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Well-posedness of the time-varying linear electromagnetic initial-boundary value problem*

Xie Li(谢 莉)† and Lei Yin-Zhao(雷银照)

Department of Electrical Engineering, School of Automation and Electrical Engineering, Beihang University, Beijing 100083, *China*

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The well-posedness of the initial-boundary value problem of the time-varying linear electromagnetic field in a multi-medium region is investigated. Function spaces are defined, with Faraday's law of electromagnetic induction and the initial-boundary conditions considered as constraints. Gauss's formula applied to a multi-medium region is used to derive the energy-estimating inequality. After converting the initial-boundary conditions into homogeneous ones and analysing the characteristics of an operator introduced according to the total current law, the existence, uniqueness and stability of the weak solution to the initial-boundary value problem of the time-varying linear electromagnetic field are proved.

Keywords: well-posedness, initial-boundary value problem, multi-medium region, Maxwell's equations

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1. Introduction

The well-posedness (i.e. the existence, uniqueness and stability of the solution) of the initial-boundary value problem is significant for analysing and calculating a time-varying linear electromagnetic field. The existence of the solutions makes the processes of solving electromagnetic problems meaningful.[1] The uniqueness of the solutions permits one to choose different appropriate methods to solve electromagnetic problems.[2] The stability of solutions ensures the reliability of calculating results when some errors occur in the initial-boundary data.[3]

The well-posedness of various initial-boundary value problems has been investigated by many researchers in the field of applied mathematics.[4,5] However, most of those results cannot provide the well-posedness information for the time-varying linear electromagnetic problem. Because the field region in engineering electromagnetics always comprises multiple media,[6,7] and their boundary conditions used frequently are different from the ones used by mathematical researchers. In the field of engineering electromagnetics, many researchers who are engaged in the numerical calculation of electromagnetic field seldom consider the well-posedness of the time-varying linear electromagnetic problem. Unless the initial-boundary value problem of the time-varying linear electromagnetic field is well-posed, the neglecting of well-posedness may have caused many results of electromagnetic problems invalid, which has not yet been found until now. After all, there exist many examples of ill-posed problems in mathematics.[8−10]

Some researchers have studied the characteristics of the solution to some kinds of electromagnetic problems.[11−18] One of Maxwell's equations was firstly considered as a constraint in the study of initial-boundary value problem in Ref.[19]. Then the method was developed in Ref.[16] to discuss the asymptotic behaviour of the solution to the electromagnetic initial-boundary value problem in a single-medium region under special boundary conditions, which provides a beneficial method for our analysis in a multi-medium region. In this paper, the results have been generalized in the following aspects.

Firstly, the generalized results are applicable to the electromagnetic problems with the inhomogeneous Dirichlet boundary conditions, which are used frequently in time-varying linear electromagnetic problems. For the problems with inhomogeneous boundary conditions, the existence of solutions strongly relies on the traces of fields on the boundaries and at $t = 0$. In

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†Corresponding author. E-mail: xieli_buaa@sina.com

this paper, we present the limits of the traces to ensure the existence of solutions to the initial-boundary value problems.

Secondly, it is proved in the paper that the method to define function spaces, with one of Maxwell's equations and the initial-boundary conditions considered as constraints, is feasible. The feasible explanation which ensures that the defined function spaces are non-empty is fundamental to the analysis.

Thirdly, the difference between the strong and weak solutions is emphasized in many parts of the paper, which sometimes is neglected by other researchers, thus causing the incompletion of analysis.

Fourthly, the field region discussed in the paper is composed of multiple media instead of single medium. In the multi-medium problem, the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ should satisfy two interface conditions of tangential components simultaneously on the interfaces between two media, and satisfy one of them on the exterior boundary, which is different from the problem in single medium region. In this paper we use Gauss's formula applied to a multi-medium region[20] for analysing the different boundary conditions.

Fifthly, the results in the paper exist when the free charge density $\rho \neq 0$. For simplicity, ρ has been assumed to be zero in the field region as in many papers.[14,16,19] The above assumption does not hold true when the relaxation time of a medium is very long.[21] Replacing the initial-boundary value problem expressed by Maxwell's equations with the one expressed by two equivalent rotation equations, the results which are applicable to the case of $\rho \neq 0$ can be derived.

2. Expressions of the initial-boundary value problem

The initial-boundary value problem of time-varying linear electromagnetic field can be expressed as follows:

Let Ω denote the fixed, bounded, three-dimensional region which consists of m kinds of linear media Ω_1, Ω_2, $\cdots$, Ω_{m}. Ω_i is a simply connected region with sufficiently smooth, closed boundary Γ_i. In Ω_i, electric field $\boldsymbol{E}_i$ and magnetic field $\boldsymbol{H}_i$ are produced by the source current density $\boldsymbol{J}_{\mathrm{s}}(x,t)$. Permittivity $\varepsilon_i(x)$, permeability $\mu_i(x)$ and conductivity $\sigma_i(x)$ in Ω_i are all continuous functions of field position x, and there exist finite positive constants $\varepsilon_{\min}$ and $\mu_{\min}$, such that $0 < \varepsilon_{\min} \leq \varepsilon_i(x) < \infty$, $0 < \mu_{\min} \leq \mu_i(x) < \infty$, $0 \leq \sigma_i(x) < \infty$, $i = 1, 2, \cdots, m$.

The exterior boundary of Ω is denoted by Γ. Suppose T to be a positive constant, and $Q = (0, T)$. Assume that a function symbol with a dot '$\cdot$' above a letter means the first-order partial derivative of the function with respect to t, then the fields $\boldsymbol{E}(x,t)$ and $\boldsymbol{H}(x,t)$ will satisfy

1) equations:

$$\nabla \times \boldsymbol{H}(x,t) - \varepsilon(x)\dot{\boldsymbol{E}}(x,t) - \sigma(x)\boldsymbol{E}(x,t) = \boldsymbol{J}_{\mathrm{s}}(x,t), \tag{1}$$

$$\nabla \times \boldsymbol{E}(x,t) + \mu(x)\dot{\boldsymbol{H}}(x,t) = \boldsymbol{0}, \tag{2}$$

$$(x,t) \in \Omega \times Q,$$

2) interface conditions:

$$\boldsymbol{n}_{ij} \times [\boldsymbol{E}_j(x,t) - \boldsymbol{E}_i(x,t)] = \boldsymbol{0}, \tag{3}$$

$$\boldsymbol{n}_{ij} \times [\boldsymbol{H}_j(x,t) - \boldsymbol{H}_i(x,t)] = \boldsymbol{K}_{ij}(x,t), \tag{4}$$

$$(x,t) \in \Gamma_{ij} \times Q, \quad i = 1, 2, \cdots, m-1,$$

$$j = 2, 3, \cdots, m, \quad i \neq j,$$

3) exterior boundary conditions:

$$\boldsymbol{n} \times \boldsymbol{E}(x,t) = \boldsymbol{F}_{\mathrm{e}}(x,t), \tag{5}$$

or

$$\boldsymbol{n} \times \boldsymbol{H}(x,t) = \boldsymbol{F}_{\mathrm{h}}(x,t), \tag{6}$$

$$(x,t) \in \Gamma \times Q,$$

4) Initial conditions:

$$\boldsymbol{E}(x,0) = \boldsymbol{E}_0(x), \tag{7}$$

$$\boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \tag{8}$$

$$\nabla \cdot [\varepsilon(x)\boldsymbol{E}(x,0)] = \rho(x), \tag{9}$$

$$\nabla \cdot [\mu(x)\boldsymbol{H}(x,0)] = 0, \tag{10}$$

$$x \in \Omega,$$

where $\boldsymbol{n}_{ij}$ is the unit normal vector pointing from medium i to medium j on the interface Γ_{ij}, $\boldsymbol{K}_{ij}(x,t)$ is the current surface density on Γ_{ij}, $\boldsymbol{n}$ is the unit normal vector outward on the exterior boundary Γ, and $\rho(x)$ is the free charge density in the field region when $t = 0$. $\boldsymbol{K}_{ij}$, $\boldsymbol{F}_{\mathrm{e}}$, $\boldsymbol{F}_{\mathrm{h}}$, $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and ρ are all known functions.

The above problem is denoted by P.

Notes:

1) In electromagnetic theory, Eq.(1) is the mathematical expression of the total current law, and Eq.(2) is the mathematical expression of Faraday's law of electromagnetic induction. Two divergence equations

in Maxwells equations are regarded as the constraints imposed on the initial data of the initial-boundary value problem. Therefore, ρ exists only in the initial conditions of the problem, which is independent of t.

2) On the exterior boundary, it is required in the problem P that one of the exterior boundary conditions (5) and (6) should be satisfied. In the following discussion, we assume condition (5) to be satisfied. It can be analysed similarly when condition (6) is satisfied.

3. Definitions of solution

Focusing on the multi-medium characteristics of the field region, we introduce the following function spaces to define the strong and weak solutions:

$$I(\Omega)=\{\boldsymbol{e}\in H^1(\Omega):\ \nabla\cdot\boldsymbol{e}=\rho,\ \rho\in L^2(\Omega)\},\quad I^0(\Omega)=\{\boldsymbol{e}\in H^1(\Omega):\ \nabla\cdot\boldsymbol{e}=\boldsymbol{0}\},$$

$$J(\Omega)=\Big\{\boldsymbol{e}\in L^2(\Omega):\ \int_\Omega \boldsymbol{e}\cdot\nabla\phi\,\mathrm{d}x=-\int_\Omega\rho\phi\,\mathrm{d}x,\ \ \rho\in L^2(\Omega),\ \ \forall\phi=\sum_{i=1}^m\phi_i,\quad \phi_i\in C^1(\Omega),\quad \mathrm{supp}\ \phi_i\subset\Omega_i\Big\},$$

$$J^0(\Omega)=\Big\{\boldsymbol{e}\in L^2(\Omega):\int_\Omega \boldsymbol{e}\cdot\nabla\phi\,\mathrm{d}x=0,\ \ \forall\phi=\sum_{i=1}^m\phi_i,\ \phi_i\in C^1(\Omega),\ \mathrm{supp}\ \phi_i\subset\Omega_i\Big\},$$

$$\mathcal{U}(\Omega,Q)=L^2(Q;H^2(\Omega))\cap H^1(Q;L^2(\Omega)),\quad \mathcal{V}(\Omega,Q)=L^2(Q;H^1(\Omega))\cap H^2(Q;L^2(\Omega)),$$

$$\begin{aligned}\mathcal{R}(\Omega,Q)=\{&\boldsymbol{E},\boldsymbol{H}\in L^2(\Omega,Q),\ \boldsymbol{E}|_{\Gamma_{ij}},\boldsymbol{H}|_{\Gamma_{ij}}\in H^{1/2}(\Gamma_{ij},Q),\ \ \boldsymbol{E}|_\Gamma,\boldsymbol{H}|_\Gamma\in H^{1/2}(\Gamma,Q):\\ &\boldsymbol{n}_{ij}\times[\boldsymbol{E}_j(x,t)-\boldsymbol{E}_i(x,t)]=\boldsymbol{0},\ \ \boldsymbol{n}_{ij}\times[\boldsymbol{H}_j(x,t)-\boldsymbol{H}_i(x,t)]=\boldsymbol{K}_{ij}(x,t),\\ &\forall(x,t)\in\Gamma_{ij}\times Q,\quad i=1,2,\cdots,m-1,\quad j=2,3,\cdots,m,\quad i\neq j,\\ &\boldsymbol{K}_{ij}\in H^{1/2}(\Gamma_{ij},Q);\quad \boldsymbol{n}\times\boldsymbol{E}(x,t)=\boldsymbol{F}_{\mathrm{e}}(x,t),\quad \forall(x,t)\in\Gamma\times Q,\quad \boldsymbol{F}_{\mathrm{e}}\in H^{1/2}(\Gamma,Q)\}.\end{aligned}$$

$C_0^1(\Omega)$ is the space of functions defined on Ω with continuous derivatives of first order and compact supports in Ω. $L^2(\Omega)$ is the space of all the square summable functions on Ω. $H^1(\Omega)$ is the space of the elements in $L^2(\Omega)$ that have square summable first-order generalized derivatives on Ω. $J(\Omega)$ is composed of all the function $\boldsymbol{e}$ which satisfies $\nabla\cdot\boldsymbol{e}=\rho$ in a weak sense. $J^0(\Omega)$ is composed of all the function $\boldsymbol{e}$ which satisfies $\nabla\cdot\boldsymbol{e}=0$ in a weak sense. The function in $\mathcal{U}(\Omega,Q)$ has second-order generalized derivatives with respect to x and first-order generalized derivatives with respect to t. The function in $\mathcal{V}(\Omega,Q)$ has first-order generalized derivatives with respect to x and second-order generalized derivatives with respect to t. $H^{1/2}(\Gamma,Q)$ is a 1/2-order Sobolev space defined on $\Gamma\times Q$. $\mathcal{R}(\Omega,Q)$ is composed of all the ($\boldsymbol{E}$, $\boldsymbol{H}$) which satisfies boundary conditions on the interfaces and the exterior boundary.

It should be noted that, in a multi-medium region, the traces of the fields on the boundaries are limited, because the function spaces which the traces belong to will affect the existence of the solution. The functions $\boldsymbol{K}_{ij}$ and $\boldsymbol{F}_{\mathrm{e}}$ in $\mathcal{R}(\Omega,Q)$ can be chosen arbitrarily under the condition that $\boldsymbol{K}_{ij}\in H^{1/2}(\Gamma_{ij},Q)$ and $\boldsymbol{F}_{\mathrm{e}}\in H^{1/2}(\Gamma,Q)$.

With the above function spaces, we can treat Eq.(2) as a constraint in defining the following function space:

$$\begin{aligned}\mathcal{K}(\Omega,Q)=\{&\boldsymbol{E}\in\mathcal{U}(\Omega,Q),\ \boldsymbol{H}\in\mathcal{V}(\Omega,Q):\ \nabla\times\boldsymbol{E}(x,t)+\mu(x)\dot{\boldsymbol{H}}(x,t)=\boldsymbol{0},\ \boldsymbol{E}(x,0)=\boldsymbol{E}_0(x),\\ &\boldsymbol{H}(x,0)=\boldsymbol{H}_0(x),\ \ \forall(x,t)\in\Omega\times Q,\ \varepsilon(x)\boldsymbol{E}_0(x)\in I(\Omega),\ \mu(x)\boldsymbol{H}_0(x)\in I^0(\Omega);\ \ (\boldsymbol{E},\boldsymbol{H})\in\mathcal{R}(\Omega,Q)\}.\end{aligned}$$

$\mathcal{K}(\Omega,Q)$ is composed of all the ($\boldsymbol{E}$, $\boldsymbol{H}$) which satisfies Eq.(2) in 1), 2), 3), 4) in the problem P. The functions $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ can be chosen arbitrarily under the condition that $\varepsilon(x)\boldsymbol{E}_0(x)\in I(\Omega)$ and $\mu(x)\boldsymbol{H}_0(x)\in I^0(\Omega)$. When Eq.(2) exists in a weak sense, the function space changes accordingly into

$$\mathcal{D}(\Omega,Q)=\Big\{\boldsymbol{E},\boldsymbol{H}\in L^2(\Omega,Q):\ \boldsymbol{E}(x,0)=\boldsymbol{E}_0(x),\ \boldsymbol{H}(x,0)=\boldsymbol{H}_0(x),\ \varepsilon(x)\boldsymbol{E}_0(x)\in J(\Omega),\mu(x)\boldsymbol{H}_0(x)\in J^0(\Omega);$$

$$\int_Q\int_\Omega[\boldsymbol{E}(x,t)\cdot\nabla\times\boldsymbol{h}(x,t)-\mu(x)\boldsymbol{H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t)]\mathrm{d}x\mathrm{d}t=\int_\Omega\mu(x)\boldsymbol{H}_0(x)\cdot\boldsymbol{h}_0(x)\,\mathrm{d}x$$
$$-\int_Q\oint_\Gamma\boldsymbol{F}_\mathrm{e}(x,t)\cdot\boldsymbol{h}(x,t)\,\mathrm{d}\Gamma\mathrm{d}t-\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\int_Q\int_{\Gamma_{ij}}\boldsymbol{n}_{ij}\times[\boldsymbol{h}_j(x,t)-\boldsymbol{h}_i(x,t)]\cdot\boldsymbol{E}_i(x,t)\,\mathrm{d}\Gamma_{ij}\mathrm{d}t,$$
$$\forall(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}(\Omega,Q);\ (\boldsymbol{E},\boldsymbol{H})\in\mathcal{R}(\Omega,Q)\Big\}.$$

The functions $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ can be chosen arbitrarily under the condition that $\varepsilon(x)\boldsymbol{E}_0(x)\in J(\Omega)$ and $\mu(x)\boldsymbol{H}_0(x)\in J^0(\Omega)$.

Here it is necessary to explain the feasibility of the method of considering Eq.(2) and the initial-boundary conditions as constraints in defining $\mathcal{K}(\Omega,Q)$ and $\mathcal{D}(\Omega,Q)$. If $\mathcal{K}(\Omega,Q)$ and $\mathcal{D}(\Omega,Q)$ were empty, the method of analysing the well-posedness presented in the paper would be invalid. It is proved in the appendix that the function space $\mathcal{K}(\Omega,Q)$ is a non-empty set for each set of given $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{F}_\mathrm{e}$, $\boldsymbol{K}_{ij}$ and ρ. So $\mathcal{D}(\Omega,Q)$ is also non-empty.

Based on these two function spaces, we introduce the following definitions suitable for the multi-medium region.

Definition 1. A pair $(\boldsymbol{E},\boldsymbol{H})\in\mathcal{K}(\Omega,Q)$ is said to be a strong solution of the problem P, with the source $\boldsymbol{J}_\mathrm{s}(x,t)\in L^2(\Omega,Q)$, the initial data $(\boldsymbol{E}_0,\boldsymbol{H}_0)$ that satisfy $\varepsilon(x)\boldsymbol{E}_0(x)\in I(\Omega)$ and $\mu(x)\boldsymbol{H}_0(x)\in I^0(\Omega)$, and the functions $\boldsymbol{F}_\mathrm{e},\boldsymbol{K}_{ij},\rho$ that satisfy $\boldsymbol{F}_\mathrm{e}(x,t)\in H^{1/2}(\Gamma,Q)$, $\boldsymbol{K}_{ij}(x,t)\in H^{1/2}(\Gamma_{ij},Q)$, $\rho(x)\in L^2(\Omega)$, if and only if it satisfies

$$\nabla\times\boldsymbol{H}(x,t)-\varepsilon(x)\dot{\boldsymbol{E}}(x,t)-\sigma(x)\boldsymbol{E}(x,t)=\boldsymbol{J}_\mathrm{s}(x,t)$$

almost everywhere in $\Omega\times Q$.

Definition 2. A pair $(\boldsymbol{E},\boldsymbol{H})\in\mathcal{D}(\Omega,Q)$ is said to be a weak solution of the problem P, with the source $\boldsymbol{J}_\mathrm{s}(x,t)\in L^2(\Omega,Q)$, the initial data $(\boldsymbol{E}_0,\ \boldsymbol{H}_0)$ that satisfy $\varepsilon(x)\boldsymbol{E}_0(x)\in J(\Omega)$ and $\mu(x)\boldsymbol{H}_0(x)\in J^0(\Omega)$, and the functions $\boldsymbol{F}_\mathrm{e},\boldsymbol{K}_{ij},\rho$ that satisfy $\boldsymbol{F}_\mathrm{e}(x,t)\in H^{1/2}(\Gamma,Q)$, $\boldsymbol{K}_{ij}(x,t)\in H^{1/2}(\Gamma_{ij},Q)$, $\rho(x)\in L^2(\Omega)$, if and only if it satisfies

$$\begin{aligned}&\int_Q\int_\Omega[\varepsilon(x)\boldsymbol{E}(x,t)\cdot\dot{\boldsymbol{e}}(x,t)-\mu(x)\boldsymbol{H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t)-\sigma(x)\boldsymbol{E}(x,t)\cdot\boldsymbol{e}(x,t)]\mathrm{d}x\mathrm{d}t\\&=\int_Q\int_\Omega\boldsymbol{J}_\mathrm{s}(x,t)\cdot\boldsymbol{e}(x,t)\,\mathrm{d}x\mathrm{d}t+\int_Q\oint_\Gamma\boldsymbol{f}_\mathrm{e}(x,t)\cdot\boldsymbol{H}(x,t)\,\mathrm{d}\Gamma\mathrm{d}t\\&\quad+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\int_Q\int_{\Gamma_{ij}}\boldsymbol{e}_i(x,t)\cdot\boldsymbol{K}_{ij}(x,t)\,\mathrm{d}\Gamma_{ij}\mathrm{d}t-\int_\Omega\varepsilon(x)\boldsymbol{E}_0(x)\cdot\boldsymbol{e}_0(x)\,\mathrm{d}x,\qquad(11)\\&\forall\,(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}(\Omega,Q),\end{aligned}$$

where $\boldsymbol{f}_\mathrm{e}(x,t)=\boldsymbol{n}\times\boldsymbol{e}(x,t)$, $\forall(x,t)\in\Gamma\times Q$.

Equation (11) is another form of Eq.(1) in a weak sense.

4. Homogenized transform of initial-boundary conditions

The problem P includes inhomogeneous initial-boundary conditions. For the convenience of analysis, the initial-boundary conditions should be changed into homogeneous ones by a suitable transform. The following method is derived from the method of making the homogenized transform of initial conditions given by Ref.[16]. We generalize it to suit for not only the homogenized transform of the initial conditions, but also the homogenized transform of the boundary conditions. Simultaneously, the feasibility of the method is proved.

Let

$$\boldsymbol{E}'(x,t)=\boldsymbol{E}(x,t)-\boldsymbol{e}'(x,t),$$

and

$$\boldsymbol{H}'(x,t)=\boldsymbol{H}(x,t)-\boldsymbol{h}'(x,t),$$

where $(\boldsymbol{e}', \boldsymbol{h}')$ is any known function in $\mathcal{K}(\Omega, Q)$, which corresponds to the same $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{F}_{\rm e}$, $\boldsymbol{K}_{ij}$ and ρ as those for $(\boldsymbol{E}, \boldsymbol{H})$. Such a $(\boldsymbol{e}', \boldsymbol{h}')$ is proved to be existential (see appendix), which ensures the feasibility of the homogenized transform. Replace $(\boldsymbol{E}', \boldsymbol{H}')$ with the old notation $(\boldsymbol{E}, \boldsymbol{H})$, then Eqs.(1) and (2) will change into

$$\begin{aligned}&\nabla \times \boldsymbol{H}(x,t) - \varepsilon(x)\dot{\boldsymbol{E}}(x,t)\\ &\quad - \sigma(x)\boldsymbol{E}(x,t) = \boldsymbol{F}(x,t),\end{aligned} \tag{12}$$

$$\nabla \times \boldsymbol{E}(x,t) + \mu(x)\dot{\boldsymbol{H}}(x,t) = \boldsymbol{0}, \tag{13}$$

where

$$\begin{aligned}\boldsymbol{F}(x,t) = {}& \boldsymbol{J}_{\rm s}(x,t) - \nabla \times \boldsymbol{h}'(x,t)\\ &+ \varepsilon(x)\dot{\boldsymbol{e}}'(x,t) + \sigma(x)\boldsymbol{e}'(x,t),\end{aligned}$$

and $\boldsymbol{F}(x,t) \in L^2(\Omega, Q)$. After the above transform, the initial conditions change into homogeneous ones

$$\boldsymbol{E}(x,0) = \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{H}(x,0) = \boldsymbol{0}\,. \tag{14}$$

Furthermore, the boundary conditions also change into homogeneous ones

$$\begin{aligned}\mathcal{R}^0(\Omega, Q) = \{&\boldsymbol{E}, \boldsymbol{H} \in L^2(\Omega, Q),\ \boldsymbol{E}|_{\Gamma_{ij}}, \boldsymbol{H}|_{\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij}, Q),\ \boldsymbol{E}|_{\Gamma}, \boldsymbol{H}|_{\Gamma} \in H^{1/2}(\Gamma, Q):\\ &\boldsymbol{n}_{ij} \times [\boldsymbol{E}_j(x,t) - \boldsymbol{E}_i(x,t)] = \boldsymbol{0},\ \ \boldsymbol{n}_{ij} \times [\boldsymbol{H}_j(x,t) - \boldsymbol{H}_i(x,t)] = \boldsymbol{0},\\ &\forall (x,t) \in \Gamma_{ij} \times Q,\ \ i = 1, 2, \cdots, m-1,\ \ j = 2, 3, \cdots, m,\ \ i \neq j;\\ &\boldsymbol{n} \times \boldsymbol{E}(x,t) = \boldsymbol{0},\ \ \forall (x,t) \in \Gamma \times Q\}.\end{aligned}$$

$\mathcal{K}(\Omega, Q)$ and $\mathcal{D}(\Omega, Q)$ change, respectively, into

$$\begin{aligned}\mathcal{K}^0(\Omega, Q) = \{&\boldsymbol{E} \in \mathcal{U}(\Omega, Q),\ \boldsymbol{H} \in \mathcal{V}(\Omega, Q):\ \nabla \times \boldsymbol{E}(x,t) + \mu(x)\dot{\boldsymbol{H}}(x,t) = \boldsymbol{0};\\ &\boldsymbol{E}(x,0) = \boldsymbol{0},\ \ \boldsymbol{H}(x,0) = \boldsymbol{0},\ \ \forall (x,t) \in \Omega \times Q;\ \ (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{R}^0(\Omega, Q)\},\\ \mathcal{D}^0(\Omega, Q) = \{&\boldsymbol{E}, \boldsymbol{H} \in L^2(\Omega, Q):\ \boldsymbol{E}(x,0) = \boldsymbol{0},\ \ \boldsymbol{H}(x,0) = \boldsymbol{0},\ \ \forall\, x \in \Omega;\\ &\int_Q \int_\Omega [\boldsymbol{E}(x,t) \cdot \nabla \times \boldsymbol{h}(x,t) - \mu(x)\boldsymbol{H}(x,t) \cdot \dot{\boldsymbol{h}}(x,t)]{\rm d}x{\rm d}t = 0\\ &\forall\ (\boldsymbol{e}, \boldsymbol{h}) \in \mathcal{K}^0(\Omega, Q);\ \ (\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{R}^0(\Omega, Q)\}.\end{aligned}$$

Equation (11) in definition 2 changes into the following form:

$$\begin{aligned}&\int_Q \int_\Omega [\varepsilon(x)\boldsymbol{E}(x,t) \cdot \dot{\boldsymbol{e}}(x,t) - \mu(x)\boldsymbol{H}(x,t) \cdot \dot{\boldsymbol{h}}(x,t) - \sigma(x)\boldsymbol{E}(x,t) \cdot \boldsymbol{e}(x,t)]{\rm d}x{\rm d}t\\ &= \int_Q \int_\Omega \boldsymbol{F}(x,t) \cdot \boldsymbol{e}(x,t){\rm d}x{\rm d}t, \quad \forall\ (\boldsymbol{e}, \boldsymbol{h}) \in \mathcal{K}^0(\Omega, Q).\end{aligned} \tag{15}$$

After the homogenized transform of the initial-boundary conditions, the problem is equivalent to the original problem P.

5. Energy-estimating inequality of the initial-boundary value problem

In this section, we combine the boundary conditions in the problem P and Gauss's formula applied to a multi-medium region with the frequently used method of energy integral in electromagnetics to derive the energy-estimating inequality of the problem P.

Theorem 5.1 Each strong solution $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{K}^0(\Omega, Q)$ to the problem P satisfies

$$\int_Q \int_\Omega [\varepsilon(x)|\boldsymbol{E}(x,t)|^2 + \mu(x)|\boldsymbol{H}(x,t)|^2]{\rm d}x{\rm d}t \leq C \int_Q \int_\Omega |\boldsymbol{F}(x,t)|^2 {\rm d}x{\rm d}t, \tag{16}$$

where C is a positive constant, which is independent of the field position x and the time t.

Proof Integrating over Ω the difference between the scalar product of Eq.(13) by $\boldsymbol{H}(x,t)$ and the scalar product of Eq.(12) by $\boldsymbol{E}(x,t)$, we have

$$\int_{\Omega} \nabla \cdot [\boldsymbol{E}(x,t) \times \boldsymbol{H}(x,t)] \mathrm{d}x = \int_{\Omega} [-\dot{\omega}(x,t) - \sigma(x)|\boldsymbol{E}(x,t)|^2 - \boldsymbol{E}(x,t) \cdot \boldsymbol{F}(x,t)] \mathrm{d}x, \tag{17}$$

where

$$\omega(x,t) = [\varepsilon(x)|\boldsymbol{E}(x,t)|^2 + \mu(x)|\boldsymbol{H}(x,t)|^2]/2.$$

According to Gauss's formula applied to a multi-medium region and the boundary conditions in $\mathcal{R}^0(\Omega, Q)$, we can obtain

$$\begin{aligned}
&\int_{\Omega} \nabla \cdot [\boldsymbol{E}(x,t) \times \boldsymbol{H}(x,t)] \mathrm{d}x = \oint_{\Gamma} [\boldsymbol{E}(x,t) \times \boldsymbol{H}(x,t)] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
&+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_{\Gamma_{ij}} [\boldsymbol{E}_i(x,t) \times \boldsymbol{H}_i(x,t) - \boldsymbol{E}_j(x,t) \times \boldsymbol{H}_j(x,t)] \cdot \boldsymbol{n}_{ij} \, \mathrm{d}\Gamma_{ij} = \oint_{\Gamma} [\boldsymbol{n} \times \boldsymbol{E}(x,t)] \cdot \boldsymbol{H}(x,t) \, \mathrm{d}\Gamma \\
&+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_{\Gamma_{ij}} [\boldsymbol{n}_{ij} \times \boldsymbol{E}_i(x,t) \cdot \boldsymbol{H}_i(x,t) - \boldsymbol{n}_{ij} \times \boldsymbol{E}_j(x,t) \cdot \boldsymbol{H}_j(x,t)] \mathrm{d}\Gamma_{ij} = \oint_{\Gamma} \boldsymbol{n} \times \boldsymbol{E}(x,t) \cdot \boldsymbol{H}(x,t) \, \mathrm{d}\Gamma \\
&+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_{\Gamma_{ij}} \boldsymbol{n}_{ij} \times [\boldsymbol{H}_j(x,t) - \boldsymbol{H}_i(x,t)] \cdot \boldsymbol{E}_i(x,t) \, \mathrm{d}\Gamma_{ij} = 0.
\end{aligned} \tag{18}$$

Letting $W(t) = \int_{\Omega} \omega(x,t) \mathrm{d}x$, we can obtain

$$\begin{aligned}
\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \int_{\Omega} [-\sigma(x)|\boldsymbol{E}(x,t)|^2 - \boldsymbol{E}(x,t) \cdot \boldsymbol{F}(x,t)] \mathrm{d}x \le \Big(\int_{\Omega} |\boldsymbol{E}(x,t)|^2 \mathrm{d}x \Big)^{1/2} \Big(\int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x \Big)^{1/2} \\
&\le \sqrt{\frac{2}{\varepsilon_{\min}}} W^{1/2}(t) \Big(\int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x \Big)^{1/2}.
\end{aligned} \tag{19}$$

Using the separation of variables, and integrating inequality (19) over $(0,t)$, we have

$$W(t) \le \frac{1}{2\varepsilon_{\min}} \Big[\int_0^t \Big(\int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x \Big)^{1/2} \mathrm{d}t \Big]^2.$$

According to Hölder's inequality,[22] there is

$$W(t) \le \frac{1}{2\varepsilon_{\min}} \Big(\int_0^t \int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x\mathrm{d}t \Big) \Big(\int_0^t \mathrm{d}t \Big) \le \frac{T}{2\varepsilon_{\min}} \int_Q \int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x\mathrm{d}t,$$

that is,

$$\int_{\Omega} [\varepsilon(x)|\boldsymbol{E}(x,t)|^2 + \mu(x)|\boldsymbol{H}(x,t)|^2] \mathrm{d}x \le \frac{T}{\varepsilon_{\min}} \int_Q \int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x\mathrm{d}t. \tag{20}$$

Integrating inequality (20) over $(0,T)$, we have

$$\int_Q \int_{\Omega} [\varepsilon(x)|\boldsymbol{E}(x,t)|^2 + \mu(x)|\boldsymbol{H}(x,t)|^2] \mathrm{d}x \le \frac{T^2}{\varepsilon_{\min}} \int_Q \int_{\Omega} |\boldsymbol{F}(x,t)|^2 \mathrm{d}x\mathrm{d}t. \tag{21}$$

6. Well-posedness of the initial-boundary value problem

In this section, an operator is introduced according to the mathematical expression of the total current law. By analysing the characteristics of the operator, the well-posedness of the problem P is proved. As stated previously, we develop the presented method from the one given in Ref.[16], and we pay much more attention to the difference between the strong and weak solutions in that it may cause the incompletion of the analysis when neglected.

The operator Λ: $\mathcal{K}^0(\Omega,Q)\to L^2(\Omega,Q)$ is introduced according to the total current law. For any $u=(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}^0(\Omega,Q)$, let

$$\Lambda u=\nabla\times\boldsymbol{h}(x,t)-\varepsilon(x)\dot{\boldsymbol{e}}(x,t)-\sigma(x)\boldsymbol{e}(x,t). \quad (22)$$

The range of the operator $R(\Lambda)\subset L^2(\Omega,Q)$.

Theorem 6.1 $R(\Lambda)$ is dense in $L^2(\Omega,Q)$.

Proof The closure of $R(\Lambda)$, denoted by $\overline{R(\Lambda)}$, is a closed linear subspace of $L^2(\Omega,Q)$. According to the theory of functional analysis,[23] if $R(\Lambda)$ were not dense, there would exist a non-zero element $\boldsymbol{J}_{\rm E}\in L^2(\Omega,Q)$, orthogonal to $\overline{R(\Lambda)}$, that is

$$\int_Q\int_\Omega[\nabla\times\boldsymbol{h}(x,t)-\varepsilon(x)\dot{\boldsymbol{e}}(x,t)-\sigma(x)\boldsymbol{e}(x,t)]\cdot\boldsymbol{J}_{\rm E}(x,t)\mathrm{d}x\mathrm{d}t=0,\quad\forall(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}^0(\Omega,Q). \quad (23)$$

Introduce a vector $\boldsymbol{J}_{\rm H}$ such that $(\boldsymbol{J}_{\rm E},\boldsymbol{J}_{\rm H})\in\mathcal{D}^0(\Omega,Q)$, then $(\boldsymbol{J}_{\rm E},\boldsymbol{J}_{\rm H})$ will satisfy

$$\int_Q\int_\Omega[\nabla\times\boldsymbol{h}(x,t)\cdot\boldsymbol{J}_{\rm E}(x,t)-\mu(x)\boldsymbol{J}_{\rm H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t)]\mathrm{d}x\mathrm{d}t=0,\quad\forall(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}^0(\Omega,Q). \quad (24)$$

Equation (23) minus Eq.(24) yields

$$\int_Q\int_\Omega[\mu(x)\boldsymbol{J}_{\rm H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t)-\varepsilon(x)\dot{\boldsymbol{e}}(x,t)\cdot\boldsymbol{J}_{\rm E}(x,t)-\sigma(x)\boldsymbol{e}(x,t)\cdot\boldsymbol{J}_{\rm E}(x,t)]\mathrm{d}x\mathrm{d}t=0,$$
$$\forall(\boldsymbol{e},\boldsymbol{h})\in\mathcal{K}^0(\Omega,Q). \quad (25)$$

Because of the arbitrariness of $(\boldsymbol{e},\boldsymbol{h})$ in $\mathcal{K}^0(\Omega,Q)$, we can choose

$$\boldsymbol{e}(x,t)=\begin{cases}(\tau-t)\boldsymbol{v}(x), & 0\le t\le\tau,\\ \boldsymbol{0}, & \tau\le t\le T,\end{cases}$$

where τ can be any value in Q. We choose

$$\boldsymbol{h}(x,t)=\begin{cases}\Big(-\dfrac{1}{2}t^2+\tau t-\dfrac{1}{2}\tau^2\Big)\boldsymbol{w}(x), & 0\le t\le\tau,\\ \boldsymbol{0}, & \tau\le t\le T\,.\end{cases}$$

The choice of function $\boldsymbol{h}$ is different from that in Ref.[16], because the pair $(\boldsymbol{e},\boldsymbol{h})$ should satisfy $\nabla\times\boldsymbol{e}(x,t)+\mu(x)\dot{\boldsymbol{h}}(x,t)=\boldsymbol{0}$. $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$ are functions in $H^1(\Omega)$, and they should satisfy

$$\nabla\times\boldsymbol{v}(x)+\mu(x)\boldsymbol{w}(x)=\boldsymbol{0}. \quad (26)$$

Substituting $(\boldsymbol{e},\boldsymbol{h})$ and their partial derivatives with respect to t into Eq.(25), the left-hand side of Eq.(25) changes into a function of τ, denoted by $g(\tau)$, which is expressed as

$$g(\tau)=-\int_0^\tau\int_\Omega[\varepsilon(x)\boldsymbol{J}_{\rm E}(x,t)\cdot\boldsymbol{v}(x)+\mu(x)(\tau-t)\boldsymbol{J}_{\rm H}(x,t)\cdot\boldsymbol{w}(x)-(\tau-t)\sigma(x)\boldsymbol{J}_{\rm E}(x,t)\cdot\boldsymbol{v}(x)]\mathrm{d}x\mathrm{d}t.$$

According to the former assumption, $\forall\,\tau\in Q=(0,T)$, we have $g(\tau)\equiv0$, that is

$$\int_0^\tau\int_\Omega[\varepsilon(x)\boldsymbol{J}_{\rm E}(x,t)\cdot\boldsymbol{v}(x)+\mu(x)(\tau-t)\boldsymbol{J}_{\rm H}(x,t)\cdot\boldsymbol{w}(x)-(\tau-t)\sigma(x)\boldsymbol{J}_{\rm E}(x,t)\cdot\boldsymbol{v}(x)]\mathrm{d}x\mathrm{d}t\equiv0. \quad (27)$$

Differentiating Eq.(27) with respect to τ, we have

$$\int_0^\tau\int_\Omega[-\sigma(x)\boldsymbol{J}_{\rm E}(x,t)\cdot\boldsymbol{v}(x)+\mu(x)\boldsymbol{J}_{\rm H}(x,t)\cdot\boldsymbol{w}(x)]\mathrm{d}x\mathrm{d}t+\int_\Omega\varepsilon(x)\boldsymbol{J}_{\rm E}(x,\tau)\cdot\boldsymbol{v}(x)\,\mathrm{d}x=0. \quad (28)$$

Let

$$\boldsymbol{J}_{\rm EI}(x,\tau)=\int_0^\tau\boldsymbol{J}_{\rm E}(x,t)\mathrm{d}t,\quad\text{and}\quad\boldsymbol{J}_{\rm HI}(x,\tau)=\int_0^\tau\boldsymbol{J}_{\rm H}(x,t)\mathrm{d}t,$$

then according to Eq.(28), $(\boldsymbol{J}_{\rm EI},\boldsymbol{J}_{\rm HI})$ will satisfy

$$\int_\Omega[\sigma(x)\boldsymbol{J}_{\rm EI}(x,\tau)\cdot\boldsymbol{v}(x)-\mu(x)\boldsymbol{J}_{\rm HI}(x,\tau)\cdot\boldsymbol{w}(x)]\mathrm{d}x=\int_\Omega\varepsilon(x)\boldsymbol{J}_{\rm E}(x,\tau)\cdot\boldsymbol{v}(x)\mathrm{d}x. \quad (29)$$

If the difference between the strong and weak solutions was ignored here, the density of $R(\Lambda)$ would be derived easily by letting $\boldsymbol{v}(x) = \boldsymbol{J}_{\mathrm{EI}}(x,\tau)$ and $\boldsymbol{w}(x) = \boldsymbol{J}_{\mathrm{H}}(x,\tau)$. However, it cannot be ignored without the consistency of the strong and weak solutions. So we try to prove it in another approach.

Because of the arbitrariness of $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$, we let $\boldsymbol{v}(x)$ in Ω_i be a function in $H^2(\Omega_i)$, $\boldsymbol{w}(x)$ in Ω_i be a function in $H^1(\Omega_i)$. They satisfy Eq.(26), and equal $\mathbf{0}$ in the other region of Ω. Denote such functions by $\boldsymbol{v}_i(x)$ and $\boldsymbol{w}_i(x)$. It is easy to validate that $\boldsymbol{v}_i(x)$ and $\boldsymbol{w}_i(x)$ can ensure that $(\boldsymbol{e},\boldsymbol{h})$ satisfies the exterior boundary condition and the interface conditions except the one on Γ_i. Substituting $\boldsymbol{v}_i(x)$ and $\boldsymbol{w}_i(x)$ into Eq.(29), we have

$$\int_{\Omega_i} [\sigma(x)\boldsymbol{J}_{\mathrm{EI}}(x,\tau)\cdot\boldsymbol{v}_i(x) - \mu(x)\boldsymbol{J}_{\mathrm{HI}}(x,\tau)\cdot\boldsymbol{w}_i(x)]\mathrm{d}x = \int_{\Omega_i} \varepsilon(x)\boldsymbol{J}_{\mathrm{E}}(x,\tau)\cdot\boldsymbol{v}_i(x)\mathrm{d}x. \tag{30}$$

As $C_0^\infty(\Omega_i)$ is dense in $L^2(\Omega_i)$,[24] we can find a $C_0^\infty(\Omega_i)$ function sufficiently close to $\boldsymbol{J}_{\mathrm{EI}}(x,\tau)$ in Ω_i. Letting $\boldsymbol{v}_i(x)$ equal the $C_0^\infty(\Omega_i)$ function in Ω_i, we can have

$$\|\boldsymbol{v}_i(x) - \boldsymbol{J}_{\mathrm{EI}}(x,\tau)\|_{L^2(\Omega_i)} \to 0. \tag{31}$$

Determining the value of $\boldsymbol{w}_i(x)$ in Ω_i according to Eq.(26), we can derive

$$\|\boldsymbol{w}_i(x) - \boldsymbol{J}_{\mathrm{H}}(x,\tau)\|_{L^2(\Omega_i)} \to 0. \tag{32}$$

Thus $\boldsymbol{v}_i(x)$ and $\boldsymbol{w}_i(x)$ also ensure that $(\boldsymbol{e},\boldsymbol{h})$ can satisfy the interface condition on Γ_i. It follows from expressions (31) and (32) that

$$\int_{\Omega_i} [\varepsilon(x)\boldsymbol{J}_{\mathrm{EI}}(x,\tau)\cdot\boldsymbol{J}_{\mathrm{E}}(x,\tau) + \mu(x)\boldsymbol{J}_{\mathrm{HI}}(x,\tau)\cdot\boldsymbol{J}_{\mathrm{H}}(x,\tau)]\mathrm{d}x \to \int_{\Omega_i} \sigma(x)J_{\mathrm{EI}}^2(x,\tau)\mathrm{d}x,$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_i} [\varepsilon(x)J_{\mathrm{EI}}^2(x,\tau) + \mu(x)J_{\mathrm{HI}}^2(x,\tau)]\mathrm{d}x \to \int_{\Omega_i} \sigma(x)J_{\mathrm{EI}}^2(x,\tau)\mathrm{d}x. \tag{33}$$

We can deduce that, if $J_{\mathrm{EI}}(x,\tau)$ and $J_{\mathrm{HI}}(x,\tau)$ do not tend to zero, the following inequality exists:

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_i} [\varepsilon(x)J_{\mathrm{EI}}^2(x,\tau) + \mu(x)J_{\mathrm{HI}}^2(x,\tau)]\mathrm{d}x\Big| \le \frac{\sigma_{\max}}{\varepsilon_{\min}}\int_{\Omega_i} [\varepsilon(x)J_{\mathrm{EI}}^2(x,\tau) + \mu(x)J_{\mathrm{HI}}^2(x,\tau)]\mathrm{d}x.$$

Let

$$W_i(\tau) = \int_{\Omega_i} [\varepsilon(x)J_{\mathrm{EI}}^2(x,\tau) + \mu(x)J_{\mathrm{HI}}^2(x,\tau)]\mathrm{d}x,$$

then the above inequality will change into

$$\frac{\mathrm{d}W_i(\tau)}{\mathrm{d}\tau} \le \frac{\sigma_{\max}}{\varepsilon_{\min}}W_i(\tau).$$

Multiplied by $\mathrm{e}^{-\tau}$ on both sides, the above inequality changes into

$$\frac{\mathrm{d}[\mathrm{e}^{-\tau}W_i(\tau)]}{\mathrm{d}\tau} \le 0.$$

Then $\mathrm{e}^{-\tau}W_i(\tau) \le W_i(0) \to 0$.

Therefore, on Ω_i, $\boldsymbol{J}_{\mathrm{EI}}(x,\tau) \to \mathbf{0}$ and $\boldsymbol{J}_{\mathrm{HI}}(x,\tau) \to \mathbf{0}$. Letting i equal 1, 2, $\cdots$, m separately, we can obtain that $\boldsymbol{J}_{\mathrm{EI}}(x,\tau) \to \mathbf{0}$ and $\boldsymbol{J}_{\mathrm{HI}}(x,\tau) \to \mathbf{0}$ on Ω. Owing to the arbitrariness of τ, $\boldsymbol{J}_{\mathrm{E}}(x,t) \to \mathbf{0}$ and $\boldsymbol{J}_{\mathrm{H}}(x,t) \to \mathbf{0}$, $\forall(x,t) \in \Omega\times Q$.

Then we can prove the existence theorem of the problem P.

Theorem 6.2 For any $\boldsymbol{F}(x,t) \in L^2(\Omega,Q)$, there exists a weak solution to the problem P.

Proof From theorem 6.1, we know that, for any $\boldsymbol{F}(x,t) \in L^2(\Omega,Q)$, there exists a sequence $\{\boldsymbol{F}^n\}_{n=1}^\infty \subset R(\Lambda)$, such that $\lim\limits_{n\to\infty}\boldsymbol{F}^n = \boldsymbol{F}$. Accordingly, there exists a sequence $(\boldsymbol{E}^n,\boldsymbol{H}^n) \in \mathcal{K}^0(\Omega,Q)$, which satisfy

$$\nabla\times\boldsymbol{H}^n(x,t) - \varepsilon(x)\dot{\boldsymbol{E}}^n(x,t) - \sigma(x)\boldsymbol{E}^n(x,t) = \boldsymbol{F}^n(x,t), \qquad n = 1,2,\cdots.$$

From Theorem 5.1, we can deduce that, for any natural number n and m, $(\boldsymbol{E}^n, \boldsymbol{H}^n)$, $(\boldsymbol{E}^m, \boldsymbol{H}^m) \in \mathcal{K}^0(\Omega, Q)$ satisfy

$$\int_Q \int_\Omega [\varepsilon(x)|\boldsymbol{E}^n(x,t) - \boldsymbol{E}^m(x,t)|^2 + \mu(x)|\boldsymbol{H}^n(x,t) - \boldsymbol{H}^m(x,t)|^2]\mathrm{d}x\mathrm{d}t \leq C \int_Q \int_\Omega |\boldsymbol{F}^n(x,t) - \boldsymbol{F}^m(x,t)|^2 \mathrm{d}x\mathrm{d}t. \quad (34)$$

$\{\boldsymbol{F}^n\}_{n=1}^\infty$ is a convergent sequence in $L^2(\Omega, Q)$, so it is a Cauchy sequence in $L^2(\Omega, Q)$. From the inequality (34), we know that $(\boldsymbol{E}^n,\ \boldsymbol{H}^n)$ $(n = 1, 2, \cdots)$ is a Cauchy sequence in $L^2(\Omega, Q)$. Owing the completeness of $L^2(\Omega, Q)$,[23] there must exist a pair $(\boldsymbol{E}, \boldsymbol{H}) \in L^2(\Omega, Q)$, such that

$$\lim_{n\to\infty} \boldsymbol{E}^n(x,t) = \boldsymbol{E}(x,t), \quad \text{and} \lim_{n\to\infty} \boldsymbol{H}^n(x,t) = \boldsymbol{H}(x,t).$$

Because $(\boldsymbol{E}^n, \boldsymbol{H}^n)$ should satisfy Eq.(15), we can obtain

$$\int_Q \int_\Omega [\varepsilon(x)\boldsymbol{E}^n(x,t)\cdot\dot{\boldsymbol{e}}(x,t) - \mu(x)\boldsymbol{H}^n(x,t)\cdot\dot{\boldsymbol{h}}(x,t) - \sigma(x)\boldsymbol{E}^n(x,t)\cdot\boldsymbol{e}(x,t)]\mathrm{d}x\mathrm{d}t = \int_Q \int_\Omega \boldsymbol{F}^n(x,t)\cdot\boldsymbol{e}(x,t)\mathrm{d}x\mathrm{d}t.$$

Let $n \to \infty$, then we will have

$$\int_Q \int_\Omega [\varepsilon(x)\boldsymbol{E}(x,t)\cdot\dot{\boldsymbol{e}}(x,t) - \mu(x)\boldsymbol{H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t) - \sigma(x)\boldsymbol{E}(x,t)\cdot\boldsymbol{e}(x,t)]\mathrm{d}x\mathrm{d}t = \int_Q \int_\Omega \boldsymbol{F}(x,t)\cdot\boldsymbol{e}(x,t)\mathrm{d}x\mathrm{d}t.$$

So it is clear that $(\boldsymbol{E}, \boldsymbol{H})$ is a weak solution of the problem P.

Thus we will prove the uniqueness theorem.

Theorem 6.3 After the homogenized transform of the initial-boundary conditions, problem P has only one weak solution in the sense of definition 2.

Proof If there were two distinct weak solutions to the problem P, with their difference denoted by $(\boldsymbol{E}, \boldsymbol{H})$, then $(\boldsymbol{E}, \boldsymbol{H}) \neq (\boldsymbol{0}, \boldsymbol{0})$. Equation (15) is reduced to

$$\int_Q \int_\Omega [\varepsilon(x)\boldsymbol{E}(x,t)\cdot\dot{\boldsymbol{e}}(x,t) - \mu(x)\boldsymbol{H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t) - \sigma(x)\boldsymbol{E}(x,t)\cdot\boldsymbol{e}(x,t)]\mathrm{d}x\mathrm{d}t = 0. \quad (35)$$

$(\boldsymbol{E}, \boldsymbol{H})$ also satisfies

$$\int_Q \int_\Omega [\boldsymbol{E}(x,t)\cdot\nabla\times\boldsymbol{h}(x,t) - \mu(x)\boldsymbol{H}(x,t)\cdot\dot{\boldsymbol{h}}(x,t)]\mathrm{d}x\mathrm{d}t = 0. \quad (36)$$

Comparing Eqs.(35) and (36) with Eqs.(23) and (24), it is obvious that the forms of the equations are same except the minus sign before $\sigma(x)$. So a similar method of proving theorem 6.1 is implemented, and a similar expression to expression (33) is derived as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_i} [\varepsilon(x)E_{\mathrm{I}}^2(x,\tau) + \mu(x)H_{\mathrm{I}}^2(x,\tau)]\mathrm{d}x \to -\int_{\Omega_i} \sigma(x)E_{\mathrm{I}}^2(x,\tau)\mathrm{d}x. \quad (37)$$

It should be noted that the difference between the strong and weak solutions is also considered in the abbreviated part.

The right-hand side of expression (37) is less than 0. If $E_{\mathrm{I}}(x,\tau)$ and $H_{\mathrm{I}}(x,\tau)$ do not tend to zero, the following inequality exists:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_i} [\varepsilon(x)E_{\mathrm{I}}^2(x,\tau) + \mu(x)H_{\mathrm{I}}^2(x,\tau)]\mathrm{d}x \leq 0.$$

Then

$$\int_{\Omega_i} [\varepsilon(x)E_{\mathrm{I}}^2(x,\tau) + \mu(x)H_{\mathrm{I}}^2(x,\tau)]\mathrm{d}x \leq \int_{\Omega_i} [\varepsilon(x)E_{\mathrm{I}}^2(x,0) + \mu(x)H_{\mathrm{I}}^2(x,0)]\mathrm{d}x = 0.$$

Therefore, on Ω_i, $\boldsymbol{E}_{\mathrm{I}}(x,\tau) \to \boldsymbol{0}$ and $\boldsymbol{H}_{\mathrm{I}}(x,\tau) \to \boldsymbol{0}$. Letting i equal 1, 2, $\cdots$, m separately, we can derive that on Ω, $\boldsymbol{E}_{\mathrm{I}}(x,\tau) \to \boldsymbol{0}$ and $\boldsymbol{H}_{\mathrm{I}}(x,\tau) \to \boldsymbol{0}$. Owing the arbitrariness of τ, $\boldsymbol{E}(x,t) \to \boldsymbol{0}$ and $\boldsymbol{H}(x,t) \to \boldsymbol{0}$, $\forall(x,t) \in \Omega \times Q$.

Then we discuss the stability of the weak solution to the problem P. According to theorem 6.2, for any $\boldsymbol{F}_1(x,t)$, $\boldsymbol{F}_2(x,t) \in L^2(\Omega, Q)$, there exist corresponding weak solutions $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2) \in L^2(\Omega, Q)$, and also exist $\{\boldsymbol{F}_1^n\}_{n=1}^\infty$, $\{\boldsymbol{F}_2^n\}_{n=1}^\infty \subset R(\Lambda)$, such that

$$\lim_{n\to\infty} \boldsymbol{F}_1^n = \boldsymbol{F}_1, \quad \lim_{n\to\infty} \boldsymbol{F}_2^n = \boldsymbol{F}_2, \quad \text{and} \quad \lim_{n\to\infty} (\boldsymbol{F}_1^n - \boldsymbol{F}_2^n) = \boldsymbol{F}_1 - \boldsymbol{F}_2.$$

There also exist sequences $(\boldsymbol{E}_1^n, \boldsymbol{H}_1^n)$, $(\boldsymbol{E}_2^n, \boldsymbol{H}_2^n) \in \mathcal{K}^0(\Omega, Q)$, $(n = 1, 2, \cdots)$. For any natural numbers n and m, the following inequalities exist:

$$\begin{aligned}
&\int_Q \int_\Omega [\varepsilon(x)|\boldsymbol{E}_1^n(x,t) - \boldsymbol{E}_1^m(x,t)|^2 + \mu(x)|\boldsymbol{H}_1^n(x,t) - \boldsymbol{H}_1^m(x,t)|^2]\mathrm{d}x\mathrm{d}t \\
&\leq C \int_Q \int_\Omega |\boldsymbol{F}_1^n(x,t) - \boldsymbol{F}_1^m(x,t)|^2 \mathrm{d}x\mathrm{d}t, \\
&\int_Q \int_\Omega [\varepsilon(x)|\boldsymbol{E}_2^n(x,t) - \boldsymbol{E}_2^m(x,t)|^2 + \mu(x)|\boldsymbol{H}_2^n(x,t) - \boldsymbol{H}_2^m(x,t)|^2]\mathrm{d}x\mathrm{d}t \\
&\leq C \int_Q \int_\Omega |\boldsymbol{F}_2^n(x,t) - \boldsymbol{F}_2^m(x,t)|^2 \mathrm{d}x\mathrm{d}t, \\
&\int_Q \int_\Omega [\varepsilon(x)|\boldsymbol{E}_1^n(x,t) - \boldsymbol{E}_2^n(x,t) - \boldsymbol{E}_1^m(x,t) + \boldsymbol{E}_2^m(x,t)|^2 \\
&+ \mu(x)|\boldsymbol{H}_1^n(x,t) - \boldsymbol{H}_2^n(x,t) - \boldsymbol{H}_1^m(x,t) + \boldsymbol{H}_2^m(x,t)|^2]\mathrm{d}x\mathrm{d}t \\
&\leq C \int_Q \int_\Omega |\boldsymbol{F}_1^n(x,t) - \boldsymbol{F}_2^n(x,t) - \boldsymbol{F}_1^m(x,t) + \boldsymbol{F}_2^m(x,t)|^2 \mathrm{d}x\mathrm{d}t. \qquad (38)
\end{aligned}$$

If $\|\boldsymbol{F}_1 - \boldsymbol{F}_2\|_{L^2(\Omega,Q)} \to 0$, when n and m are sufficiently large, the right-hand side of inequality (38) tends to 0. Accordingly, the left-hand side also tends to 0. So

$$\begin{aligned}
\|\boldsymbol{E}_1 - \boldsymbol{E}_2\|_{L^2(\Omega,Q)} &= \|\lim_{n\to\infty}(\boldsymbol{E}_1^n - \boldsymbol{E}_2^n)\|_{L^2(\Omega,Q)} \to 0, \\
\|\boldsymbol{H}_1 - \boldsymbol{H}_2\|_{L^2(\Omega,Q)} &= \|\lim_{n\to\infty}(\boldsymbol{H}_1^n - \boldsymbol{H}_2^n)\|_{L^2(\Omega,Q)} \to 0.
\end{aligned}$$

That is, the weak solution to the problem P is stable.

7. Application example

As an application of the initial-boundary value problem of time-varying linear electromagnetic field, we derive the analytical solution of the electric field produced by a single pulse of current in this section.

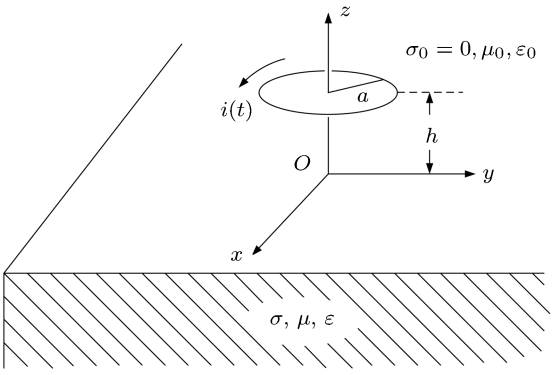


Fig.1. The plane of circular loop parallel to the conducting medium occupying half-space.

It should be noted that even one thousand examples cannot verify the previous well-posedness results, but only one counter example is sufficient to defeat them. The purpose of the following example is only to explain how the well-posedness results affect the analysis of an example.

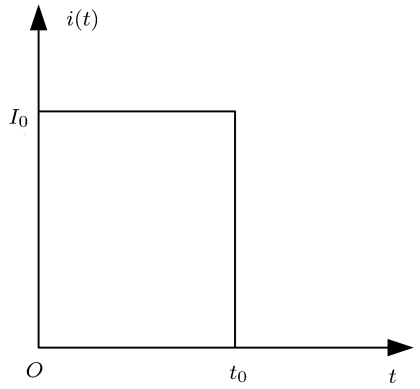


Fig.2. The waveform of the exciting current.

A similar example has been analysed in Ref.[25]. However, in their initial-boundary value problem, the equations do not include the displacement current density, and the interface condition (3) is replaced by the interface condition of the normal component of the magnetic flux density, which distinguish the

eddy-current electromagnetic problem from the time-varying electromagnetic problem.

In the case of Fig.1, a circular current loop is located in the half-space with zero conductivity, constant permeability μ_0 and permittivity ε_0. The other half-space is occupied by a conducting medium with constant electromagnetic parameters ε, σ and μ. The plane of the circular loop is parallel to the interface and an alternating current is flowing in it. The radius of the circular loop is a, and the distance between the centre of the circular loop and the interface is h. The cylindrical system of coordinates $O-\rho\phi z$ is suitable for analysis. The origin O is chosen to be fixed on the interface, the z-axis coincides with the symmetrical axis of the circular loop, and the relationship between the positive direction of the axis z and the reference direction of the current $i(t)$ in the loop obeys the right-hand screw rule. The waveform of $i(t)$ is shown in Fig.2.

According to the axisymmetrical feature of the field region, the electric field $\boldsymbol{E}$ is independent of coordinate variable ϕ, and there exists $\boldsymbol{E} = E_\phi \boldsymbol{e}_\phi$. Dividing the field region into three parts, and supposing the initial value of electric field to be zero, we can derive the initial-boundary value problem of Fig.1 according to Eqs.(1)–(10).

$$\begin{cases} \nabla \times \boldsymbol{H}_1 - \sigma \boldsymbol{E}_1 - \varepsilon \dot{\boldsymbol{E}}_1 = \boldsymbol{0}, & (39) \\ \nabla \times \boldsymbol{E}_1 + \mu \dot{\boldsymbol{H}}_1 = \boldsymbol{0}, \quad z < 0, & (40) \end{cases}$$

$$\begin{cases} \nabla \times \boldsymbol{H}_2 - \varepsilon_0 \dot{\boldsymbol{E}}_2 = \boldsymbol{0}, & (41) \\ \nabla \times \boldsymbol{E}_2 + \mu_0 \dot{\boldsymbol{H}}_2 = \boldsymbol{0}, \quad 0 < z < h, & (42) \end{cases}$$

$$\begin{cases} \nabla \times \boldsymbol{H}_3 - \varepsilon_0 \dot{\boldsymbol{E}}_3 = \boldsymbol{0}, & (43) \\ \nabla \times \boldsymbol{E}_3 + \mu_0 \dot{\boldsymbol{H}}_3 = \boldsymbol{0}, \quad z > h, & (44) \end{cases}$$

$$\begin{cases} \boldsymbol{e}_z \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}, & (45) \\ \boldsymbol{e}_z \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0}, \quad z = 0, & (46) \end{cases}$$

$$\begin{cases} \boldsymbol{e}_z \times (\boldsymbol{E}_3 - \boldsymbol{E}_2) = \boldsymbol{0}, & (47) \\ \boldsymbol{e}_z \times (\boldsymbol{H}_3 - \boldsymbol{H}_2) = I_0[u(t) - u(t-t_0)]\delta(\rho - a)\boldsymbol{e}_\phi, \quad z = h, & (48) \end{cases}$$

$$\lim_{r \to \infty} \boldsymbol{E}_i = \boldsymbol{0}, \quad (49)$$

$$\boldsymbol{E}_i|_{t=0} = \boldsymbol{0}, \quad (50)$$

$$\boldsymbol{H}_i|_{t=0} = \boldsymbol{0}, \quad (51)$$

where $i = 1, 2, 3$, $u(t)$ is a unit step function, δ is the Dirac δ-function, and $r = \sqrt{\rho^2 + z^2}$.

The derived well-posedness results indicate that the above problem can be solved, and the solution is unique. Therefore, the solution can be calculated by any appropriate method, including analytical and numerical methods. In addition, when the accuracy of measurements or the word length limit of computers causes some errors in the initial data, the results of numerical calculation will not vary greatly from the accurate ones.

Next, a simple method is chosen to derive the analytical solution.

Firstly, according to the initial conditions (50) and (51), the Laplace transforms of Eqs.(39)–(49) with respect to t can be obtained.

Secondly, by an equivalent transformation, the initial-boundary value problem changes into

$$\nabla^2 \boldsymbol{E}_i + k_i^2 \boldsymbol{E}_i = \boldsymbol{0}, \quad (52)$$

$$\nabla \cdot \boldsymbol{E}_i = 0, \quad (53)$$

$$\begin{cases} \boldsymbol{e}_z \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}, & (54) \\ \boldsymbol{e}_z \times \left(\dfrac{1}{\mu}\nabla \times \boldsymbol{E}_1 - \dfrac{1}{\mu_0}\nabla \times \boldsymbol{E}_2\right) = \boldsymbol{0}, & z = 0, \quad (55) \end{cases}$$

$$\begin{cases} \boldsymbol{e}_z \times (\boldsymbol{E}_3 - \boldsymbol{E}_2) = \boldsymbol{0}, & (56) \\ \boldsymbol{e}_z \times (\nabla \times \boldsymbol{E}_2 - \nabla \times \boldsymbol{E}_3) = sI(s)\mu_0\delta(\rho - a)\boldsymbol{e}_\phi, & z = h, \quad (57) \end{cases}$$

$$\lim_{r\to\infty} \boldsymbol{E}_i = \boldsymbol{0}, \quad (58)$$

where $i = 1, 2, 3$, $k_1^2 = -\varepsilon\mu s^2 - \mu\sigma s$, $k_2^2 = k_3^2 = -\varepsilon_0\mu_0 s^2$, and $I(s) = (I_0/s)(1 - \mathrm{e}^{-t_0 s})$.

Thirdly, supposing $u_1 = (\lambda^2 + \varepsilon\mu s^2 + \mu\sigma s)^{1/2}$ and $u_2 = (\lambda^2 + \varepsilon_0\mu_0 s^2)^{1/2}$, and using the separation of variables, the general solution expressions of the electric field components can be obtained as follows:

$$E_{1\phi} = \int_0^\infty C_1 \mathrm{J}_1(\lambda\rho)\mathrm{e}^{u_1 z}\mathrm{d}\lambda, \quad z < 0, \quad (59)$$

$$E_{2\phi} = \int_0^\infty (C_{21}\mathrm{e}^{u_2 z} + C_{22}\mathrm{e}^{-u_2 z})\mathrm{J}_1(\lambda\rho)\mathrm{d}\lambda, \quad 0 < z < h, \quad (60)$$

$$E_{3\phi} = \int_0^\infty C_3 \mathrm{J}_1(\lambda\rho)\mathrm{e}^{-u_2 z}\mathrm{d}\lambda, \quad z > h. \quad (61)$$

After deriving the coefficients C_1, C_{21}, C_{22}, C_3 by using boundary conditions (54)–(57), the expressions of electric field on complex number field are obtained as follows:

$$\boldsymbol{E}_1 = -\mu a s I(s) \int_0^\infty \lambda \mathrm{J}_1(\lambda a)\mathrm{J}_1(\lambda\rho)\frac{\mathrm{e}^{u_1 z - u_2 h}}{\mu_r u_2 + u_1}\mathrm{d}\lambda\boldsymbol{e}_\phi, \quad z < 0, \quad (62)$$

$$\boldsymbol{E}_2 = -\frac{\mu_0 a s I(s)}{2} \int_0^\infty \lambda \mathrm{J}_1(\lambda a)\mathrm{J}_1(\lambda\rho)\mathrm{e}^{-u_2 h}\Big[\frac{\mathrm{e}^{-u_2 z}(\mu_r u_2 - u_1)}{u_2(\mu_r u_2 + u_1)} + \frac{\mathrm{e}^{u_2 z}}{u_2}\Big]\mathrm{d}\lambda\boldsymbol{e}_\phi, \quad 0 < z < h, \quad (63)$$

$$\boldsymbol{E}_3 = -\frac{\mu_0 a s I(s)}{2} \int_0^\infty \lambda \mathrm{J}_1(\lambda a)\mathrm{J}_1(\lambda\rho)\mathrm{e}^{-u_2 z}\Big[\frac{\mathrm{e}^{-u_2 h}(\mu_r u_2 - u_1)}{u_2(\mu_r u_2 + u_1)} + \frac{\mathrm{e}^{u_2 h}}{u_2}\Big]\mathrm{d}\lambda\boldsymbol{e}_\phi, \quad z > h. \quad (64)$$

Lastly, after the inverse Laplace transform,[26] the expression of the electric field in time domain can be derived. All the solutions (62)–(64) have the uniform structure $\boldsymbol{E}(s) = I(s)\boldsymbol{M}(s)$, where $I(s) = \mathrm{L}[\mathrm{i}(t)]$. Therefore, $\boldsymbol{e}(t) = \mathrm{L}^{-1}[\boldsymbol{E}(s)] = i(t) * \boldsymbol{m}(t)$, where $\boldsymbol{m}(t) = \mathrm{L}^{-1}[\boldsymbol{M}(s)]$, and the convolution algorithm '$*$' can be calculated by the approach given in Ref.[27].

We can easily obtain the following results from the solution. The first one is that the derived solution satisfies Eqs.(39)–(51), implying the existence of the solution to the problem. The second one is that the coefficients of the solutions are uniquely determined in the solving process, showing the uniqueness of the solution. The last one is that the fields will vary slightly when the range of the pulse I_0 or the width of pulse t_0 varies slightly, indicating the stability of the solution.

8. Conclusion

In this paper, the initial-boundary value problem of the time-varying linear electromagnetic field with different inhomogeneous boundary conditions in the multi-medium region has been studied. On the basis of the solution definitions, homogenized transform of initial-boundary conditions and introduction of a linear operator Λ, it is proved that the solution to a time-varying linear electromagnetic problem is existential, unique and stable. The results can provide theoretical support for the analysis and calculation of the time-varying linear electromagnetic field.

Appendix: Existence of $\mathcal{K}(\Omega, Q)$

A function space is said to be existential if it is non-empty.

According to the mathematical form of space $\mathcal{K}(\Omega, Q)$ in Section 2, its element should satisfy the following constraint equation:

$$\nabla \times \boldsymbol{E}(x,t) + \mu(x)\dot{\boldsymbol{H}}(x,t) = \boldsymbol{0}, \qquad (x,t) \in \Omega \times Q. \tag{A1}$$

Integrating over Ω the scalar product of Eq.(A1) by $\boldsymbol{H}(x,t)$, yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mu(x)H^2(x,t)\,\mathrm{d}x + \int_{\Omega} [\nabla \times \boldsymbol{E}(x,t)] \cdot \boldsymbol{H}(x,t)\,\mathrm{d}x = 0. \tag{A2}$$

After its rearrangement, Eq.(A2) changes into

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mu H^2\,\mathrm{d}x + \int_{\Omega} (\nabla \times \boldsymbol{H}) \cdot \boldsymbol{E}\,\mathrm{d}x + \oint_{\Gamma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n}\,\mathrm{d}x \\ &+ \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_{\Gamma_{ij}} (\boldsymbol{E}_i \times \boldsymbol{H}_i - \boldsymbol{E}_j \times \boldsymbol{H}_j) \cdot \boldsymbol{n}_{ij}\,\mathrm{d}x = 0. \end{aligned} \tag{A3}$$

If the existence of $\mathcal{K}(\Omega, Q)$ still can be proved when some conditions which are uncontradicted with $\mathcal{K}(\Omega, Q)$ are added, the primary $\mathcal{K}(\Omega, Q)$ exists definitely. Therefore, letting $\nabla \times \boldsymbol{H} = \varepsilon\dot{\boldsymbol{E}} + \sigma\boldsymbol{E}$, and substituting it into Eq.(A3), we can derive

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mu H^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \varepsilon E^2\,\mathrm{d}x + \int_{\Omega} \sigma E^2\,\mathrm{d}x \\ &+ \oint_{\Gamma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n}\,\mathrm{d}x + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_{\Gamma_{ij}} (\boldsymbol{E}_i \times \boldsymbol{H}_i - \boldsymbol{E}_j \times \boldsymbol{H}_j) \cdot \boldsymbol{n}_{ij}\,\mathrm{d}x = 0. \end{aligned} \tag{A4}$$

Integrating both sides of Eq.(A4) with respect to t over $(0,t)$ gives

$$\begin{aligned} \frac{1}{2}\int_{\Omega} \mu H^2\,\mathrm{d}x = &-\frac{1}{2}\int_{\Omega} \varepsilon E^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega} \mu H_0^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega} \varepsilon E_0^2\,\mathrm{d}x - \int_0^t \int_{\Omega} \sigma E^2\,\mathrm{d}x\mathrm{d}t \\ &- \int_0^t \oint_{\Gamma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n}\,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_0^t \int_{\Gamma_{ij}} (\boldsymbol{E}_i \times \boldsymbol{H}_i - \boldsymbol{E}_j \times \boldsymbol{H}_j) \cdot \boldsymbol{n}_{ij}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A5}$$

When $\boldsymbol{E}|_{\Gamma_{ij}}, \boldsymbol{H}|_{\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij}, Q)$, $\boldsymbol{E}|_{\Gamma}, \boldsymbol{H}|_{\Gamma} \in H^{1/2}(\Gamma, Q)$, there exist

$$\Big| \int_0^t \oint_{\Gamma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n}\,\mathrm{d}x\mathrm{d}t \Big| < \infty, \quad \text{and} \quad \Big| \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_0^t \int_{\Gamma_{ij}} (\boldsymbol{E}_i \times \boldsymbol{H}_i - \boldsymbol{E}_j \times \boldsymbol{H}_j) \cdot \boldsymbol{n}_{ij}\,\mathrm{d}x\mathrm{d}t \Big| < \infty.$$

In $\mathcal{K}(\Omega, Q)$, $(\boldsymbol{E}, \boldsymbol{H}) \in \mathcal{R}(\Omega, Q)$. Substituting the conditions in $\mathcal{R}(\Omega, Q)$ into Eq.(A5), we have

$$\begin{aligned} \frac{1}{2}\int_{\Omega} \mu H^2\,\mathrm{d}x = &-\frac{1}{2}\int_{\Omega} \varepsilon E^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega} \mu H_0^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega} \varepsilon E_0^2\,\mathrm{d}x - \int_0^t \int_{\Omega} \sigma E^2\,\mathrm{d}x\mathrm{d}t \\ &- \int_0^t \oint_{\Gamma} \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{H}\,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_0^t \int_{\Gamma_{ij}} \boldsymbol{E}_i \cdot \boldsymbol{K}_{ij}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A6}$$

The functions $\boldsymbol{K}_{ij}$ and $\boldsymbol{F}_{\mathrm{e}}$ can be chosen arbitrarily under the condition that $\boldsymbol{K}_{ij} \in H^{1/2}(\Gamma_{ij}, Q)$ and $\boldsymbol{F}_{\mathrm{e}} \in H^{1/2}(\Gamma, Q)$. So

$$\int_0^t \oint_{\Gamma} \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{H}\,\mathrm{d}x\mathrm{d}t < \infty,$$

and

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \int_0^t \int_{\Gamma_{ij}} \boldsymbol{E}_i \cdot \boldsymbol{K}_{ij}\,\mathrm{d}x\mathrm{d}t < \infty.$$

$\varepsilon(x)\boldsymbol{E}_0(x) \in I(\Omega)$, $\mu(x)\boldsymbol{H}_0(x) \in I^0(\Omega)$, and ε_i, μ_i are

continuous functions in Ω_i, so

$$\int_\Omega \mu H_0^2 \mathrm{d}x < \infty,$$

and

$$\int_\Omega \varepsilon E_0^2 \mathrm{d}x < \infty.$$

By treating t as a parameter, if $\boldsymbol{E} \in L^2(\Omega)$, there exist

$$\int_\Omega \varepsilon E^2 \mathrm{d}x < \infty,$$

and

$$\int_0^t \int_\Omega \sigma E^2 \mathrm{d}x\mathrm{d}t \leq \int_0^T \int_\Omega \sigma E^2 \mathrm{d}x\mathrm{d}t < \infty.$$

We can know that $\boldsymbol{H} \in L^2(\Omega)$ according to Eq.(A6).

From the above analysis, we can find that the constraints on $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{E}|_{\Gamma_{ij}}$, $\boldsymbol{H}|_{\Gamma_{ij}}$, $\boldsymbol{E}|_\Gamma$, $\boldsymbol{H}|_\Gamma$, $\boldsymbol{K}_{ij}$ and $\boldsymbol{F}_\mathrm{e}$ in $\mathcal{K}(\Omega, Q)$ all to ensure $\boldsymbol{H} \in L^2(\Omega)$.

Integrating both sides of Eq.(A1) with respect to t over $(0, t)$, yields

$$\boldsymbol{H}(x,t) = \boldsymbol{H}_0(x) - \frac{1}{\mu(x)} \int_0^t \nabla \times \boldsymbol{E}(x,t) \mathrm{d}x. \quad \text{(A7)}$$

If $\boldsymbol{H}_0$ and $\boldsymbol{E}$ are given in the inner region of Ω except the exterior boundary and the interfaces, $\boldsymbol{H}$ can be determined by Eq.(A7). When $\boldsymbol{E} \in \mathcal{U}(\Omega, Q)$ and $\boldsymbol{H}_0 \in H^1(\Omega)$, the function $\boldsymbol{H}$ determined by them is a function in $\mathcal{V}(\Omega, Q)$.

According to the above analysis, we can give a method to construct the elements of $\mathcal{K}(\Omega, Q)$. The procedure is as follows: firstly, let $(\boldsymbol{E}, \boldsymbol{H})$ equal $(\boldsymbol{E}_0, \boldsymbol{H}_0)$ at the initial moment; secondly, let $\boldsymbol{E}$ equal any $H^{1/2}(\Gamma, Q)$ function on the exterior boundary which satisfies $\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{F}_\mathrm{e}$, and equal any $H^{1/2}(\Gamma_{ij}, Q)$ function on the interfaces which satisfies $\boldsymbol{n}_{ij} \times (\boldsymbol{E}_j - \boldsymbol{E}_i) = \boldsymbol{0}$; thirdly, let $\boldsymbol{H}$ equal any $H^{1/2}(\Gamma, Q)$ function on the exterior boundary, and equal any $H^{1/2}(\Gamma_{ij}, Q)$ function on the interfaces which satisfies $\boldsymbol{n}_{ij} \times (\boldsymbol{H}_j - \boldsymbol{H}_i) = \boldsymbol{K}_{ij}$; Lastly, given any $\boldsymbol{E} \in \mathcal{U}(\Omega, Q)$ in the inner region of Ω except the exterior boundary and the interfaces, the value of $\boldsymbol{H}$ in the inner region of Ω except the exterior boundary and the interfaces can be determined according to Eq.(A7).

Thus, the elements of $\mathcal{K}(\Omega, Q)$ can be constructed, which means that $\mathcal{K}(\Omega, Q)$ is existential.

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