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Some new exact solutions to the Burgers–Fisher equation and generalized Burgers–Fisher equation

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Some new exact solutions of the Burgers–Fisher equation and generalized Burgers–Fisher equation have been obtained by using the first integral method. These solutions include exponential function solutions, singular solitary wave solutions and some more complex solutions whose figures are given in the article. The result shows that the first integral method is one of the most effective approaches to obtain the solutions of the nonlinear partial differential equations.

Keywords: the first integral method, Burgers–Fisher equation, exact solutions
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1. Introduction

In recent years, many new methods for finding the analytical solutions of the nonlinear partial differential equations have been developed such as variable separation method,[1] the tanh function expansion method,[2,3] Jacobi elliptic function expansion method,[4−6] the homogeneous balance method,[7,8] the Hirota bilinear method[9] and the first integral method,[10,11] etc.

In this paper, according to the first integral method, we deal with the Burgers–Fisher (BF) equation[12]

$$u_t + auu_x - du_{xx} + bu(u-1) = 0, \qquad (1)$$

and generalized Burgers–Fisher (GBF) equation[7]

$$u_t + auu_x - du_{xx} + bu(u^2-1) = 0, \qquad (2)$$

where a, b and d are non-negative parameters. Supposing $b = 0$, these two equations were originally introduced by Burgers[13] to model one-dimensional turbulence and to describe waves in fluid-filled viscous elastic tubes and magneto hydrodynamic waves in a medium with finite electrical conductivity.[14] They were also used to investigate sound waves in a viscous medium by Lighthill.[15] With all three parameters positive, Eq.(1) corresponds to the Burgers equation with nonlinear reaction. On the other hand, Eq.(1) can be considered as a modified Fisher equation[16]

$$u_t - du_{xx} + bu(u-1) = 0.$$

As can be seen, the BF equation and GBF equation include the effects of nonlinear advection, linear diffusion and non-linear logistic reaction. With different combinations of parameters, we can obtain six equations which play fundamental roles in the mathematical modelling in a range of important physical and engineering phenomena. For example, $b = 0$ gives the Burgers equation; $a = 0$ leads to the Fisher equation; $d = 0$ leads to the diffusionless Burgers equation with nonlinear reaction; for $a = 0$ and $d = 0$ Eqs.(1) and (2) correspond to logistic ordinary differential equation; $a = 0$ and $b = 0$ leads to the linear diffusion equation; and $d = 0$ and $b = 0$ in Eq.(1) leads to the diffusionless Burgers equation. Consequently, any finite difference in the scheme constructed for Eq.(1) also provides corresponding discrete model for the above indicated six ordinary and partial differential equations.

Chen and Zhang[17] derived some travelling wave solutions of Eq.(2) with the tanh method, and Kaya and Salah[18] gave a numerical simulation and explicit solutions of GBF by the adomian decomposition method (ADM). Construction and analysis for the solutions of BF was given by Micken and Gumel.[12] Several exact solutions containing some parameters for the BF equation and GBF equation were obtained by Wang and Bai.[19] In the following paragraphs, by us-

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ing the first integral method[20] which was introduced by Feng[10] in 2002, we gain some new exact solutions of BF and GBF equations, these new results have not been presented previously.

The main idea of the first integral method is to apply the theory of commutative algebra to the search for the solutions of nonlinear partial equations. Assume that Eq.(1) has the travelling wave solution in the form

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{3}$$

where c is the velocity of the travelling wave. Substituting Eq.(3) into Eq.(1) yields

$$u''(\xi) = -\frac{c}{d}u'(\xi) + \frac{a}{d}uu'(\xi) + \frac{b}{d}u^2(\xi) - \frac{b}{d}u(\xi). \tag{4}$$

Let $\nu = u, y = u'$, then Eq.(4) is equivalent to

$$\begin{aligned} &\nu' = y, \\ &y' = Ay + B\nu y + D\nu^2 - D\nu, \end{aligned} \tag{5}$$

in which $A = -\frac{c}{d}, B = \frac{a}{d}, D = \frac{b}{d}$. If we can find the two first integrals of Eq.(5) under the same conditions, the general solution of Eq.(5) can be obtained directly. But, in general, it is rather difficult to achieve, even for one first integral. The idea of the first integral method is using the Division Theorem to seek one first integral of Eq.(5), which can reduce Eq.(4) to a first-order integrable ordinary differential equation, and then we can obtain the exact solutions of Eq.(5) by means of the direct integral method.

2. Exact solutions of BF equation

According to the first integral method as described above, we suppose that $\nu = \nu(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to Eq.(5), and $S(\nu, y) = \sum_{i=0}^{m} a_i(\nu)y^i$ is an irreducible polynomial in the complex domain $C[\nu, y]$, such that

$$S[\nu(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(\nu)y^i = 0, \tag{6}$$

where $a_i(\nu)(i = 0, 1, \ldots, m)$ are polynomials of ν and $a_m(\nu) \neq 0$. Equation (6) is called the first integral of Eq.(5). Generally, we assume $m = 2$ in Eq.(6). Note that $\mathrm{d}S/\mathrm{d}\xi$ is a polynomial of ν and y, and according to Eq.(6), we know $\mathrm{d}S/\mathrm{d}\xi = 0$. Due to the Division Theorem, there exists a polynomial $h(\nu, y) = m(\nu) + n(\nu)y$ in the complex domain $C[\nu, y]$, such that

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}\xi} &= \frac{\partial S}{\partial \nu} \cdot \frac{\partial \nu}{\partial \xi} + \frac{\partial S}{\partial y} \cdot \frac{\partial y}{\partial \xi} \\ &= [m(\nu) + n(\nu)y] \left[\sum_{i=0}^{2} a_i(\nu)y^i \right], \end{aligned} \tag{7}$$

where $m(\nu)$ and $n(\nu)$ are polynomials of ν. Substituting Eq.(5) into Eq.(7), and comparing the coefficients of $y^i(i = 0, 1, 2, 3)$ on both sides of this equation, we obtain

$$\begin{aligned} &a_2'(\nu) - n(x)a_2(\nu) = 0, \\ &-n(\nu)a_1(\nu) + 2Aa_2(\nu) + 2B\nu a_2(\nu) - m(\nu)a_2(\nu) + a_1'(\nu) = 0, \\ &a_0'(\nu) + B\nu a_1(\nu) - n(\nu)a_0(\nu) - 2D\nu a_2(\nu) + Aa_1(\nu) - m(\nu)a_1(\nu) + 2D\nu^2 a_2(\nu) = 0, \\ &D\nu^2 a_1(\nu) - D\nu a_1(\nu) - m(\nu)a_0(\nu) = 0. \end{aligned} \tag{8}$$

Since $a_i(\nu), \quad n(\nu)$ are polynomials, we obtain that $n(\nu) = 0$, and $a_2(\nu)$ is a constant from the first equation of Eq.(8), and without loss of generality we take $a_2(\nu) = 1$. Inserting $n(\nu) = 0$ and $a_2(\nu) = 1$ into Eq.(8) yields

$$2A + 2B\nu - m(\nu) + a_1'(\nu) = 0, \tag{9a}$$

$$a_0'(\nu) + B\nu a_1(\nu) - 2D\nu + Aa_1(\nu) - m(\nu)a_1(\nu) + 2D\nu^2 = 0, \tag{9b}$$

$$D\nu^2 a_1(\nu) - D\nu a_1(\nu) - m(\nu)a_0(\nu) = 0. \tag{9c}$$

Balancing the degrees of $a_1(\nu)$ and $m(\nu)$ according to Eq.(9a), we educe that when $\deg a_1(\nu) = 1$, then $\deg m(\nu) = 1$; when $\deg a_1(\nu) = 2$, then $\deg m(\nu) = 0$ or 1. Next we discuss the three cases separately.

2.1. In the case of $\deg m(\nu) = 0$ and $\deg a_1(\nu) = 2$

In this case, we assume that

$$m(\nu) = m, \quad a_1(\nu) = k_2\nu^2 + k_1\nu + k_0 (k_2 \neq 0), \tag{10}$$

where m, k_2, k_1, k_0 are constants to be determined. Substituting Eq.(10) into Eqs.(9a) and (9b), we obtain $m = k_1 + 2A$, $k_2 = B$ and

$$\begin{aligned} a_0(\nu) = & -\frac{1}{4}B^2\nu^4 + \frac{1}{3}(BA - 2D)\nu^3 + \frac{1}{2}(Ak_1 - Bk_0 + 2D + k_1^2)\nu^2 \\ & +(Ak_0 + k_1k_0)\nu + d_0, \end{aligned} \tag{11}$$

with d_0 an arbitrary integral constant. Substituting Eqs.(10) and (11) with $m(\nu) = m = k_1 + 2A$ into Eq.(9c), setting the coefficients of the same powers of ν to zero, we can obtain the following set of equations:

$$\begin{aligned} & \frac{1}{2}AB^2 + \frac{1}{4}k_1B^2 + DB = 0, \\ & -BD - \frac{2}{3}BA^2 + \frac{4}{3}AD + \frac{5}{3}Dk_1 - \frac{1}{3}k_1BA = 0, \\ & Dk_0 - \frac{1}{2}k_1^3 - k_1A^2 + ABk_0 - \frac{3}{2}Ak_1^2 - 2Dk_1 + \frac{1}{2}k_1k_0B - 2AD = 0, \\ & -Dk_0 - k_1^2k_0 - 2A^2k_0 - 3Ak_1k_0 = 0, \\ & -k_1d_0 - 2Ad_0 = 0. \end{aligned} \tag{12}$$

Solving these algebraic equations, we can obtain three groups of solutions

$$k_1 = -A, \quad k_0 = d_0 = 0, \quad B = A, \quad D = -\frac{A^2}{4}, \tag{13}$$

$$k_1 = -4A, \quad k_0 = d_0 = 0, \quad B = -4A, \quad D = -2A^2, \tag{14}$$

$$k_1 = -\frac{4}{7}A, \quad k_0 = -\frac{8A}{7}, \quad B = \frac{12}{7}A, \quad D = -\frac{30}{49}A^2, \quad d_0 = 0. \tag{15}$$

Inserting Eqs.(10), (11) and (13) and $a_2(\nu) = 1$ into Eq.(6), we obtain the solution of $y(\nu)$

$$y_1(\nu) = \frac{(-1 \pm \sqrt{2})A(\nu - 1)\nu}{2}. \tag{16}$$

Expression (16) is the first integral of Eq.(5). Combining Eq.(5) with $\nu = u, y = u'$ and Eq.(16), we have the travelling wave solution of Eq.(4):

$$u(\xi) = \frac{1}{1 + C_1 \exp\left(\dfrac{(-1 \pm \sqrt{2})A}{2}\xi\right)}, \tag{17}$$

where C_1 is an arbitrary integral constant. Noting that $A = -\dfrac{c}{d}, B = \dfrac{a}{d}, D = \dfrac{b}{d}$ and $\xi = x - ct$, the two exact solutions of Eq.(1) are obtained as follows:

$$u_1(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left\{\frac{1 - \sqrt{2}}{4} \times \frac{c}{d}(x - ct + \alpha_0)\right\}, \tag{18}$$

$$u_2(x,t) = \frac{1}{2} + \frac{1}{2} \times \tanh\left\{\frac{-1 - \sqrt{2}}{4} \cdot \frac{c}{d}(x - ct + \beta_0)\right\}, \tag{19}$$

where $\exp\left(\dfrac{\sqrt{2} - 1}{2} \cdot \dfrac{c}{d}\alpha_0\right) = C_1$ or $\exp\left(-\dfrac{\sqrt{2} + 1}{2} \cdot \dfrac{c}{d}\beta_0\right) = C_1$.

As for the case of Eq.(14), according to Eq.(6), we have

$$y_2(\nu) = (2\nu + 2 + 2\sqrt{2\nu^2 + 2\nu})A\nu. \tag{20}$$

So, the exact solution of Eq.(1) is

$$u_3(x,t) = \frac{2\exp[2(Ax - Act + AC_1)]}{8\exp(Ax - Act + AC_1) - 1 - 14\exp[2(Ax - Act + AC_1)]}, \tag{21}$$

where C_1 is an arbitrary integral constant, and $A = -c/d$.

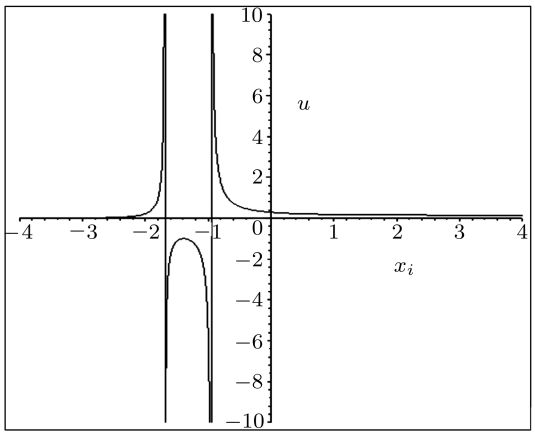


Fig.1. The plot of the travelling wave solution (21) for the FB equation with $A = 1$ and $C_1 = 0$.

Exponential function solution (21) is a new solution to Eq.(1) obtained by means of the present method, which has two singular points: $\xi = \ln\left(\frac{2}{7} \pm \frac{\sqrt{2}}{14}\right)$. Figure 1 shows the travelling wave solution Eq.(21) with $A = 1$ and $C_1 = 0$.

For the case of Eq.(15), the solution of $y(\nu)$ is

$$y_3(\nu) = \left(-\frac{6v}{7} - \frac{4}{7} \pm \frac{2\sqrt{18\nu^2 + 18\nu + 4}}{7}\right) A(v-1), \tag{22}$$

and in this case the explicit solution of Eq.(1) is difficult to be obtained by combining $\nu = u, y = u'$ and Eq.(22), but the implicit solution is

$$30A\xi - 35\ln\left(5\tan\left(\frac{1}{2}\operatorname{arcsec}(6u(\xi)+3)\right)^2 - 4\right) + 70\ln\left(2\tan\left(\frac{1}{2}\operatorname{arcsec}(6u(\xi)+3)\right) \pm \sqrt{2}\right)$$
$$\pm 28\sqrt{10}\operatorname{arctanh}\left(\frac{\sqrt{5}}{2}\tan\left(\frac{1}{2}\operatorname{arcsec}(6u(\xi)+3)\right)\right) + 30AC_1 = 0, \tag{23}$$

where C_1 is an arbitrary integral constant and $\xi = x - ct$, $A = -c/d$.

2.2. In the case of $\deg m(\nu) = 1$ and $\deg a_1(\nu) = 2$

In this case, we assume

$$m(\nu) = m\nu + n, \quad a_1(\nu) = k_2\nu^2 + k_1\nu + k_0 (k_2 \neq 0). \tag{24}$$

Inserting Eq.(24) into Eqs.(9a) and (9b), we deduce that $m(\nu) = (2k_2 + 2B)\nu + k_1 + 2A$ and

$$a_0(\nu) = \left(\frac{1}{2}k_2^2 + \frac{1}{4}Bk_2\right)\nu^4 + \left(\frac{1}{3}Ak_2 + k_1k_2 + \frac{1}{3}Bk_1 - \frac{2}{3}D\right)\nu^3$$
$$+ \left(\frac{1}{2}Ak_1 + k_0k_2 + \frac{1}{2}Bk_0 + D + \frac{1}{2}k_1^2\right)\nu^2 + (Ak_0 + k_0k_1)\nu + d_0, \tag{25}$$

where d_0 is an arbitrary integral constant. Substituting Eqs.(24), (25) and $m(\nu) = (2k_2 + 2B)v + k_1 + 2A$ into Eq.(9c), and setting the coefficients of the same powers of v to zero, yields

$$\begin{aligned}
&-\frac{1}{2}B^2k_2 - k_2^3 - \frac{3}{2}k_2^2B = 0,\\
&-\frac{5}{3}k_2^2A - \frac{5}{2}k_2^2k_1 + \frac{7}{3}Dk_2 - \frac{7}{6}BAk_2 - \frac{2}{3}B^2k_1 - \frac{35}{12}k_1k_2B + \frac{4}{3}BD = 0,\\
&-3k_2k_0B + \frac{5}{3}Dk_1 - \frac{10}{3}Ak_1k_2 - \frac{4}{3}Bk_1^2 - \frac{2}{3}A^2k_2 - \frac{5}{3}BAk_1\\
&-2k_0k_2^2 - 2k_2k_1^2 - 2BD + \frac{4}{3}AD - B^2k_0 - 3Dk_2 = 0,\\
&-\frac{1}{2}k_1^3 - 2AD - 3k_0k_1k_2 - A^2k_1 - \frac{3}{2}Ak_1^2 - 2Dk_1 - \frac{5}{2}Bk_0k_1\\
&+Dk_0 - 4k_0k_2A - 3BAk_0 = 0,\\
&-Dk_0 - 2A^2k_0 - 2Bd_0 - 3k_1k_0A - 2k_2d_0 - k_1^2k_0 = 0,\\
&-2Ad_0 - k_1d_0 = 0.
\end{aligned} \tag{26}$$

Solving Eqs.(26), we have four groups of solutions as

$$k_2 = -B, \quad k_1 = B, \quad k_0 = 0, \quad D = -\frac{1}{2}AB - \frac{1}{4}B^2, \quad d_0 = 0, \tag{27}$$

$$k_2 = \frac{3}{5}A, \quad k_1 = -\frac{4}{5}A, \quad k_0 = d_0 = 0, \quad D = \frac{6}{25}A^2, \quad B = -\frac{6}{5}A, \tag{28}$$

$$k_2 = \frac{1}{3}A, \quad k_1 = -\frac{2}{3}A, \quad k_0 = d_0 = 0, \quad D = \frac{2}{9}A^2, \quad B = -\frac{2}{3}A, \tag{29}$$

$$k_2 = 3A, \quad k_1 = -2A, \quad k_0 = -A, \quad D = -6A^2, \quad B = -6A, \quad d_0 = A^2. \tag{30}$$

Substituting Eqs.(27)–(30) into Eq.(6) in turn, we gain

$$y_4(\nu) = -\frac{1}{2}B\nu + \frac{1}{2}B\nu^2, \tag{31}$$

$$y_5(\nu) = \left(\frac{2}{5} - \frac{3\nu}{10} \pm \frac{\sqrt{-12\nu + 9\nu^2}}{10}\right) A\nu, \tag{32}$$

$$y_6(\nu) = \frac{A\nu}{3}, \quad y_7(\nu) = -\frac{1}{3}A\nu^2 + \frac{1}{3}A\nu, \tag{33}$$

$$y_8(\nu) = -A + A\nu, \quad y_9(\nu) = A\nu^2 - A\nu, \tag{34}$$

$$y_{10}(\nu) = \left(-\frac{1}{2} - \frac{3\nu}{2} \pm \frac{\sqrt{9\nu^2 - 6\nu - 3}}{2}\right) A(\nu - 1). \tag{35}$$

Expressions (31)–(35) are the first-integrals for Eq.(5). Integrating Eqs.(31)–(35), we obtain some exact solutions of Eq.(1) as follows.

For Eq.(31), we obtain

$$u_4(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{a}{4d}(x - ct + \xi_0)\right]. \tag{36}$$

For Eq.(32), we have

$$u_5(x,t) = -\frac{4}{3}\frac{\exp\left(-\frac{c}{5d}(x - ct + \xi_0)\right)}{\exp\left(\frac{c}{5d}(x - ct + \xi_0)\right) - 2}, \tag{37a}$$

$$u_5(x,t) = -\frac{4}{3}\frac{\exp\left(-\frac{c}{5d}(x - ct + \xi_0)\right)}{\exp\left(\frac{c}{5d}(x - ct + \xi_0)\right) + 2}. \tag{37b}$$

Figure 2 shows the travelling wave solution Eq.(37) with $c = d$, that is $A = 1$ and $\xi_0 = 0$.

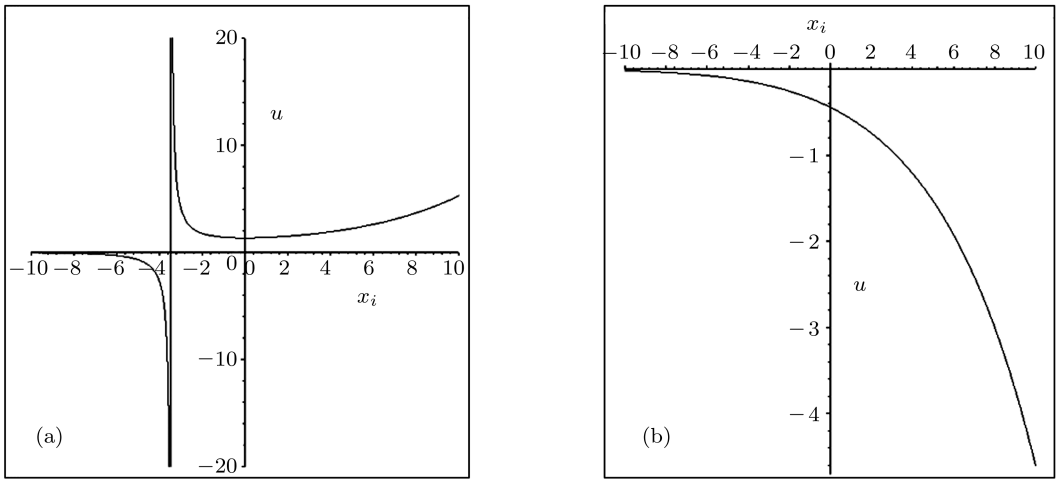


Fig.2. Plots of the travelling wave solutions (37a) and (37b) for the FB equation with $A = 1, C_1 = 0$ respectively, where (a) is for solution (37a), and (b) is for solution (37b).

For Eq.(33), we obtain

$$u_6(x,t) = C_1 \exp\left(-\frac{c}{d}(x-ct)\right) \tag{38}$$

and the solution $u_7(x,t)$ is the same as Eq.(36).

For Eq.(34), we obtain

$$u_8(x,t) = 1 + C_1 \exp\left(-\frac{c}{d}(x-ct)\right), \tag{39}$$

and the solution $u_9(x,t)$ is the same as Eq.(36).

For Eq.(35), we have

$$u_9(x,t) = \frac{1}{3}\frac{3\exp\left(-\frac{c}{d}(x-ct+\xi_0)\right) - 6 + 4\exp\left(\frac{c}{d}(x-ct+\xi_0)\right)}{\exp\left(-\frac{c}{d}(x-ct+\xi_0)\right) - 2}, \tag{40}$$

$$u_{10}(x,t) = -\frac{1}{6}\frac{\exp\left(-\frac{4c}{d}(x-ct+\xi_0)\right) + 3}{\exp\left(-\frac{2c}{d}(x-ct+\xi_0)\right) + 1}, \tag{41}$$

where $\xi_0,\ C_1$ are arbitrary integral constants. Equations (36)–(41) are all new solutions to Eq.(1).

Figure 3 shows the travelling wave solutions (40) and (41) with $c = d,\ \xi_0 = 0$. The solutions (38) and (39) are exponential function solutions, and Eq.(36) is a singular solitary wave solution. The solutions (37), (40), (41) are more complicated than the others, and we have plotted the two-dimensional figures such as Figs.1 and 2 by means of Maple.

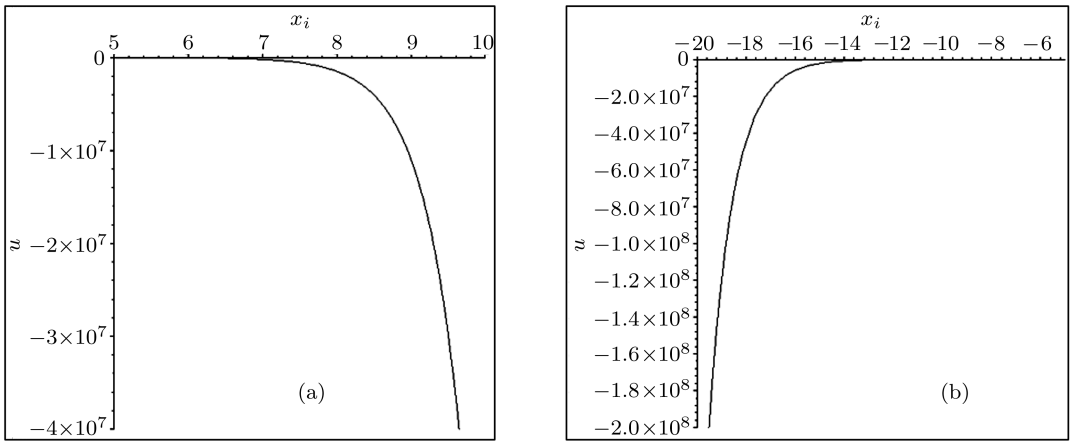


Fig.3. Plots of the travelling wave solutions (40) and (41) for BF equation with $c = d, \xi_0 = 0$, where (a) is for solution (40), and (b) is for solution (41).

2.3. In the case of $\deg m(\nu) = 1$ and $\deg a_1(\nu) = 1$

In this case, assume

$$m(\nu) = m\nu + n, \quad a_1(\nu) = a_1\nu + k_1, \tag{42}$$

where m, n, a_1, k_1 are constants to be determined. Substituting Eq.(42) into Eqs.(9a) and (9b), we have $m = 2B, n = a_1 + 2A$, that is $m(\nu) = 2B\nu + a_1 + 2A$ and

$$a_0(\nu) = -\frac{1}{3}(2D - Ba_1)\nu^3 - \frac{1}{2}(-a_1^2 - Bk_1 - 2D - Aa_1)\nu^2 + (Ak_1 + a_1k_1)\nu + d_0, \tag{43}$$

with d_0 an arbitrary integral constant. Substituting Eqs.(42) and (43) and $m(\nu) = 2B\nu + a_1 + 2A$ into Eq.(9c), setting the coefficients of the same powers of ν to zero, we obtain the following set of equations:

$$\begin{aligned}
&\frac{4}{3}BD - \frac{2}{3}B^2a_1 = 0,\\
&-B^2k_1 + \frac{4}{3}AD - \frac{5}{3}BAa_1 - \frac{4}{3}Ba_1^2 + \frac{5}{3}Da_1 - 2BD = 0,\\
&-A^2a_1 - 2Da_1 - \frac{3}{2}Aa_1^2 - \frac{5}{2}Ba_1k_1 - \frac{1}{2}a_1^3 - 3BAk_1 - 2AD + Dk_1 = 0,\\
&-Dk_1 - a_1^2k_1 - 2A^2k_1 - 2Bd_0 - 3Aa_1k_1 = 0,\\
&-a_1d_0 - 2Ad_0 = 0.
\end{aligned} \tag{44}$$

Solving these algebraic equations, we have two groups of results as follows:

$$a_1 = \frac{2D}{B}, \quad k_1 = -\frac{2D}{B}, \quad A = -\frac{D}{B}, \quad d_0 = \frac{D^2}{B^2}. \tag{45}$$

$$a_1 = \frac{2D}{B}, \quad k_1 = 0, \quad A = -\frac{B^2 + D}{B}, \quad d_0 = 0. \tag{46}$$

Substituting Eqs.(45) and (46) into Eq.(6), we obtain the solutions of $y(\nu)$

$$y_1(\nu) = -\frac{D(v-1)}{B} \quad \text{and} \quad y_2(\nu) = -\frac{D}{B}v. \tag{47}$$

Equations (47) are the first-integral to Eq.(5). Integrating Eq.(47), the solutions of Eq.(1) are obtained:

$$u_1(x,t) = 1 + \exp\left(-\frac{b}{a}(x - ct + \xi_0)\right), \tag{48}$$

$$u_2(x,t) = \exp\left(-\frac{b}{a}(x - ct + \xi_0)\right), \tag{49}$$

where ξ_0, C_1 are arbitrary integral constants. The solutions (48) and (49) are both the exponential function solutions to Eq.(1).

3. Exact solutions of GBF equation

The GBF equation is different from the BF equation, which has only one case. In fact, in the case of $\deg m(\nu) = 1$, $\deg a_1(\nu) = 1$ and $\deg m(\nu) = 0$, $\deg a_1(\nu) = 2$, we have the result of $B = 0$ or $D = 0$, that is $a = 0$ or $b = 0$. These results are contradictory to Eq.(2). So we just consider the case of $\deg m(\nu) = 1$ and $\deg a_1(\nu) = 2$.

In this case, we assume

$$m(\nu) = m\nu + n, \quad a_1(\nu) = k_2\nu^2 + k_1\nu + k_0 (k_2 \neq 0). \tag{50}$$

According to Eqs.(9a) and (9b), we deduce that $m(\nu) = (2k_2 + 2B)\nu + k_1 + 2A$ and

$$\begin{aligned}
a_0(\nu) = &\left(\frac{1}{2}k_2^2 + \frac{1}{4}Bk_2 - \frac{D}{2}\right)\nu^4 + \left(\frac{1}{3}Ak_2 + k_1k_2 + \frac{1}{3}Bk_1\right)\nu^3\\
&+ \left(\frac{1}{2}Ak_1 + k_0k_2 + \frac{1}{2}Bk_0 + D + \frac{1}{2}k_1^2\right)\nu^2 + (Ak_0 + k_0k_1)\nu + d_0,
\end{aligned} \tag{51}$$

where d_0 is an arbitrary integral constant. Substituting Eqs.(50), (51) and $m(\nu)$ into Eq.(9c), and setting the coefficients of the same powers of v to zero, yields

$$
\begin{aligned}
&-\frac{1}{2}B^2k_2-k_2^3-\frac{3}{2}k_2^2B+2Dk_2+BD=0,\\
&-\frac{5}{3}k_2^2A-\frac{5}{2}k_2^2k_1-\frac{7}{6}BAk_2-\frac{2}{3}B^2k_1-\frac{35}{12}k_1k_2B+\frac{3}{2}Dk_1+AD=0,\\
&-\frac{1}{2}k_1^3-2AD-3k_0k_1k_2-A^2k_1-\frac{3}{2}Ak_1^2-2Dk_1-\frac{5}{2}Bk_0k_1-4k_0k_2A-3BAk_0=0,\\
&-3k_2k_0B+\frac{5ABk_1}{3}-\frac{10Ak_1k_2}{3}-\frac{4Bk_1^2}{3}-\frac{2A^2k_2}{3}-2k_0k_2^2-2k_2k_1^2-2BD-B^2k_0-3Dk_2=0,\\
&-Dk_0-2A^2k_0-2Bd_0-3k_1k_0A-2k_2d_0-k_1^2k_0=0,\\
&-2Ad_0-k_1d_0=0.
\end{aligned}
\tag{52}
$$

Solving these algebraic equations, we can obtain some results as follows:

$$k_2=-\frac{2A}{3}-\frac{2B}{3},\quad k_1=-\frac{2A}{3}-\frac{2B}{3},\quad k_0=d_0=0,\quad D=\frac{2}{9}A^2+\frac{1}{9}AB-\frac{1}{9}B^2. \tag{53}$$

$$k_2=-3A,\quad k_1=-2A,\quad k_0=A,\quad B=6A,\quad D=-3A^2,\quad d_0=\frac{A^2}{2}. \tag{54}$$

$$k_2=\frac{3A}{4},\quad k_1=-\frac{A}{2},\quad k_0=-\frac{5A}{4},\quad B=-\frac{3A}{2},\quad D=-\frac{3}{4}A^2,\quad d_0=0. \tag{55}$$

$$k_2=\frac{2A}{3},\quad k_1=-A,\quad k_0=d_0=0,\quad B=-\frac{7A}{6},\quad D=-\frac{1}{6}A^2. \tag{56}$$

$$k_2=\frac{4A}{5},\quad k_1=-\frac{4A}{5},\quad k_0=-\frac{8A}{5},\quad B=-\frac{7A}{5},\quad D=-\frac{6}{25}A^2,\quad d_0=0. \tag{57}$$

$$k_2=4A,\quad k_1=-2A,\quad k_0=-2A,\quad B=-7A,\quad D=-6A^2,\quad d_0=2A^2. \tag{58}$$

Substituting Eqs.(53), (54) and (55) into Eq.(6), we have the solutions of $y(\nu)$

$$y_1(\nu)=\frac{1}{3}(A+B)\nu+\frac{1}{3}(A+B)\nu^2, \tag{59}$$

$$y_2(\nu)=\left(\frac{3\nu}{2}-\frac{1}{2}\pm\frac{\sqrt{3\nu^2+2\nu-1}}{2}\right)A(\nu+1), \tag{60}$$

$$y_3(\nu)=\left(\frac{5}{8}-\frac{3\nu}{8}\pm\frac{\sqrt{-15\nu^2+10\nu+25}}{8}\right)A(\nu+1). \tag{61}$$

Equations (59), (60) and (61) are the first-integral of Eq.(5). By integrating these three expressions, the solutions of Eq.(1) are obtained

$$u_1(x,t)=-\frac{1}{2}-\frac{1}{2}\coth\left(\frac{c-a}{6d}(x-ct+\xi_0)\right), \tag{62}$$

$$u_2(x,t)=\frac{6+\exp\left(-2\dfrac{c}{d}(x-ct+\xi_0)\right)-2\sqrt{3}\exp\left(-\dfrac{c}{d}(x-ct+\xi_0)\right)}{6-\exp\left(-2\dfrac{c}{d}(x-ct+\xi_0)\right)+2\sqrt{3}\exp\left(-\dfrac{c}{d}(x-ct+\xi_0)\right)}, \tag{63a}$$

$$u_2(x,t)=\frac{6+\exp\left(-2\dfrac{c}{d}(x-ct+\xi_0)\right)+2\sqrt{3}\exp\left(-\dfrac{c}{d}(x-ct+\xi_0)\right)}{6-\exp\left(-2\dfrac{c}{d}(x-ct+\xi_0)\right)-2\sqrt{3}\exp\left(-\dfrac{c}{d}(x-ct+\xi_0)\right)}, \tag{63b}$$

$$
\begin{aligned}
u_3(x,t)=\ &\frac{4}{3}\tan\left[\pm2\sqrt{35}\arctan\left(\pm\frac{5}{2}+\frac{\sqrt{15}}{2}+\frac{1}{2}\tanh\left(\frac{\sqrt{21}A}{12}(x-ct)+\frac{\sqrt{21}A}{12}C_1\right)\right)\right.\\
&\left.\pm\frac{\sqrt{21}}{2}\tanh\left(\frac{\sqrt{21}A}{12}(x-ct)+\frac{\sqrt{21}A}{12}C_1\right)\right]+\frac{1}{3},
\end{aligned}
\tag{64}
$$

where all ξ_0 are arbitrary integral constants. The solutions (62), (63) and (64) are all new solutions to Eq.(2) obtained in this work. Figure 4 shows the travelling wave solution (63) with $c=d,\xi_0=0$.

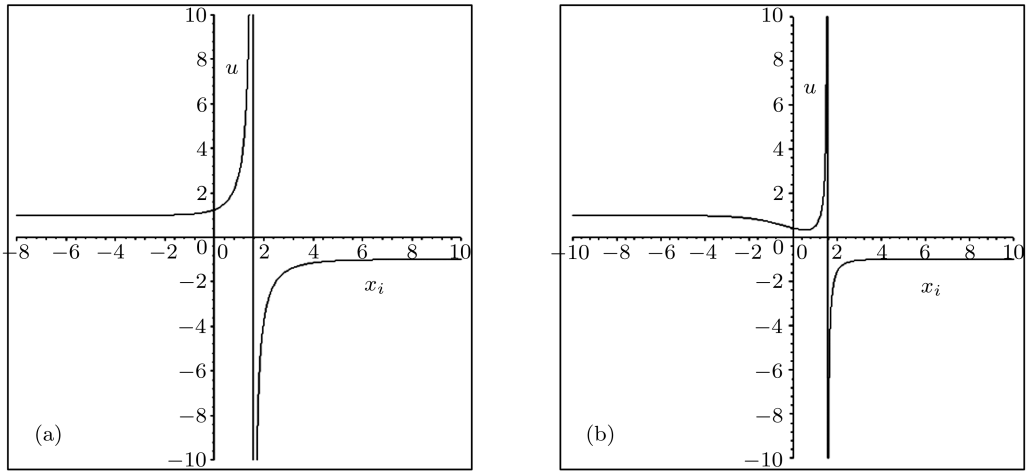


Fig.4. Plots of the travelling wave solution (63) for GBF equation with $c = d, \xi_0 = 0$, where (a) is for solution (63a), and (b) is for solution (63b).

As for the case of Eqs.(56)–(58), from Eq.(6), we obtain

$$y_4(\nu) = \left(\frac{1}{2} - \frac{\nu}{3} \pm \frac{\sqrt{15-10\nu}}{6}\right) A\nu, \tag{65}$$

$$y_5(\nu) = \left(\frac{4}{5} - \frac{2\nu}{5} \pm \frac{2\sqrt{4-2\nu}}{5}\right) A(\nu+1), \tag{66}$$

$$y_6(\nu) = (-2\nu - 1 + \sqrt{-2\nu-1})A(\nu-1). \tag{67}$$

Unfortunately, it is not convenient to derive the explicit solutions for GBF equation from Eqs.(65)–(67), so we omit these implicit solutions in such cases.

4. Conclusion

Using the first integral technique, we have obtained various kinds of new solutions of the BF equation and GBF equation. These solutions include exponential function solutions, singular solitary wave solutions, some more complex solutions, whose figures have been given in the text, and some implicit solutions. From these results, we can see that the first integral method is one of the most effective methods to obtain exact solutions, especially for non-integrable equations.

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