 2.

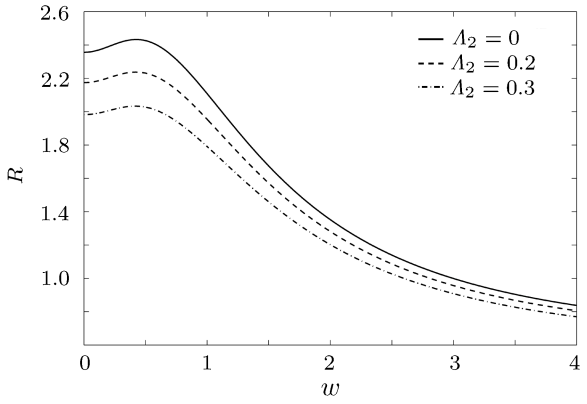


Fig.3. SNR as a function of the frequency w for $a = 1$, $\lambda = 1$, $\sigma_1 = \sigma_2 = 1$, $\Lambda_1 = A = 1$ with varied Λ_2.

Figure 4 is a plot of SNR as a function of the multiplicative noise intensity σ_1 for different values of the asymmetry of the multiplicative noise Λ_1. The SNR decreases with increasing σ_1 first, then increases with increasing σ_1, and then decreases with increasing σ_1 again. The curve of SNR exhibits a maximum and a minimum. The conventional SR appears in the case and the position of the peak shifts to the right with the increasing Λ_1.

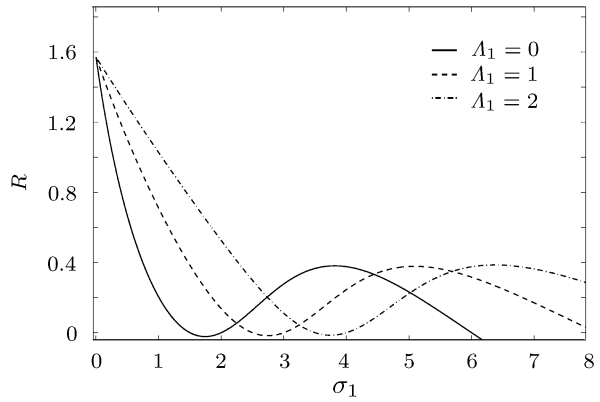


Fig.4. SNR as a function of the multiplicative noise intensity σ_1 for $a = 1$, $\lambda = 1$, $w = \sigma_2 = 1$, $\Lambda_2 = A = 1$ with varied Λ_1.

The curve of SNR R as a function of the correlation time $\tau = \lambda^{-1}$ for different values of the multiplicative noise intensity σ_1 is shown in Fig.5. There is a resonant peak appearing in the curve. If the non-monotonic behaviour of the SNR as a function of some characteristics of the noise is considered as the SR in a broad sense, then SR in a broad sense occurs for the case. The SNR is a non-monotonic function of the multiplicative noise intensity σ_1, as shown in Fig.4. The maximum of the peak moves to the right with increasing σ_1.

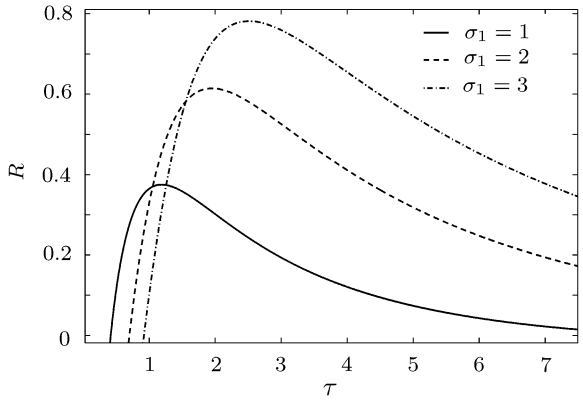


Fig.5. SNR as a function of the correlation time $\tau = \lambda^{-1}$ for $a = 0.1$, $w = \sigma_2 = 1$, $A = \Lambda_1 = \Lambda_2 = 1$ with varied σ_1.

In Fig.6, we plot the curve of SNR R as a function of the correlation time $\tau = \lambda^{-1}$ for different values of the additive noise intensity σ_2. As shown in Fig.6, the SNR is a non-monotonic function of the correlation time τ and SR in a broad sense occurs. The maximum of the peak decreases with increasing σ_2. By comparing Fig.5 with Fig.6, we can see that the effects of the additive noise intensity σ_2 on the SNR is different from that of the multiplicative noise intensity σ_1, since the latter maximum of the peak increases with increasing σ_1.

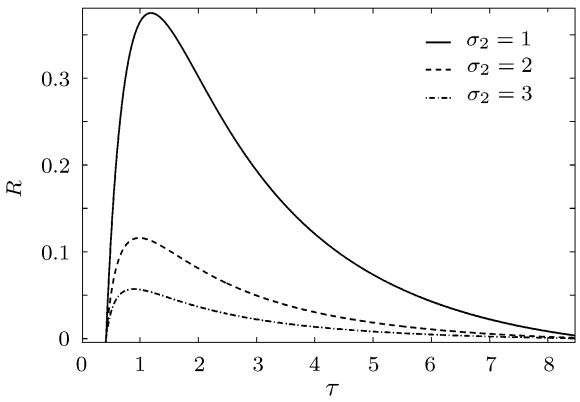


Fig.6. SNR as a function of the correlation time $\tau = \lambda^{-1}$ for $a = 0.1$, $w = \sigma_1 = 1$, $A = \Lambda_1 = \Lambda_2 = 1$ with varied σ_2.

In conclusion, the bona fide SR, the conventional SR and SR in the broad sense all appeared in the linear system driven by multiplicative noise and additive quadratic noise. The effect of the asymmetry of the multiplicative noise on the SNR is different from that of the additive noise and the effect of multiplicative noise and additive noise on SNR is different.

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