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Asymptotical solutions of coupled nonlinear Schrödinger equations with perturbations*

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In this paper Lou's direct perturbation method is applied to the perturbed coupled nonlinear Schrödinger equations to obtain their asymptotical solutions, which include not only the zero-order solutions but also the first-order modifications. Based on the asymptotical solutions, the effects of perturbations on soliton parameters and the collision between two solitons are then discussed in brief. Furthermore, we directly simulate the perturbed coupled nonlinear Schrödinger equations by split-step Fourier method to check the validity of the direct perturbation method. It turns out that our analytical results are well supported by the numerical calculations.

Keywords: direct perturbation method, perturbed coupled nonlinear Schrödinger equations, solitons, asymptotical solutions

PACC: 0260, 0340K, 4735

1. Introduction

The general form of the perturbed coupled nonlinear Schrödinger equations can be written as

$$\begin{aligned} &\mathrm{i}u_t + \alpha_1 u_{xx} + (\sigma_1|u|^2 + \gamma_1|v|^2)u \\ &= \epsilon U(u, v, u_x, v_x, u_t, v_t, \cdots), \\ &\mathrm{i}v_t + \alpha_2 v_{xx} + (\sigma_2|v|^2 + \gamma_2|u|^2)v \\ &= \epsilon V(v, u, v_x, u_x, u_t, v_t, \cdots). \end{aligned} \tag{1}$$

Equation (1) has been applied in various physical fields, such as nonlinear optics,[1−3] condense matter physics,[4,5] geophysics fluid dynamics,[6] plasma physics,[7,8] etc. In the context of nonlinear optics, u and v in Eq.(1) describe the envelopes of the waves with mutually orthogonal linear polarization, x and t correspond to normalized distance and time, respectively. The parameters $\alpha_{1,2}$ are the dispersion coefficients, $\sigma_{1,2}$ the Landau constants which describe the self-modulation of the wave packets, and $\gamma_{1,2}$ the wave-wave interaction coefficients which describe the cross-modulations of the wave packets. ϵU and ϵV are perturbations of arbitrary form, and ϵ is a small parameter characterizing the magnitude of these perturbations. When $\epsilon = 0$, Eq.(1) reduces into the coupled nonlinear Schrödinger equation (CNLSE). Manakov[9] was the first to show that Eq.(1) is integrable in terms of the method of inverse scattering transform (IST) for the case of $\alpha_1 = \alpha_2 = 1$, $\sigma_1 = \sigma_2 = \gamma_1 = \gamma_2$. Later, Zakharov and Schulman[10] studied the integrability of system (1) thoroughly and they concluded that only when $\alpha_1 = \alpha_2$, $\sigma_1 = \sigma_2 = \gamma_1 = \gamma_2$ or $\alpha_1 = -\alpha_2$, $\sigma_1 = \sigma_2 = -\gamma_1 = -\gamma_2$, system (1) is integrable, i.e. it can be solved by the IST method.

To study the perturbed CNLSE, several effective perturbation methods have been well developed.[11] One of the most powerful techniques is the perturbed IST method.[12] This technique requires the unperturbed equations to be exactly solvable by the IST, which restricts the range of applications, but enables one to solve the most sophisticated dynamical problems. However, it is rather inconvenient for one who is not familiar with IST. Another famous perturbation method is the direct perturbation method,[13,14] which does not require the unperturbed equations to be exactly integrable. But as a rule, direct perturbation method is applicable to one-soliton problems only, it cannot be used to solve the perturbation problems of multi-soliton. Recently, Lou[15] proposed a new direct perturbation method. This method relies on no tools of IST and is easy to understand. In the framework of this method, different types of perturbation solutions can be obtained at the same time, instead of the

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concrete one-soliton or multi-soliton solutions. So far, this direct method has been successfully applied to the perturbed NLSE and the NLSE in critical dimension with a small time-dispersion.[16]

The motivation of this paper is to apply Lou's direct perturbation method to perturbed CNLSE, to obtain asymptotical solutions. In addition, this method also permits us to take into account the effects of linear damping on solitons, where the wave propagation is subject to small perturbations which break integrability. To identify the validity of this direct perturbation method, we compare the results of analytical method with those of direct numerical calculations.

2. Direct perturbation method for perturbed CNLSE

For simplicity and convenience, we take the coefficients $\alpha_1 = \alpha_2 = \sigma_1 = \sigma_2 = \gamma_1 = \gamma_2 = 1$ in Eq.(1), and $\epsilon U(u, v, u_x, v_x, \cdots) = -\mathrm{i}\epsilon u$, $\epsilon V(v, u, v_x, u_x, \cdots) = -\mathrm{i}\epsilon v$, corresponding to the action of linear damping in a fibre system.[17] System (1) is rewritten as

$$\mathrm{i}u_t + u_{xx} + (|u|^2 + |v|^2)u = -\mathrm{i}\epsilon u,$$
$$\mathrm{i}v_t + v_{xx} + (|v|^2 + |u|^2)v = -\mathrm{i}\epsilon v. \tag{2}$$

As we know, several perturbation methods[18] have been employed to obtain the zero-order solutions of Eq.(2), but perturbation methods for deriving the first-order modifications have still been lacking.

Since the zero-order solutions of Eq.(2) should be the adiabatic ones, we expand u and v as follows:

$$\begin{aligned} u &= \mathrm{e}^{\epsilon(a+\mathrm{i}b)}u'(\xi, \tau, \epsilon) \\ &= \mathrm{e}^{\epsilon(a+\mathrm{i}b)}[u_0(\xi, \tau) + \epsilon u_1(\xi, \tau) + O(\epsilon^2)], \\ v &= \mathrm{e}^{\epsilon(c+\mathrm{i}d)}v'(\xi, \tau, \epsilon) \\ &= \mathrm{e}^{\epsilon(c+\mathrm{i}d)}[v_0(\xi, \tau) + \epsilon v_1(\xi, \tau) + O(\epsilon^2)], \end{aligned} \tag{3}$$

where $a \equiv a(\ x,\ t)$, $b \equiv b(\ x,\ t)$, $c \equiv c(\ x,\ t)$, $d \equiv d(\ x,\ t)$, $\xi \equiv \xi(\ x,\ t,\ \epsilon)$, $\eta \equiv \eta(\ x,\ t,\ \epsilon)$ are all real functions of indicated variables. ξ, τ should satisfy the relation

$$\{\ \xi,\ \tau\} \overset{\epsilon\to 0}{\longrightarrow} \{\ x,\ t\}, \tag{4}$$

such that u_0, v_0 would be the exact solutions of the CNLSE as $\epsilon \to 0$.

Substituting expansions (3) into Eq.(2), we obtain

$$\begin{aligned} &\mathrm{i}u_{0\tau}\tau_t + u_{0\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon a}|u_0|^2u_0 + \mathrm{e}^{2\epsilon c}|v_0|^2u_0 + \epsilon[\mathrm{i}u_{1\tau}\tau_t + u_{1\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon a}({u_1^*}{u_0}^2 + 2|u_0|^2u_1) + \mathrm{e}^{2\epsilon c}(u_1|v_0|^2 \\ &+u_0v_1v_0^* + u_0v_0v_1^*)] + \mathrm{i}\epsilon(a_t + \mathrm{i}b_t)u_0 + \epsilon(a_{xx} + \mathrm{i}b_{xx})u_0 + \mathrm{i}\epsilon u_0 + \mathrm{i}\xi_t u_{0\xi} + \xi_{xx}u_{0\xi} + 2\epsilon(a_x + \mathrm{i}b_x)\xi_x u_{0\xi} \\ &+2\xi_x\tau_x u_{0\xi\tau} + \tau_{xx}u_{0\tau} = O(\epsilon^2), \\ &\mathrm{i}v_{0\tau}\tau_t + v_{0\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon c}|v_0|^2v_0 + \mathrm{e}^{2\epsilon a}|u_0|^2v_0 + \epsilon[\mathrm{i}v_{1\tau}\tau_t + v_{1\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon c}({v_1^*}{v_0}^2 + 2|v_0|^2v_1) \\ &+\mathrm{e}^{2\epsilon a}(v_1|u_0|^2 + v_0u_1u_0^* + v_0u_0u_1^*)] + \mathrm{i}\epsilon(c_t + \mathrm{i}d_t)v_0 + \epsilon(c_{xx} + \mathrm{i}d_{xx})v_0 + \mathrm{i}\epsilon v_0 + \mathrm{i}\xi_t v_{0\xi} + \xi_{xx}v_{0\xi} \\ &+2\epsilon(c_x + \mathrm{i}d_x)\xi_x v_{0\xi} + 2\xi_x\tau_x v_{0\xi\tau} + \tau_{xx}v_{0\tau} = O(\epsilon^2), \end{aligned} \tag{5}$$

where the asterisk stands for complex conjugation. We have considered the fact that ξ_t, τ_x and ξ_{xx} are all in ϵ order because of Eq.(4). Vanishing the coefficients of ϵ^0 and ϵ^1 orders in Eq.(5), we have

$$\begin{aligned} &\mathrm{i}u_{0\tau}\tau_t + u_{0\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon a}|u_0|^2u_0 + \mathrm{e}^{2\epsilon c}|v_0|^2u_0 = 0, \\ &\mathrm{i}v_{0\tau}\tau_t + v_{0\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon c}|v_0|^2v_0 + \mathrm{e}^{2\epsilon a}|u_0|^2v_0 = 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\epsilon[\mathrm{i}u_{1\tau}\tau_t + u_{1\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon a}({u_1^*}{u_0}^2 + 2|u_0|^2u_1) + \mathrm{e}^{2\epsilon c}(u_1|v_0|^2 + u_0v_1v_0^* + u_0v_0v_1^*)] + \mathrm{i}\epsilon(a_t + \mathrm{i}b_t)u_0 \\ &+\epsilon(a_{xx} + \mathrm{i}b_{xx})u_0 + \mathrm{i}\epsilon u_0 + \mathrm{i}\xi_t u_{0\xi} + \xi_{xx}u_{0\xi} + 2\epsilon(a_x + \mathrm{i}b_x)\xi_x u_{0\xi} + 2\xi_x\tau_x u_{0\xi\tau} + \tau_{xx}u_{0\tau} = 0, \\ &\epsilon[\mathrm{i}v_{1\tau}\tau_t + v_{1\xi\xi}{\xi_x}^2 + \mathrm{e}^{2\epsilon c}({v_1^*}{v_0}^2 + 2|v_0|^2v_1) + \mathrm{e}^{2\epsilon a}(v_1|u_0|^2 + v_0u_1u_0^* + v_0u_0u_1^*)] + \mathrm{i}\epsilon(c_t + \mathrm{i}d_t)v_0 \\ &+\epsilon(c_{xx} + \mathrm{i}d_{xx})v_0 + \mathrm{i}\epsilon v_0 + \mathrm{i}\xi_t v_{0\xi} + \xi_{xx}v_{0\xi} + 2\epsilon(c_x + \mathrm{i}d_x)\xi_x v_{0\xi} + 2\xi_x\tau_x v_{0\xi\tau} + \tau_{xx}v_{0\tau} = 0. \end{aligned} \tag{7}$$

Based on the fact that u_0 and v_0 in Eq.(6) are not explicitly dependent on ϵ and the limiting property of Eq.(4) of ξ and τ, it can be seen that the adiabatic variables ξ and τ are related to functions a and c by

$$\xi_x = \mathrm{e}^{\epsilon a} = \mathrm{e}^{\epsilon c} \quad \text{and} \quad \tau_t = \mathrm{e}^{2\epsilon a} = \mathrm{e}^{2\epsilon c}. \tag{8}$$

This implies that $a = c$. Therefore, the functions u_0 and v_0 should be the exact solutions of

$$\begin{aligned} &\mathrm{i}u_{0\tau} + u_{0\xi\xi} + (|u_0|^2 + |v_0|^2)u_0 = 0, \\ &\mathrm{i}v_{0\tau} + v_{0\xi\xi} + (|v_0|^2 + |u_0|^2)v_0 = 0. \end{aligned} \tag{9}$$

From Eq.(3), u_1 and v_1 are just the symmetries of the adiabatic CNLSE (9), i.e., they are the solutions of the linearized equations of Eq.(9),

$$\begin{aligned} &\mathrm{i}u_{1\tau} + u_{1\xi\xi} + u_1^* {u_0}^2 + 2|u_0|^2 u_1 + u_1|v_0|^2 \\ &+u_0 v_1 v_0^* + u_0 v_0 v_1^* = 0, \\ &\mathrm{i}v_{1\tau} + v_{1\xi\xi} + v_1^* {v_0}^2 + 2|v_0|^2 v_1 + v_1|u_0|^2 \\ &+v_0 u_1 u_0^* + v_0 u_0 u_1^* = 0. \end{aligned} \tag{10}$$

It is known that the two simplest solutions of Eq.(10) can be written as

$$u_1 = u_{0\xi},\ v_1 = v_{0\xi} \quad \text{and} \quad u_1 = u_{0\tau},\ v_1 = v_{0\tau}, \tag{11}$$

which are related to the variables ξ and τ.

Equations (7) and (10) with Eq.(8) are consistent with each other at any space x and time t for the given nontrivial solutions u_0 and v_0, which leads to that the rest terms of Eq.(7) should be all zero, i.e.,

$$b_{xx} + a_t + 1 = 0, \quad a_{xx} - b_t = 0, \tag{12}$$

$$\xi_{xx} + 2\epsilon a_x \xi_x = 0, \quad \xi_t + 2\epsilon b_x \xi_x = 0, \quad \tau_x = 0, \tag{13}$$

$$d_{xx} + c_t + 1 = 0, \quad c_{xx} - d_t = 0, \tag{14}$$

$$\xi_{xx} + 2\epsilon c_x \xi_x = 0, \quad \xi_t + 2\epsilon d_x \xi_x = 0, \quad \tau_x = 0. \tag{15}$$

By solving Eqs.(12)–(15) and (8), the following expressions for parameters $a,\ b,\ c,\ d,\ \xi,\ \tau$ can be obtained

$$\begin{aligned} &a = c = -2t, \quad b = \frac{x^2}{2} + c_1, \\ &d = \frac{x^2}{2} + c_2, \quad \xi = x\mathrm{e}^{-2\epsilon t}, \quad \tau = \frac{1 - \mathrm{e}^{-4\epsilon t}}{4\epsilon}, \end{aligned} \tag{16}$$

where $c_1,\ c_2$ are arbitrary constants.

The asymptotical solutions of perturbed CNLSE (2) can now be written as

$$\begin{aligned} u &= \mathrm{e}^{-\epsilon(2t - \mathrm{i}\frac{x^2}{2})}(u_0 + \epsilon u_1), \\ v &= \mathrm{e}^{-\epsilon(2t - \mathrm{i}\frac{x^2}{2})}(v_0 + \epsilon v_1), \end{aligned} \tag{17}$$

where we have taken $c_1 = c_2 = 0$.

In the usual perturbation methods, one has to treat different types of solutions (e.g., solutions with different boundary conditions) separately; however, in the present direct perturbation method, we can obtain all the perturbation solutions at the same time. u_0 and v_0 appearing in Eq.(17) can be any one of the exact solutions of the adiabatic CNLSE (9), and the first-order modifications u_1 and v_1 can be any kind of the symmetries of Eq.(9). As is well known, the solutions of the CNLSE have been well studied.[19−22] There are one-soliton solutions, multi-soliton solutions, periodic solutions, etc..

The one-soliton solutions of Eq.(9) have the form

$$\begin{aligned} u_0 &= \cos(\theta)\mathrm{e}^{\mathrm{i}\delta}\sqrt{2}p\mathrm{e}^{\mathrm{i}\{q\xi + (p^2 - q^2)\tau + \zeta_0\}}\mathrm{sech}(p(\xi - 2q\tau) + \gamma_0), \\ v_0 &= \sin(\theta)\mathrm{e}^{-\mathrm{i}\delta}\sqrt{2}p\mathrm{e}^{\mathrm{i}\{q\xi + (p^2 - q^2)\tau + \zeta_0\}}\mathrm{sech}(p(\xi - 2q\tau) + \gamma_0), \end{aligned} \tag{18}$$

where $p,\ q,\ \theta,\ \delta,\ \gamma_0,\ \zeta_0$ are all real arbitrary constants. We can see that the soliton has initial central position $-\gamma_0/p$, velocity $2q$, amplitude $\sqrt{2}p$, etc.. When $u_1 = u_{0\tau}$ in Eq.(17), the corresponding asymptotical solutions of perturbed CNLSE (2) take the form

$$\begin{aligned} u =\ & \cos\theta\mathrm{e}^{\mathrm{i}\delta}\sqrt{2}p\mathrm{e}^{-2\epsilon t}[1 + 2\epsilon pq\tanh\phi + \mathrm{i}\epsilon(p^2 - q^2)]\mathrm{sech}\phi \\ &\times\exp\left\{\frac{\mathrm{i}(2x^2\epsilon^2 + 4qx\mathrm{e}^{-2\epsilon t}\epsilon + (p^2 - q^2)(1 - \mathrm{e}^{-4\epsilon t}) + 4\epsilon\zeta_0)}{4\epsilon}\right\}, \\ v =\ & \sin\theta\mathrm{e}^{-\mathrm{i}\delta}\sqrt{2}p\mathrm{e}^{-2\epsilon t}[1 + 2\epsilon pq\tanh\phi + \epsilon(p^2 - q^2)\mathrm{i}]\mathrm{sech}\phi \\ &\times\exp\left\{\frac{\mathrm{i}(2x^2\epsilon^2 + 4qx\mathrm{e}^{-2\epsilon t}\epsilon + (p^2 - q^2)(1 - \mathrm{e}^{-4\epsilon t}) + 4\epsilon\zeta_0)}{4\epsilon}\right\}, \end{aligned} \tag{19}$$

where

$$\phi(x,t) = \frac{2px\epsilon\mathrm{e}^{-2\epsilon t} - pq + pq\mathrm{e}^{-4\epsilon t} + 2\gamma_0\epsilon}{2\epsilon}.$$

Comparing Eq.(19) with Eq.(18), we find that soliton amplitudes and phases have been changed due to the small perturbations. By expanding ϕ in Taylor series up to ϵ order, we obtain

$$\phi(x,t) \approx px - 2pqt + \gamma_0 + (4pqt^2 - 2pxt)\epsilon = p(1-2\epsilon t)(x-2qt) + \gamma_0. \tag{20}$$

It is not difficult to find that the soliton velocities $2q$ do not change, but the wave numbers and the initial central positions are shifted by linear damping. The shifted values can be obtained directly from Eq.(20).

The most general expressions for two-soliton solutions of adiabatic CNLSE (9) are

$$u_0 = \frac{\alpha_1 \mathrm{e}^{\eta_1} + \alpha_2 \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+S_1} + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*+S_2}}{1 + \mathrm{e}^{\eta_1+\eta_1^*+R_1} + \mathrm{e}^{\eta_1+\eta_2^*+S_0} + \mathrm{e}^{\eta_1^*+\eta_2+S_0^*} + \mathrm{e}^{\eta_2+\eta_2^*+R_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}}, \tag{21}$$

$$v_0 = \frac{\beta_1 \mathrm{e}^{\eta_1} + \beta_2 \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+S_1'} + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*+S_2'}}{1 + \mathrm{e}^{\eta_1+\eta_1^*+R_1} + \mathrm{e}^{\eta_1+\eta_2^*+S_0} + \mathrm{e}^{\eta_1^*+\eta_2+S_0^*} + \mathrm{e}^{\eta_2+\eta_2^*+R_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}}, \tag{22}$$

where

$$\alpha_s = \cos(\theta_s)\mathrm{e}^{\mathrm{i}\delta_s}, \quad \beta_s = \sin(\theta_s)\mathrm{e}^{-\mathrm{i}\delta_s}, \quad \eta_s = k_s\xi + \mathrm{i}k_s^2\tau + \eta_{s0}, \quad k_s = p_s + \mathrm{i}q_s, \tag{23}$$

$$\eta_{s0} = \gamma_{s0} + \mathrm{i}\zeta_{s0}, \quad \mathrm{e}^{S_0} = \frac{\kappa_{12}}{k_1 + k_2^*}, \quad \mathrm{e}^{R_1} = \frac{\kappa_{11}}{k_1 + k_1^*}, \quad \mathrm{e}^{R_2} = \frac{\kappa_{22}}{k_2 + k_2^*}, \tag{24}$$

$$\mathrm{e}^{S_1} = \frac{k_1 - k_2}{(k_1 + k_1^*)(k_1 + k_2^*)}(\alpha_1\kappa_{21} - \alpha_2\kappa_{11}), \quad \mathrm{e}^{S_1'} = \frac{k_1 - k_2}{(k_1 + k_1^*)(k_1^* + k_2)}(\beta_1\kappa_{21} - \beta_2\kappa_{11}), \tag{25}$$

$$\mathrm{e}^{S_2} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_2^*)}(\alpha_2\kappa_{12} - \alpha_1\kappa_{22}), \quad \mathrm{e}^{S_2'} = \frac{k_2 - k_1}{(k_2 + k_2^*)(k_1 + k_2^*)}(\beta_2\kappa_{12} - \beta_1\kappa_{22}), \tag{26}$$

$$\mathrm{e}^{R_3} = \frac{|k_1 - k_2|^2}{(k_1 + k_1^*)(k_2 + k_2^*)|k_1 + k_2|^2}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}), \quad \kappa_{lj} = \frac{\alpha_l\alpha_j^* + \beta_l\beta_j^*}{2(\alpha_l + \alpha_j^*)} \quad (l, j = 1, 2), \tag{27}$$

and p_s, q_s, θ_s, δ_s, γ_{s0}, ζ_{s0} $(s = 1, 2)$ are all real constants. Once these actual parameters of the initial solitons are given, the exact soliton solutions can then be well determined. For further studying, let us consider the effects of perturbations on two-soliton solutions of CNLSE with the help of Maple.

Figure 1(a) plots the evolution of intensity profile $|u_0|^2$ of two-soliton solution; the parameters in Eqs.(23)–(27) are set at $k_1 = 0.5 + 0.5\mathrm{i}$, $k_2 = 0.55 - 0.25\mathrm{i}$, $\alpha_1 = \alpha_2 = \beta_1 = 1$, and $\beta_2 = \eta_{10} = \eta_{20} = 0$. It is an inelastic collision for these parameters. Some general behaviours of solitons in this case can be found in Ref.[19]. The profiles of the two solitons are split between two components initially, but after collision, one of them possesses only one component. Figure 1(b) illustrates the evolution of intensity profile $|u|^2$ of solution of the perturbed CNLSE (2) with $\epsilon = 0.02$ (we have taken $u_1 = u_{0\tau}$ in Eq.(17)). Because of the action of linear damping, the two-soliton solution decays with time t during the whole collision. For larger ϵ values, the soliton amplitudes decay faster.

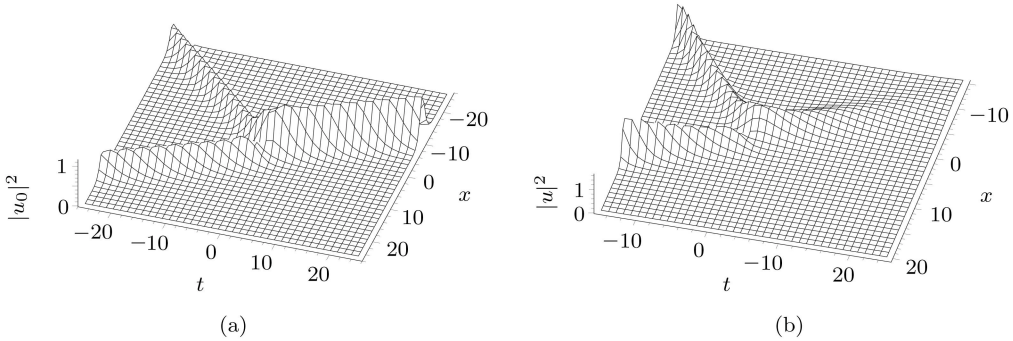


Fig.1. (a) Evolution of intensity profile $|u_0|^2$ of two-soliton solutions of CNLSE and (b) intensity profile $|u|^2$ of solution of perturbed CNLSE with $\epsilon = 0.02$. The parameters in Eqs.(23)–(27) are chosen to be $k_1 = 0.5 + 0.5\mathrm{i}$, $k_2 = 0.55 - 0.25\mathrm{i}$, $\alpha_1 = \alpha_2 = \beta_1 = 1$, $\beta_2 = \eta_{10} = \eta_{20} = 0$.

Proceeding in a similar manner, the elastic collision can be obtained by taking the parameters in Eqs.(23)–(27) as $k_1 = 1 + \mathrm{i},\ k_2 = 1 - 0.5\mathrm{i}$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, and $\eta_{10} = \eta_{20} = 0$. This is shown in Fig.2(a). The most apparent feature in this case is that, there is no intensity redistribution among the modes both before and after collision. However, under the action of the small linear damping terms in Eq.(2), the slow decay of the amplitudes of each bright solitons throughout the whole collision. This can be seen from Fig.2(b). One more feature in Fig.2(b) is that, the perturbations do not introduce any change in the collision itself, i.e., the intensities between the two components are the same at any time t instead of being redistributed. Moreover, the effects of perturbations on other types of solutions of CNLSE (e.g., the periodic solutions) can also be analysed in a similar manner.

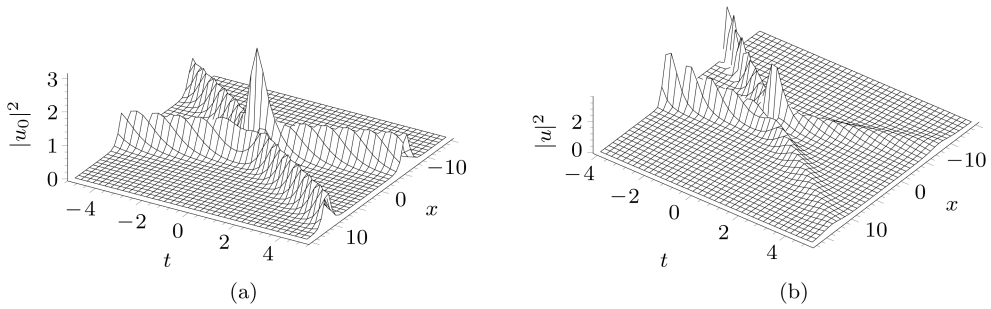


Fig.2. (a) Evolution of intensity profile $|u_0|^2$ of two-soliton solutions of CNLSE and (b) intensity profile $|u|^2$ of solution of perturbed CNLSE with $\epsilon = 0.08$. The parameters are set at $k_1 = 1 + \mathrm{i},\ k_2 = 1 - 0.5\mathrm{i}$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, $\eta_{10} = \eta_{20} = 0$.

3. Numerical simulation and discussion

In order to verify the validity of the direct perturbation method, we simulate directly the perturbed CNLSE (2) via the split-step Fourier method, which is one of the most popular numerical methods for solving nonlinear equations.[23]

Figure 3 illustrates the time evolution of the amplitudes of solutions of Eq.(2) with different ϵ values, where the solid lines are from analytical calculations, and the dashed ones are from numerical simulations. Figure 3(a) is obtained for the perturbations of linear damping to one-soliton solutions. The initial conditions are given by Eq.(17), and the parameters in Eq.(18) are chosen to be $p = q = 0.5$, $\theta = \dfrac{\pi}{4}$, and $\delta = \gamma_0 = \zeta_0 = 0$. As predicted, the decay of amplitudes of solutions with the increase of time t is observed. And the agreement between the analytical and numerical values is very good. The numerical calculation also turns out that even in the presence of relatively large perturbations, the direct perturbation method is able to reproduce the numerically observed behaviour of solutions. For example, the mean absolute error of the amplitudes between the analytical and numerical results is about 0.0012 when $\epsilon = 0.1$.

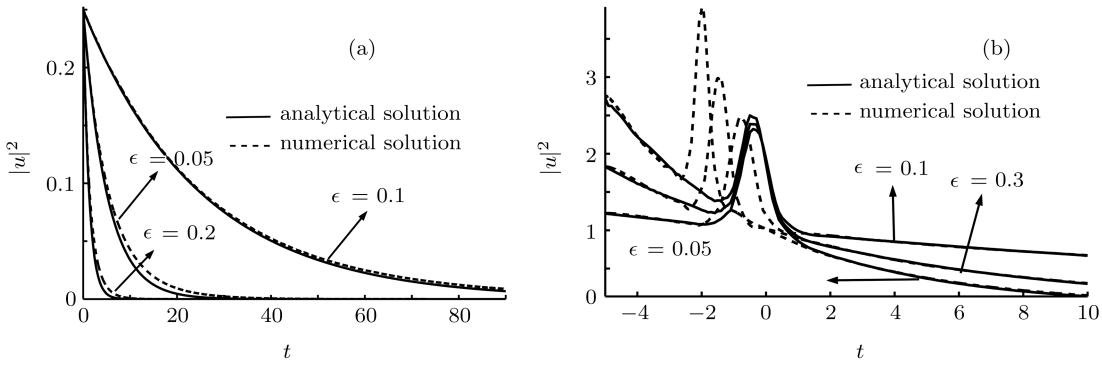


Fig.3. The time evolution of amplitudes of one-soliton solutions (a) and two-soliton solutions (b) of Eq.(2) with different ϵ values. The solid lines are from analytical calculations, and the dashed ones are from numerical simulations.

Figure 3(b) is obtained for the perturbations of linear damping to two-soliton solutions of CNLSE, where the initial values are taken as Eq.(17), and $k_1 = 1 + \mathrm{i}$, $k_2 = 1 - 0.5\mathrm{i}$, $\alpha_1 = \beta_1 = \eta_{10} = 1$, $\alpha_2 = \beta_2 = \dfrac{\sqrt{2}}{2}$, and $\eta_{20} = 2$ in Eq.(18). It is an elastic collision for these parameters. Moreover, the amplitudes of the two components are always the same at any time. So they can be described by one line in Fig.3(b). It is clear that there is a qualitative agreement between the analytical and numerical values both before and after collision. The mismatch of the soliton amplitudes increases when the solitons come into collision. For instance, the agreement between the analytical and numerical results remains good when $\epsilon = 0.01$, but a slight change of ϵ value introduces a progressively larger shift between analytical and numerical results. The amplitudes of numerical solutions are larger than those of analytical ones. Furthermore, the shifted values of the collision positions between analytical and numerical results have a slow increase with the increase of ϵ values.

As is known, the loss coefficients in fibres can be very small,[24] for example, it may be 0.15dB/km at $\lambda = 1.55\mu$m. We hope that our theoretical results will be checked in the fibre system and other physical systems by experimental physicists, and the new direct perturbation method can then play its role in real physical applications. However, the mismatch between analytical and numerical values gradually increases when ϵ becomes larger, and the amplitudes tend to zero more quickly, thus the higher-order modifications have to be taken into account and more effective perturbation theories are required.

4. Conclusions

In this paper, we have successfully applied Lou's direct perturbation method to the CNLSE with perturbations of $-\mathrm{i}\epsilon u$ and $-\mathrm{i}\epsilon v$, and obtained the asymptotical solutions including not only the zero-order solutions but also the first-order ones. In the asymptotical solutions, u_0 and v_0 can be any solutions of the CNLSE, such as the one-soliton solutions, the multi-soliton solutions, the periodic solutions, etc, and u_1 and v_1 can be any symmetries of CNLSE. So the solutions obtained by this direct perturbation method are more abundant than those obtained by other methods. In usual perturbation methods, one has to treat different types of solutions (e.g., solutions with different boundary conditions) separately, but in our treatment, all the perturbation solutions can be obtained at the same time. For further studying, we take the soliton solutions of perturbed CNLSE (2) as typical examples to analyse the effects of perturbations on solitons. It is shown that, in the presence of perturbations of linear damping, the amplitudes, the phases and the initial positions of solitons are changed, but there is no change in the soliton velocities. To verify the validity of the direct perturbation method, we have solved the perturbed CNLSE numerically. It turns out that our analytical results agree well with the numerical calculations (see Fig.3). In addition, we believe that Lou's direct perturbation method can also be extended to the N-components NLSE. And how to apply the direct perturbation method to system (1) with other different perturbations, such as $\epsilon|u|^4u$, $\epsilon u(|u|^2)_t$, ϵu_{xxx}, etc., and other complicated perturbed nonlinear equations (such as sine-Gordon equation (SGE), Kadomtsev–Petviashvili equation (KPE), etc.) are also worthy of further study.

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