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Density matrix of two interacting particles with kinetic coupling derived in bipartite entangled state representation*

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A density matrix is usually obtained by solving the Bloch equation, however only a few Hamiltonians' density matrices can be analytically derived. The density matrix for two interacting particles with kinetic coupling is hard to derive by the usual method due to this coupling; this paper solves this problem by using the bipartite entangled state representation.

Keywords: density matrix, the bipartite entangled state representation, kinetic coupling
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1. Introduction

Calculation of density matrix $\rho = \mathrm{e}^{-\beta H}$ by solving the Bloch equation[1,2] is one of the major topics in quantum statistics,[3−6] however only a few Hamiltonians' $\mathrm{e}^{-\beta H}$ can be analytically derived. For example, Feynman[3] presented the density matrix in the coordinate representation for a free particle with the Hamiltonian $H = p^2/2m$,

$$\rho_1(x, x'; \beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \exp\Big[-\frac{m}{2\hbar^2\beta}(x - x')^2\Big], \quad (1)$$

and for a harmonic oscillator Feynman derived,[3]

$$\rho_2(x, x'; \beta) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega/kT)}} \times \exp\Big\{\frac{-m\omega}{2\hbar \sinh(\hbar\omega/kT)} \times \Big[(x^2 + x'^2)\cosh\Big(\frac{\hbar\omega}{kT}\Big) - 2xx'\Big]\Big\}. \quad (2)$$

Recently some other methods are presented to calculate the density matrix for some special physical system: e.g., in Ref.[7] the density matrix for an electron confined in a quantum dot under uniform magnetic field and static electric field is calculated. In this paper we aim at calculating the density matrix for two interacting particles with kinetic coupling except for the potential $V(x_1 - x_2)$, the total Hamiltonian is

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + kP_1P_2 + V(x_1 - x_2), \quad (3)$$

where V is the function of relative position $x_1 - x_2$, kP_1P_2 is named as kinetic coupling, such kind of interacting term often appears as describing internal potential in molecular physics theory, the mutual magnetic inductance between two quantized coupling circuits, etc. Due to the existence of kP_1P_2, the corresponding density matrix seems hard to be derived. The difficulty lies in that we can not separate the motion of centre-of-mass off. However, we find that when working in the bipartite entangled state representation $\langle\xi|$, then we can fulfil this task easily.

In this work we intend to apply the bipartite entangled state representation $|\xi\rangle$ to study density matrix. The work is arranged as follows. In Section 2 we briefly introduce the $\langle\xi|$ state, in Section 3 we deduce the Schrödinger equation of H in $\langle\xi|$ representation and in Section 4 we solve it and obtain the eigenfunctions of H, in Section 5 we calculate the density matrix for H. In doing so a complete theory is set up for solving the density matrix of two interacting particles with kinetic coupling.

2. The properties of $|\xi\rangle$

In recent years quantum entanglement and entangled states[8−14] have been paid much attention by physicists due to their wide use in quantum information and quantum communication. The quantum en-

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tanglement involved in a bipartite state of continuum variables can be traced back to the pioneering work of Einstein–Podolsky–Rosen (EPR)[15] in which they noticed that two particles' relative coordinate operator $X_1 - X_2$ commutes with the total momentum operator $P_1 + P_2$. On the basis of EPR's analysis the bipartite entangled state of continuum variables, denoted as $|\eta\rangle$, is constructed which is the common eigenvector of $(X_1 - X_2)$ and $(P_1 + P_2)$.[16] More and more entangled states are constructed by Fan, such as the bipartite entangled state $|\zeta\rangle$ which is the common eigenvector of $(X_1 + X_2)$ and $(P_1 - P_2)$,[17] and the mass-dependent entangled state $|\xi\rangle$[18] which is the common eigenvector of two particles' centre-of-mass coordinate $Q_{\rm cm}$,

$$Q_{\rm cm} = \mu_1 x_1 + \mu_2 x_2, \tag{4}$$

and the mass-weighted relative momentum $P_{\rm r}$,

$$P_{\rm r} = \mu_2 P_1 - \mu_1 P_2, \tag{5}$$

where $\mu_i = m_i/M$ $(i = 1, 2)$, $M = m_1 + m_2$ is the total mass.

In two-mode Fock space the mass-dependent entangled state $|\xi\rangle$ is defined as[18]

$$\begin{aligned}|\xi\rangle = \exp\Big\{&-\frac{1}{2}|\xi|^2 + \frac{1}{\sqrt{\lambda}}[\xi + (\mu_1 - \mu_2)\xi^*]a_1^\dagger \\ &+ \frac{1}{\sqrt{\lambda}}[\xi^* - (\mu_1 - \mu_2)\xi]a_2^\dagger \\ &+ \frac{1}{\lambda}[(\mu_2 - \mu_1)(a_1^{\dagger 2} - a_2^{\dagger 2}) - 4\mu_1\mu_2 a_1^\dagger a_2^\dagger]\Big\}|00\rangle, \end{aligned} \tag{6}$$

where $\xi = \xi_1 + {\rm i}\xi_2$, $\lambda \equiv 2(\mu_1^2 + \mu_2^2)$. Operating a_1 and a_2 on $|\xi\rangle$ state respectively, we have the following equations:

$$\begin{aligned}a_1|\xi\rangle = \Big[&\frac{2}{\sqrt{\lambda}}(\mu_1\xi_1 + {\rm i}\mu_2\xi_2) - 4\frac{\mu_1\mu_2}{\lambda}a_2^\dagger \\ &+ \frac{2}{\lambda}(\mu_2 - \mu_1)a_1^\dagger\Big]|\xi\rangle, \end{aligned} \tag{7}$$

$$\begin{aligned}a_2|\xi\rangle = \Big[&\frac{2}{\sqrt{\lambda}}(\mu_2\xi_1 - {\rm i}\mu_1\xi_2) - 4\frac{\mu_1\mu_2}{\lambda}a_1^\dagger \\ &- \frac{2}{\lambda}(\mu_2 - \mu_1)a_2^\dagger\Big]|\xi\rangle. \end{aligned} \tag{8}$$

Expressing

$$Q_{\rm cm} = \frac{1}{\sqrt{2}}[\mu_1(a_1 + a_1^\dagger) + \mu_2(a_2 + a_2^\dagger)], \tag{9}$$

$$P_{\rm r} = \frac{\rm i}{\sqrt{2}}[\mu_1(a_2 - a_2^\dagger) - \mu_2(a_1 - a_1^\dagger)], \tag{10}$$

obviously, we notice that there is the commutation relation between $Q_{\rm cm}$ and $P_{\rm r}$: $[Q_{\rm cm}, P_{\rm r}] = 0$. By using Eqs.(7) and (8), we see that $|\xi\rangle$ obeys the following eigenvector equations:

$$Q_{\rm cm}|\xi\rangle = \sqrt{\frac{\lambda}{2}}\,\xi_1|\xi\rangle = \sqrt{\mu_1^2 + \mu_2^2}\,\xi_1|\xi\rangle, \tag{11}$$

$$P_{\rm r}|\xi\rangle = \sqrt{\frac{\lambda}{2}}\,\xi_2|\xi\rangle = \sqrt{\mu_1^2 + \mu_2^2}\,\xi_2|\xi\rangle. \tag{12}$$

Notice that ξ_2 corresponds to the momentum operator $P_{\rm r}$. By using the technique of integral within an ordered products, we have verified the completeness relation of the entangled state $|\xi\rangle$,

$$\begin{aligned}\int \frac{{\rm d}^2\xi}{\pi}|\xi\rangle\langle\xi| = \int \frac{{\rm d}^2\xi}{\pi}:\exp\Big\{&-|\xi|^2 + \frac{\xi}{\sqrt{\lambda}}[(\mu_1 + \mu_2) \\ &\times(a_1^\dagger + a_2) + (\mu_1 - \mu_2)(a_1 - a_2^\dagger)] \\ &+ \frac{\xi^*}{\sqrt{\lambda}}[(\mu_1 - \mu_2)(a_1^\dagger - a_2) + (\mu_1 + \mu_2)(a_1 + a_2^\dagger)] \\ &+ \frac{1}{\lambda}(\mu_2 - \mu_1)(a_1^{\dagger 2} - a_2^{\dagger 2} + a_1^2 - a_2^2) \\ &- a_1^\dagger a_1 - a_2^\dagger a_2 - \frac{4}{\lambda}\mu_1\mu_2(a_1^\dagger a_2^\dagger + a_1 a_2)\Big\}: \;= 1, \end{aligned} \tag{13}$$

and the orthogonality,

$$\langle\xi'|\xi\rangle = \pi\delta^{(2)}(\xi' - \xi). \tag{14}$$

3. The Schrödinger equation of H in $\langle\xi|$ representation

Noting

$$\begin{aligned}&x_{\rm r} = x_1 - x_2, \quad P_1 + P_2 = P, \quad [x_{\rm r}, P] = 0, \\ &P_1 = P_{\rm r} + \mu_1 P, \quad P_2 = \mu_2 P - P_{\rm r}, \\ &x_1 = Q_{\rm cm} + \mu_2 x_{\rm r}, \quad x_2 = Q_{\rm cm} - \mu_1 x_{\rm r}, \end{aligned} \tag{15}$$

$$[Q_{\rm cm}, P_{\rm r}] = 0, \quad [x_{\rm r}, P_{\rm r}] = {\rm i}, \quad [P, Q_{\rm cm}] = -{\rm i}, \tag{16}$$

we can rewrite H as

$$\begin{aligned}H = &\Big(\frac{1}{2M} + k\mu_1\mu_2\Big)P^2 + \Big(\frac{1}{2\mu} - k\Big)P_{\rm r}^2 \\ &+ k(\mu_2 - \mu_1)PP_{\rm r} + V(x_{\rm r}), \end{aligned} \tag{17}$$

where $\mu = m_1 m_2/M$. The existence of $PP_{\rm r}$ hinders the separation of total momentum term from H, so we appeal to use the $\langle\xi|$ representation by noting

$$\langle\xi|P = \sqrt{\frac{2}{\lambda}}\Big[-{\rm i}\frac{\partial}{\partial\xi_1} - (\mu_1 - \mu_2)\xi_2\Big]\langle\xi|, \tag{18}$$

$$\langle\xi|x_{\rm r} = \sqrt{\frac{2}{\lambda}}\Big[{\rm i}\frac{\partial}{\partial\xi_2} + (\mu_1 - \mu_2)\xi_1\Big]\langle\xi|. \tag{19}$$

Because the total momentum is conservative, $[P, H] = 0$, we assume that their common eigenvector is $|p, E_n\rangle$, i.e.

$$H|p, E_n\rangle = E_n|p, E_n\rangle, \quad P|p, E_n\rangle = p|p, E_n\rangle. \quad (20)$$

Next we employ the mass-dependent bipartite entangled state representation to calculate it. Considering Eqs.(12) and (17) we obtain the following equations:

$$\begin{aligned} E_n\langle\xi|p, E_n\rangle = &\Big[\Big(\frac{1}{2M}+k\mu_1\mu_2\Big)p^2+k(\mu_2-\mu_1)\sqrt{\frac{\lambda}{2}}\xi_2 p\Big] \\ &\times\langle\xi|p, E_n\rangle + \Big(\frac{1}{2\mu}-k\Big)\frac{\lambda}{2}\xi_2^2\langle\xi|p, E_n\rangle \\ &+\langle\xi|V(x_{\rm r})p, E_n\rangle, \end{aligned} \quad (21)$$

$$p\langle\xi|p, E_n\rangle = \langle\xi|P|p, E_n\rangle. \quad (22)$$

Using Eqs.(18) and (19) and combining Eq.(21) with Eq.(22), we obtain

$$\begin{aligned} &\Big\{V\Big[{\rm i}\sqrt{\frac{2}{\lambda}}\Big(\frac{\partial}{\partial\xi_2}-{\rm i}(\mu_1-\mu_2)\xi_1\Big)\Big] \\ &+\Big(\frac{1}{2M}+k\mu_1\mu_2\Big)p^2+k(\mu_2-\mu_1)\sqrt{\frac{\lambda}{2}}\xi_2 p \\ &+\Big(\frac{1}{2\mu}-k\Big)\frac{\lambda}{2}\xi_2^2-E_n\Big\}\langle\xi|p, E_n\rangle = 0, \end{aligned} \quad (23)$$

and

$$\Big\{-{\rm i}\sqrt{\frac{2}{\lambda}}\Big[\frac{\partial}{\partial\xi_1}-{\rm i}(\mu_1-\mu_2)\xi_2\Big]-p\Big\}\langle\xi|p, E_n\rangle = 0. \quad (24)$$

4. Eigenfunctions of H

Making the ansatz that the wavefunction $\langle\xi|p, E_n\rangle$ is

$$\langle\xi|p, E_n\rangle = \Psi_n(\xi){\rm exp}[{\rm i}(\mu_1-\mu_2)\xi_1\xi_2], \quad (25)$$

and substituting Eq.(25) into Eq.(23), we obtain

$$\begin{aligned} &\Big\{V\Big({\rm i}\sqrt{\frac{2}{\lambda}}\frac{\partial}{\partial\xi_2}\Big)+\Big(\frac{1}{2\mu}-k\Big)\frac{\lambda}{2}(\xi_2-\xi_0)^2 \\ &+T-E_n\Big\}\Psi_n(\xi) = 0, \end{aligned} \quad (26)$$

where

$$\begin{aligned} &\xi_0 = \frac{\sqrt{2}\mu k(\mu_1-\mu_2)}{(1-2\mu k)\sqrt{\lambda}}p = bp, \\ &b \equiv \frac{\sqrt{2}\mu k(\mu_1-\mu_2)}{(1-2\mu k)\sqrt{\lambda}}, \\ &T = \Big(\frac{1}{2M}+k\mu_1\mu_2\Big)p^2-\Big(\frac{1}{2\mu}-k\Big)\frac{\lambda}{2}\xi_0^2 = sp^2, \\ &s \equiv \frac{1-\mu Mk^2}{1-2\mu k}\frac{1}{2M}. \end{aligned} \quad (27)$$

Substituting Eq.(25) into Eq.(24), we obtain

$$\Big(-{\rm i}\sqrt{\frac{2}{\lambda}}\frac{\partial}{\partial\xi_1}-p\Big)\Psi_n(\xi) = 0, \quad (29)$$

its solution is

$$\Psi_n(\xi) = {\rm exp}\Big\{{\rm i}\sqrt{\frac{\lambda}{2}}p\xi_1\Big\}\chi_n(\xi_2), \quad (30)$$

where χ_n is irrelative to ξ_1. Substituting Eq.(30) into Eq.(26), and setting $V(x) = (\mu/2)\omega^2x^2$, a harmonic oscillator potential, we obtain the differential equation for χ_n,

$$\begin{aligned} &\Big\{-\frac{1}{\lambda}\mu\omega^2\frac{\partial^2}{\partial\xi_2^2}+\frac{\lambda}{2}\Big(\frac{1}{2\mu}-k\Big)(\xi_2-\xi_0)^2 \\ &+T-E_n\Big\}\chi_n(\xi_2) = 0. \end{aligned} \quad (31)$$

Comparing Eq.(31) with the standard Hermite equation (its solution is Hermite polynomials H_n) we obtain the solution of $\chi_n(\xi_2)$,

$$\chi_n(\xi_2) = N_n{\rm exp}\Big\{-\frac{g}{2}(\xi_2-\xi_0)^2\Big\}H_n\{\sqrt{g}(\xi_2-\xi_0)\Big\}, \quad (32)$$

here N_n is the normalization coefficient,

$$\begin{aligned} 1 = &\int {\rm d}\xi_2\chi_n^2(\xi_2) \\ = &\frac{1}{\sqrt{g}}N_n^2\int {\rm d}(\sqrt{g}\xi_2){\rm exp}\{-g(\xi_2-\xi_0)^2\} \\ &\times H_n^2\{\sqrt{g}(\xi_2-\xi_0)\} = \frac{1}{\sqrt{g}}N_n^2\sqrt{\pi}2^n n! \end{aligned}$$

so

$$N_n = \sqrt[4]{g}(\sqrt{\pi}2^n n!)^{-1/2}, \quad g \equiv \frac{\lambda\sqrt{1-2\mu k}}{2\mu\omega}. \quad (33)$$

From Eq.(31), we can also obtain the energy of the system,

$$E_n = \Big(n+\frac{1}{2}\Big)\sqrt{1-2\mu k}\omega + T. \quad (34)$$

Substituting Eq.(30) and (32) into Eq.(25), we obtain the eigenfunction of H in $\langle\xi|$ representation.

$$\langle\xi|p, E_n\rangle = {\rm exp}\Big[{\rm i}(\mu_1-\mu_2)\xi_1\xi_2+{\rm i}\sqrt{\frac{\lambda}{2}}p\xi_1\Big]\chi_n(\xi_2). \quad (35)$$

Equation (35) will be used in calculating the density matrix for H.

5. Density matrix for H in $|\xi\rangle$ representation

Now let's calculate the density matrix for H in $|\xi\rangle$ representation. We know that the density matrix for a system in $|\xi\rangle$ representation is defined as

$$\rho(\xi',\xi)=\langle\xi'|\mathrm{e}^{-\beta H}|\xi\rangle=\int_{-\infty}^{\infty}\mathrm{d}p\sum_{n=0}^{\infty}\langle\xi'|\mathrm{e}^{-\beta H}|p,E_n\rangle\langle p,E_n|\xi\rangle, \tag{36}$$

where we have inserted the completeness relation $\int_{-\infty}^{\infty}\mathrm{d}p\sum\limits_{n=0}^{\infty}|p,E_n\rangle\langle p,E_n|=1$. Using the eigenvector equation (20) and the eigenfunction (35), we have

$$\begin{aligned}\rho(\xi',\xi)=&\int_{-\infty}^{\infty}\mathrm{d}p\sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_n}\exp\Big[\mathrm{i}(\mu_1-\mu_2)\xi_1'\xi_2'+\mathrm{i}\sqrt{\frac{\lambda}{2}}p\xi_1'\Big]\chi_n(\xi_2')\\&\times\exp\Big[-\mathrm{i}(\mu_1-\mu_2)\xi_1\xi_2-\mathrm{i}\sqrt{\frac{\lambda}{2}}p\xi_1\Big]\chi_n^*(\xi_2).\end{aligned} \tag{37}$$

Substituting the expressions of E_n, $\chi_n(\xi_2)$ and N_n into Eq.(37), we obtain

$$\begin{aligned}\rho(\xi',\xi)=&|N_n|^2\int_{-\infty}^{\infty}\mathrm{d}p\sum_{n=0}^{\infty}\mathrm{e}^{-\beta(n+1/2)\sqrt{1-2\mu k}\omega}\mathrm{e}^{-\beta sp^2}\mathrm{e}^{\mathrm{i}\sqrt{\lambda/2}\,p(\xi_1'-\xi_1)}\mathrm{e}^{\mathrm{i}(\mu_1-\mu_2)(\xi_1'\xi_2'-\xi_1\xi_2)}\\&\times H_n[\sqrt{g}(\xi_2'-\xi_0)]H_n[\sqrt{g}(\xi_2-\xi_0)]\exp\Big\{-\frac{g}{2}[(\xi_2'-\xi_0)^2+(\xi_2-\xi_0)^2]\Big\}\\=&A\int_{-\infty}^{\infty}\mathrm{d}p\mathrm{e}^{-\beta sp^2}\mathrm{e}^{\mathrm{i}\sqrt{\lambda/2}\,p(\xi_1'-\xi_1)}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\beta\sqrt{1-2\mu k}\omega n}}{2^n n!}H_n[\sqrt{g}(\xi_2'-\xi_0)]\\&\times H_n[\sqrt{g}(\xi_2-\xi_0)]\mathrm{e}^{-(g/2)[(\xi_2'-\xi_0)^2+(\xi_2-\xi_0)^2]},\end{aligned} \tag{38}$$

where

$$A=\sqrt{\frac{g}{\pi}}\mathrm{e}^{-(\beta/2)\sqrt{1-2\mu k}\omega}\mathrm{e}^{\mathrm{i}(\mu_1-\mu_2)(\xi_1'\xi_2'-\xi_1\xi_2)}. \tag{39}$$

Then using the generating functions of the Hermite polynomial[19]

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{2}z)^n}{n!}H_n(x)H_n(y)=(1-z^2)^{-1/2}\exp\Big\{\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\Big\}, \tag{40}$$

and letting

$$\beta\sqrt{1-2\mu k}\omega\equiv 2f, \tag{41}$$

we can express

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\beta\sqrt{1-2\mu k}\omega n}}{2^n n!}H_n[\sqrt{g}(\xi_2'-\xi_0)]H_n[\sqrt{g}(\xi_2-\xi_0)]\\&=(1-\mathrm{e}^{-4f})^{-1/2}\exp\Big\{g(\xi_2'-\xi_0)(\xi_2-\xi_0)\mathrm{csch}(2f)-\frac{g}{2}[(\xi_2'-\xi_0)^2+(\xi_2-\xi_0)^2]\frac{\mathrm{e}^{-2f}}{\sinh(2f)}\Big\}.\end{aligned} \tag{42}$$

Substituting it into Eq.(38) and noting $\xi_0=bp$ (see Eq.(27)) we have

$$\begin{aligned}\rho(\xi',\xi)=&A(1-\mathrm{e}^{-4f})^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}p\mathrm{e}^{-\beta sp^2}\mathrm{e}^{\mathrm{i}\sqrt{\lambda/2}\,p(\xi_1'-\xi_1)}\\&\times\exp\Big\{g(\xi_2'-bp)(\xi_2-bp)\mathrm{csch}(2f)-\frac{g}{2}\coth(2f)[(\xi_2'-bp)^2+(\xi_2-bp)^2]\Big\}\\=&A(1-\mathrm{e}^{-4f})^{-1/2}\mathrm{e}^{g\xi_2'\xi_2\mathrm{csch}(2f)-\frac{g}{2}(\xi_2^2+\xi_2'^2)\coth(2f)}\int_{-\infty}^{\infty}\mathrm{d}p\mathrm{e}^{-\beta sp^2}\mathrm{e}^{\mathrm{i}\sqrt{\lambda/2}\,p(\xi_1'-\xi_1)+[b(\xi_2+\xi_2')p-b^2p^2]g\tanh(f)}.\end{aligned} \tag{43}$$

Integrating Eq.(43) and substituting the expression of A into Eq.(43), we obtain

$$\rho(\xi',\xi)=C\mathrm{e}^{\mathrm{i}(\mu_1-\mu_2)(\xi_1'\xi_2'-\xi_1\xi_2)}\mathrm{e}^{g\xi_2'\xi_2\mathrm{csch}(2f)-\frac{g}{2}(\xi_2^2+\xi_2'^2)\coth(2f)}\exp\Big\{\frac{[\mathrm{i}\sqrt{\lambda/2}(\xi_1'-\xi_1)+gb(\xi_2+\xi_2')\tanh(f)]^2}{4(\beta s+gb^2\tanh(f))}\Big\}, \tag{44}$$

where

$$C\equiv\mathrm{e}^{-f}(1-\mathrm{e}^{-4f})^{-1/2}\sqrt{\frac{g}{\beta s+gb^2\tanh(f)}}=\sqrt{\frac{g\mathrm{csch}(2f)}{2(\beta s+gb^2\tanh(f))}}\ . \tag{45}$$

It is the final form of the density matrix $\rho(\xi',\xi)$, the cross term in Eq.(44) represents the coupling interaction between two particles. We can check its correctness by letting $k=0$, then

$$s=\frac{1}{2M},\quad b=0,\quad g=\frac{\lambda}{2\mu\omega},\quad 2f=\beta\omega,\quad C=\sqrt{\frac{g}{2\beta s\sinh(\beta\omega)}}, \tag{46}$$

and

$$\rho(\xi',\xi)|_{k=0}=\mathrm{e}^{\mathrm{i}(\mu_1-\mu_2)(\xi_1'\xi_2'-\xi_1\xi_2)}\sqrt{\frac{g}{2\beta s\sinh(\beta\omega)}}\exp\Big[\frac{-\lambda(\xi_1'-\xi_1)^2}{8\beta s}\Big]$$
$$\times\exp\Big\{-\frac{g}{2\sinh(\beta\omega)}[(\xi_2^2+\xi_2'^2)\cosh(\beta\omega)-2\xi_2'\xi_2]\Big\}=\pi\sqrt{\frac{8}{\lambda}}\mathrm{e}^{\mathrm{i}(\mu_1-\mu_2)(\xi_1'\xi_2'-\xi_1\xi_2)}\rho(\xi_1',\xi_1)\rho(\xi_2',\xi_2), \tag{47}$$

where

$$\rho(\xi_1',\xi_1)\equiv\sqrt{\frac{\lambda}{8\pi\beta s}}\exp\Big[\frac{-\lambda(\xi_1'-\xi_1)^2}{8\beta s}\Big],$$
$$\rho(\xi_2',\xi_2)\equiv\sqrt{\frac{1}{2\pi(2\mu/\lambda)\omega\sinh(\beta\omega)}}\exp\Big\{-\frac{1}{2(2\mu/\lambda)\omega\sinh(\beta\omega)}[(\xi_2^2+\xi_2'^2)\cosh(\beta\omega)-2\xi_2'\xi_2]\Big\}. \tag{48}$$

Equation (47) represents the density matrix for H $(H=P_1^2/(2m_1)+P_2^2/(2m_2)+(\mu/2)\omega^2(x_1-x_2)^2)$ in $\langle\xi|$ representation. The $\rho(\xi_1',\xi_1)$ term represents the contribution of a one-dimensional free particle with the mass $(m_1^2+m_2^2)/(m_1+m_2)$ in the coordinate representation, and the $\rho(\xi_2',\xi_2)$ term represents the contribution of a linear harmonic oscillator with the mass $2\mu/\lambda$ in the momentum representation.

Further, when $k=0$, $m_1=m_2=m$, $(m_1^2+m_2^2)/(m_1+m_2)=m$, $2\mu/\lambda=m$, then

$$\rho(\xi',\xi)|_{k=0,m_1=m_2}=\sqrt{8}\pi\sqrt{\frac{m}{2\pi\beta}}\exp\Big[\frac{-m}{2\beta}(\xi_1'-\xi_1)^2\Big]\sqrt{\frac{1}{2\pi m\omega\sinh(\beta\omega)}}$$
$$\times\exp\Big\{-\frac{1}{2m\omega\sinh(\beta\omega)}[(\xi_2^2+\xi_2'^2)\cosh(\beta\omega)-2\xi_2'\xi_2]\Big\}, \tag{49}$$

which just represents the combining contribution of a one-dimensional free particle in the coordinate representation and a linear harmonic oscillator in the momentum representation.

In summary, by establishing the Schrödinger equation in $\langle\xi|$ representation and obtaining the eigenfunction of H, we have calculated the density matrix in $\langle\xi|$ representation. This approach again shows the importance of entangled state representation,[16–18] and it may also work for multipartite cases.

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